Remarks on neighborhood star-Lindelöf spaces

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Abstract. A space $X$ is said to be neighborhood star-Lindelöf if for every open cover $U$ of $X$ there exists a countable subset $A$ of $X$ such that for every open $O \supseteq A$, $X = St(O, U)$. In this paper, we continue to investigate the relationship between neighborhood star-Lindelöf spaces and related spaces, and study topological properties of neighborhood star-Lindelöf spaces.

1. Introduction

By a space, we mean a topological space. In the rest of this section, we give definitions of terms which are used in this paper. Let $X$ be a space and $U$ a collection of subsets of $X$. For $A \subseteq X$, let $St(A, U) = \bigcup \{ U \in U : U \cap A \neq \emptyset \}$. As usual, we write $St(x, U)$ for $St(\{x\}, U)$.

Recall that a space $X$ is strongly starcompact (see [5,7,8] - under different name) if for every open cover $U$ of $X$ there exists a finite subset $A$ of $X$ such that $X = St(A, U)$; A space $X$ is strongly star-Lindelöf (see [1, 2, 5, 8, 9] - under different name) if for every open cover $U$ of $X$ there exists a countable subset $A$ of $X$ such that $X = St(A, U)$; A space $X$ is starcompact (resp., star-Lindelöf)(see [5, 8] - under different name) if for every open cover $U$ of $X$ there exists a finite (resp., countable) subset $V$ of $U$ such that $X = St(\bigcup V, U)$. Clearly, every strongly starcompact space is strongly star-Lindelöf, every strongly starcompact space starcompact, every strongly star-Lindelöf space is star-Lindelöf and every strongly star-Lindelöf space is star-Lindelöf. It is known that every countably compact space is strongly starcompact, and every Hausdorff strongly starcompact space is countably compact (see [5, 8]).

It is natural in this context to introduce the following definitions:

**Definition 1.1.** ([3]) A space $X$ is said to be weakly starcompact if for every open cover $U$ of $X$ there exists a finite subset $A$ of $X$ such that for every open $O \supseteq A$, $X = St(O, U)$.

**Definition 1.2.** ([4]) A space $X$ is said to be neighborhood star-Lindelöf if for every open cover $U$ of $X$ there exists a countable subset $A$ of $X$ such that for every open $O \supseteq A$, $X = St(O, U)$.

From the definitions, it is clear that every weakly starcompact space is neighborhood star-Lindelöf, every strongly star-Lindelöf space is neighborhood star-Lindelöf space and every neighborhood star-Lindelöf space is star-Lindelöf.
The purpose of this note is to investigate the relationship between neighborhood star-Lindelöf spaces and related spaces, and study topological properties of neighborhood star-Lindelöf spaces.

Throughout this paper, let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $\kappa$ the cardinality of the set of all real numbers. For a cardinal $\kappa$, let $\kappa^+$ be the smallest cardinal greater than $\kappa$. For each pair of ordinals $\alpha, \beta$ with $\alpha < \beta$, we write $[\alpha, \beta) = \{y : \alpha \leq y < \beta\}$, $(\alpha, \beta] = \{y : \alpha < y \leq \beta\}$, $(\alpha, \beta) = \{y : \alpha < y < \beta\}$ and $[\alpha, \beta] = \{y : \alpha \leq y \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [6].

2. Neighborhood Star-Lindelöf spaces and Related Spaces

In this section, we give some examples to clarify the relationship between neighborhood star-Lindelöf spaces and related spaces. Recall that a space is called Urysohn if every two distinct points have neighborhood with disjoint closures. Clearly, the property is between the Hausdorff condition and regularity. Bonanzinga et al. in [3] showed that the three properties, countable compactness, strongly starcompactness, and weak starcompactness are equivalent for Urysohn spaces.

**Example 2.1.** There exists a Tychonoff neighborhood star-Lindelöf space $X$ that is not weakly starcompact.

**Proof.** Let $X = \omega$ be the countably infinite discrete space. Clearly, $X$ is not weakly starcompact. Since $X$ is countable, then $X$ is strongly star-Lindelöf, hence $X$ is neighborhood star-Lindelöf, since every strongly star-Lindelöf space is neighborhood star-Lindelöf. $\square$

For the next example, we need the following Lemmas.

