Some Bounds for the Pseudocharacter of the Space $C_\alpha (X, Y)$

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Abstract. Let $C_\alpha (X, Y)$ be the set of all continuous functions from $X$ into $Y$ endowed with the set-open topology where $\alpha$ is a hereditarily closed, compact network on $X$. We obtain that:

\[ i) \psi(f, C_\alpha (X, Y)) \leq \text{wac}(X) \cdot \sup_{A \in \alpha} (\psi(f(A), Y)) \cdot \sup_{A \in \alpha} (\text{w}(f(A))) \]

\[ ii) \psi(f, C_\alpha (X, Y)) \leq \text{wac}(X) \cdot \text{psw}(f(X), Y). \]

1. Introduction and Terminology

Let $X$ and $Y$ be topological spaces, and let $C(X, Y)$ denote the set of all continuous mappings from $X$ into $Y$. Let $\alpha$ be a collection of subsets of $X$. The topology having subbase $[[A, V] : A \in \alpha$ and $V$ is an open subset of $Y]$ on the set $C(X, Y)$ is denoted by $C_\alpha (X, Y)$ where $[A, V] = \{ f \in C(X, Y) : f(A) \subseteq V \}$. If $\alpha$ consists of all finite subsets of $X$, then the set $C(X, Y)$ endowed with that topology is called pointwise convergence topology and denoted by $C_\alpha (X, Y)$.

The cardinality and the closure of a set is denoted by $|A|$ and $cl(A)$, respectively. The restriction of a mapping $f : X \rightarrow Y$ to a subset $A$ of $X$ is denoted by $f_A$. $T(X)$ denotes the set of all non-empty open subsets of a topological space $X$. $ord(x, A)$ is the cardinality of the collection $\{ A \in A : x \in A \}$. Throughout this paper $X$ and $Y$ are regular topological spaces, and $\alpha$ is a hereditarily closed, compact network on the domain space $X$. (i.e., $\alpha$ is a network on $X$ such that each member of it is compact and each closed subset of a member of it is a member of $\alpha$.) Without loss of generality, we may assume that $\alpha$ is closed under finite unions. Recall that the weak $\alpha$-covering number of $X$ is defined to be $\text{wac}(X) = \min \{|B| : B \subseteq \alpha$ and $