**Lemma 2.2.** A space $X$ having a dense Lindelöf subspace is star-Lindelöf.

**Proof.** Let $X$ have a dense Lindelöf subspace $D$. We show that $X$ is star-Lindelöf. Let $\mathcal{U}$ be an open cover of $X$. Since $D$ is a dense Lindelöf subset of $X$, then there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $D \subseteq \bigcup \mathcal{V}$. Hence $\text{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$, which shows that $X$ is star-Lindelöf. $\square$

**Lemma 2.3.** ([4]) A space $X$ is neighborhood star-Lindelöf if and only if for every open cover $\mathcal{U}$ of $X$ there exists a countable subset $A$ of $X$ such that $\text{St}(A, \mathcal{U}) \cap A \neq \emptyset$ for each $x \in X$.

**Example 2.4.** There exists a Tychonoff star-Lindelöf space that is not neighborhood star-Lindelöf.

**Proof.** Let $D = \{d_\alpha : \alpha < \xi\}$ be a discrete space of cardinality $\xi$ and let $Y = D \cup \{y_0\}$ be one-point compactification of $D$.

Let $X = (Y \times [0, \omega)) \cup (D \times [\omega])$

be the subspace of the product space $Y \times [0, \omega]$. Then $X$ is star-Lindelöf by lemma 2.2, since $Y \times [0, \omega)$ is a dense Lindelöf subset of $X$.

Next we show that $X$ is not neighborhood star-Lindelöf. For each $\alpha < \xi$, let $U_\alpha = \{d_\alpha\} \times [0, \omega]$.

Then $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$.

Let $\mathcal{U} = \{U_\alpha : \alpha < \xi\} \cup \{Y \times [0, \omega]\}$.

Then $\mathcal{U}$ is an open cover of $X$. Let us consider the open cover $\mathcal{U}$ of $X$. It suffices to show that for any countable subset $A$ of $X$, there exists a point $x \in X$ such that $\text{St}(x, \mathcal{U}) \cap A = \emptyset$ by Lemma 2.3. Let $A$ be any countable subset of $X$. Then $\{\alpha : A \cap U_\alpha \neq \emptyset\}$ is countable. Pick $a_0 < \xi$ such that $A \cap U_{a_0} = \emptyset$. Since $U_{a_0}$ is the only element of $\mathcal{U}$ containing the point $(d_{a_0}, \omega)$, then $\text{St}((d_{a_0}, \omega), \mathcal{U}) = U_{a_0}$. By the constructions of the topology of $X$ and the open cover $\mathcal{U}$, we have $\text{St}((d_{a_0}, \omega), \mathcal{U}) = U_{a_0}$. Thus we complete the proof. $\square$
Remark 2.5. Bonanzinga et al. in [4] showed that there exists a Urysohn neighborhood star-Lindelöf space that is not strongly star-Lindelöf. But the author does not know if there exists a Tychonoff example.

3. Properties of neighborhood star-Lindelöf spaces

In this section, we study topological properties of neighborhood star-Lindelöf spaces. The Isbell-Mrówka space is $X = \omega \cup R$ (see [10]), where $R$ is a maximal almost disjoint family of infinite subsets of $\omega$ with $|R| = \omega$. The space $X$ is strongly star-Lindelöf, since $\omega$ is a countable dense subset of $X$. Thus $X$ is neighborhood star-Lindelöf. The space $X$ shows that a closed subset of a Tychonoff neighborhood star-Lindelöf space $X$ need not be neighborhood star-Lindelöf, since $\mathbb{R}$ is a discrete closed subset of cardinality $\omega$. Now we give a stronger example showing that a regular-closed subset of a Tychonoff neighborhood star-Lindelöf space $X$ need not be neighborhood star-Lindelöf. Here a subset $A$ of a space $X$ is said to be regular-closed in $X$ if $\text{cl}_{X}\text{int}_{X}A = A$.

Example 3.1. There exists a Tychonoff neighborhood star-Lindelöf space having a regular-closed subspace which is not neighborhood star-Lindelöf.

Proof. Let $S_1$ be the same space $X$ in the proof of Example 2.4. Then $S_1$ is Tychonoff, not neighborhood star-Lindelöf.

Let $R$ be a maximal almost disjoint family of infinite subsets of $\omega$ with $|R| = \omega$. Let

$$S_2 = R \cup \{[0, c^*) \times \alpha \}.$$

We topologize $S_2$ as follows: $[0, c^*) \times \alpha$ has the usual product topology and is an open subspace of $X$, and a basic neighborhood of $r \in R$ takes the form

$$G_{\beta,K}(r) = ([\alpha : \beta < \alpha < c^*) \times (r \setminus K)) \cup \{r\}$$

for $\beta < c^*$ and a finite subset $K$ of $\omega$. To show that $S_2$ is neighborhood star-Lindelöf. We need only show that $S_2$ is strongly star-Lindelöf, since every strongly star-Lindelöf space is neighborhood star-Lindelöf. To this end, let $U$ be an open cover of $S_2$. For each $n \in \omega$, since $[0, c^*) \times \{n\}$ is countably compact, there exists a finite subset $F_n \subseteq [0, c^*) \times \{n\}$ such that

$$[0, c^*) \times \{n\} \subseteq \text{St}(F_n, U).$$

Let $F' = \bigcup_{n \in \omega} F_n$. Then

$$[0, c^*) \times \omega \subseteq \text{St}(F', U).$$

On the other hand, for each $r \in R$, take $U_r \in U$ with $r \in U_r$, and fix $\alpha_r < c^*$ and $n_r \in \omega$ such that

$$\{(\alpha_r, n_r) : \alpha_r < c^* \} \subseteq U_r.$$

For each $n \in \omega$, let

$$R_n = \{r \in R : n_r = n\}$$

and $\alpha'_n = \sup\{\alpha_r : r \in R_n\}$. Then $\alpha'_n < c$, since $|R_n| \leq \omega$. Pick $\alpha_n > \alpha'_n$. Then $R_n \subseteq \text{St}(\{\alpha_n, n\}, U)$. Thus, if we put $F'' = \{\langle \alpha_n, n \rangle : n \in \omega\}$, then $R \subseteq \text{St}(F'', U)$. Let $F = F' \cup F''$. Then $F$ is a countable subset of $S_2$ such that $S_2 = \text{St}(F, U)$, which completes the proof.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\omega\} \to R$ be a bijection. Let $X$ be the quotient image of the disjoint sum $S_1 \oplus S_2$ obtained by identifying $\langle d_{\alpha}, \omega \rangle$ of $S_1$ with $\pi((d_{\alpha}, \omega)))$ of $S_2$ for every $\alpha < c$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_1)$ is a regular-closed subspace of $X$ which is not neighborhood star-Lindelöf, since it is homeomorphic to $S_1$.

Finally we show that $X$ is neighborhood star-Lindelöf. We need only show that $X$ is strongly star-Lindelöf. To this end, let $U$ be an open covers of $X$. Since $\varphi(S_2)$ is homeomorphic to $S_2$, then $\varphi(S_2)$ is strongly star-Lindelöf, there exists a countable subset $F' \subseteq \varphi(S_2)$ such that

$$\varphi(S_2) \subseteq \text{St}(F', U).$$
On the other hand, for each \( n \in \omega \), since \( \varphi(Y \times \{n\}) \) is homeomorphic to \( Y \times \{n\} \), then \( \varphi(Y \times \{n\}) \) is compact, we can find a finite subset \( F_n \subseteq \varphi(Y \times \{n\}) \) such that

\[
\varphi(Y \times \{n\}) \subseteq St(F_n, \mathcal{U}).
\]

Let \( F = F_0 \cup \bigcup_{n \in \omega} F_n \). Then \( F \) is a countable subset of \( X \) such that \( X = St(F, \mathcal{U}) \), which completes the proof. \( \square \)

It is known that a continuous image of a strongly star-Lindelöf space is strongly star-Lindelöf. Similarly, we show that neighborhood star-Lindelöfness is preserved by continuous mappings.

**Theorem 3.2.** A continuous image of a neighborhood star-Lindelöf space is neighborhood star-Lindelöf.

**Proof.** Let \( f : X \to Y \) be a continuous mapping from a neighborhood star-Lindelöf space \( X \) onto a space \( Y \). Let \( \mathcal{U} \) be an open cover of \( Y \). Then \( f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\} \) is an open cover of \( X \). Since \( X \) is neighborhood star-Lindelöf, there exists a countable subset \( A \) of \( X \) such that for every open \( O \supseteq A \), \( X = St(O, f^{-1}(\mathcal{U})) \). Then \( f(A) \) is a countable subset of \( Y \) such that for every open \( W \supseteq f(A) \), \( Y = St(W, \mathcal{U}) \). In fact, let \( y \in Y \). Then there is \( x \in X \) such that \( f(x) = y \). Let \( W \) be an open subset of \( Y \) such that \( f(A) \subseteq W \). Then \( f^{-1}(W) \) is an open subset of \( X \) such that \( A \subseteq f^{-1}(W), St(f^{-1}(W), f^{-1}(\mathcal{U})) = X \). Hence there exists \( U \in \mathcal{U} \) such that \( x \in f^{-1}(U) \) and \( f^{-1}(U) \cap f^{-1}(W) \neq \emptyset \). Thus \( y = f(x) \in f(f^{-1}(U)) = U \) and \( U \cap W \neq \emptyset \). This means that \( y \in St(W, \mathcal{U}) \). \( \square \)

Next we turn to consider preimages. To show that the preimage of a neighborhood star-Lindelöf space under a closed 2-to-1 continuous map need not be neighborhood star-Lindelöf, we use the the Alexandorff duplicate \( A(X) \) of a space \( X \). The underlying set \( A(X) \) is \( X \times [0, 1] \); each point of \( X \times [1] \) is isolated and a basic neighborhood of \((x, 0) \in X \times [0] \) is a set of the form \((U \times [0]) \cup ((U \times [1]) \setminus \{(x, 0)\})\), where \( U \) is a neighborhood of \( x \) in \( X \).

**Example 3.3.** There exists a closed 2-to-1 continuous map \( f : X \to Y \) such that \( Y \) is a neighborhood star-Lindelöf space, but \( X \) is not neighborhood star-Lindelöf.

**Proof.** Let \( Y \) be the same space \( S_2 \) in the proof of Example 3.1. As we proved in Example 3.1 above, \( Y \) is neighborhood star-Lindelöf. Let \( X \) be the Alexandorff duplicate \( A(Y) \) of the space \( Y \). Then \( X \) is not neighborhood star-Lindelöf. In fact, let \( A = \{\langle r, 1 \rangle : r \in \mathcal{R}\} \). Then \( A \) is an open and closed subset of \( X \) with \(|A| = \omega \), and each point \((r, 1) \) is isolated. Hence \( A(X) \) is not neighborhood star-Lindelöf, since every open and closed subset of a neighborhood star-Lindelöf space is neighborhood star-Lindelöf and \( A \) is not neighborhood star-Lindelöf. Let \( f : X \to Y \) be the projection. Then \( f \) is a closed 2-to-1 continuous map, which completes the proof. \( \square \)

**Example 3.4.** There exists a neighborhood star-Lindelöf space \( X \) and a compact space \( Y \) such that \( X \times Y \) is not neighborhood star-Lindelöf.

**Proof.** Let \( X = \omega \cup \mathcal{R} \) be the Isbell-Mrówka space \([10]\), where \( \mathcal{R} \) is a maximal almost disjoint family of infinite subsets of \( \omega \) with \(|\mathcal{R}| = \omega \). Then \( X \) is neighborhood star-Lindelöf.

Let \( D = \{d_\alpha : \alpha < \omega \} \) be a discrete space of cardinality \( \omega \) and let \( Y = D \cup \{y_\omega\} \) be the one-point compactification of \( D \).

We show that \( X \times Y \) is not neighborhood star-Lindelöf. Since \(|\mathcal{R}| = \omega \), we can enumerate \( \mathcal{R} \) as \( \{r_\alpha : \alpha < \omega \} \).

Let

\[
U_n = [n] \times Y \text{ for each } n \in \omega,
\]

\[
V_\alpha = X \times \{d_\alpha\} \text{ for each } \alpha < \omega
\]

and

\[
W_\alpha = [(r_\alpha) \cup \omega] \times (Y \setminus \{d_\alpha\}) \text{ for each } \alpha < \omega.
\]

Let

\[
\mathcal{U} = \{U_n : n \in \omega\} \cup \{V_\alpha : \alpha < \omega\} \cup \{W_\alpha : \alpha < \omega\}.
\]
Then $\mathcal{U}$ is an open cover of $X \times Y$. Observe that $(r_\alpha, d_\alpha) \in U \in \mathcal{U}$ if and only if $U = V_\alpha$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. It suffices to show that for any countable subset $A$ of $X \times Y$, there exists a point $\langle x, y \rangle \in X \times Y$ such that $\text{St}(\langle x, y \rangle, \mathcal{U}) \cap A = \emptyset$ by Lemma 2.3. Let $A$ be any countable subset of $X \times Y$. Then there exists $\alpha < \beta$ such that $A \cap V_\alpha = \emptyset$. Since $V_\alpha$ is the only element of $\mathcal{U}$ containing the point $(r_\alpha, d_\alpha)$, then $\text{St}(\langle r_\alpha, d_\alpha \rangle, \mathcal{U}) = V_\alpha$. By the constructions of the topology of $X \times Y$ and the open cover $\mathcal{U}$, we have $\text{St}(\langle r_\alpha, d_\alpha \rangle, \mathcal{U}) = V_\alpha$, which shows that $X \times Y$ is not neighborhood star-Lindelöf. Thus we complete the proof.

**Remark 3.5.** Example 3.4 shows that the preimage of a neighborhood star-Lindelöf space under an open perfect map need not be neighborhood star-Lindelöf.

The following well-known example shows that the product of two countably compact (and hence neighborhood star-Lindelöf) spaces need not be neighborhood star-Lindelöf. Here we give the proof roughly for the sake of completeness. For a Tychonoff space $X$, let $\beta X$ denote the Čech-Stone compactification of $X$.

**Example 3.6.** There exist two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not neighborhood star-Lindelöf.

**Proof.** Let $D$ be a discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$ and $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta D$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

1. $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;
3. Every infinite subset of $E_\alpha$ (resp., $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta D$ has cardinality $2^\mathfrak{c}$ (see [11]). Then $X \times Y$ is not neighborhood star-Lindelöf, because the diagonal $\langle (d, d) : d \in D \rangle$ is a discrete open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and the open and closed subsets of neighborhood star-Lindelöf spaces are neighborhood star-Lindelöf. \hfill $\square$

In [5, Example 3.3.3], van Douwen-Reed-Roscoe-Tree gave an example showing that there exist a countably compact space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not strongly star-Lindelöf. Now, we shall show that the product space $X \times Y$ is not neighborhood star-Lindelöf.

**Example 3.7.** There exist a countably compact (and neighborhood star-Lindelöf) space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not neighborhood star-Lindelöf.

**Proof.** Let $X = [0, \omega_1)$ with the usual order topology. Then $X$ is countably compact. Let $Y = [0, \omega_1)$ with the following topology: each point $\alpha$ with $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then $Y$ is Lindelöf.

Now, we show that $X \times Y$ is not neighborhood star-Lindelöf. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha) \times [\alpha, \omega_1) \text{ and } V_\alpha = (\alpha, \omega_1) \times \{\alpha\}.$$ Then

$$V_\alpha \cap V_\beta = \emptyset \text{ if } \alpha \neq \alpha' \text{ and } U_\alpha \cap V_\beta = \emptyset \text{ for any } \alpha < \mathfrak{c}, \beta < \mathfrak{c}.$$ Let

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$ Then $\mathcal{U}$ is an open cover of $X \times Y$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. It suffices to show that for any countable subset $A$ of $X \times Y$, there exists a point $\langle x, y \rangle \in X \times Y$ such that $\text{St}(\langle x, y \rangle, \mathcal{U}) \cap A = \emptyset$ by Lemma 2.3. Let $A$ be any countable subset of $X \times Y$. Then there exists $\alpha < \mathfrak{c}$ such that $A \cap V_\alpha = \emptyset$. Since $V_\alpha$ is the only element of $\mathcal{U}$ containing the point $(\alpha + 1, \alpha)$, then $\text{St}(\langle \alpha + 1, \alpha \rangle, \mathcal{U}) = V_\alpha$ and $V_\alpha \cap A = \emptyset$. By the constructions of the topology of $X$ and the open cover $\mathcal{U}$, we have $\text{St}(\langle \alpha + 1, \alpha \rangle, \mathcal{U}) = V_\alpha$, which shows that $X \times Y$ is not neighborhood star-Lindelöf. Thus we complete the proof. \hfill $\square$
Now we give some conditions under which neighborhood star-Lindelöfness implies strongly star-Lindelöfness. Recall that a space $X$ is paraLindelöf if every open cover $\mathcal{U}$ of $X$ has a locally countable open refinement.

**Theorem 3.8.** Every paraLindelöf neighborhood star-Lindelöf space is Lindelöf (hence star-Lindelöf).

**Proof.** Let $X$ be a paraLindelöf neighborhood star-Lindelöf space and $\mathcal{U}$ be an open cover of $X$. Then there exists a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists an open neighborhood $V_x$ of $x$ such that $V_x \subseteq V$ for some $V \in \mathcal{V}$ and $\{V \in \mathcal{V} : V \cap V_x \neq \emptyset\}$ is countable. Let $\mathcal{V}' = \{V_x : x \in X\}$. Then $\mathcal{V}'$ is a refinement of $\mathcal{V}$. Since $X$ is neighborhood star-Lindelöf, there exists a countable subset $A$ of $X$ such that for every open $O \supseteq A$, $X = St(O, \mathcal{V})$.

Let $$O = \bigcup\{V_x \in \mathcal{V}' : x \in A\}.$$ Then $O$ is an open subset of $X$ and $A \subseteq O$. Thus $St(O, \mathcal{V}) = X$.

Let $$\mathcal{V}'' = \{V \in \mathcal{V} : V \cap O \neq \emptyset\}.$$ Then $\mathcal{V}''$ is a countable open cover of $X$. For each $V \in \mathcal{V}''$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{V}''\}$ is a countable subcover of $\mathcal{U}$, which shows that $X$ is Lindelöf. Thus we complete the proof.

Since every strongly star-Lindelöf space is neighborhood star-Lindelöf, the following corollary follows from Theorem 3.8.

**Corollary 3.9.** A paraLindelöf space $X$ is neighborhood star-Lindelöf iff $X$ is strongly star-Lindelöf.

Since every paracompact space is paraLindelöf, the following Corollary follows from Corollary 3.9.

**Corollary 3.10.** A paracompact space $X$ is neighborhood star-Lindelöf iff $X$ is strongly star-Lindelöf.

Recall that a space $X$ is locally separable if $x$ has a separable neighborhood at every point $x \in X$.

**Theorem 3.11.** Every locally separable neighborhood star-Lindelöf space is star-Lindelöf.

**Proof.** Let $X$ be a locally separable neighborhood star-Lindelöf space and $\mathcal{U}$ be an open cover of $X$. For each $x \in X$, there exists an open separable subspace $V_x$ of $X$ such that $x \in V_x \subseteq U$ for some $U \in \mathcal{U}$, since $X$ is locally separable. Let $\mathcal{V} = \{V_x : x \in X\}$. Then $\mathcal{V}$ is an open cover of $X$. Since $X$ is neighborhood star-Lindelöf, there exists a countable subset $A$ of $X$ such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$.

Let $$O = \bigcup\{V_x \in \mathcal{V} : x \in A\}.$$ Then $O$ is an open subset of $X$ and $A \subseteq O$. Thus $St(O, \mathcal{U}) = X$. For each $x \in A$, since $V_x$ is separable, there exists a countable dense subset $D_x$ of $V_x$.

Let $$F = \bigcup\{D_x : x \in A\}.$$ Then $F$ is a countable subset of $X$ and $St(F, \mathcal{U}) = X$, which shows that $X$ is star-Lindelöf.

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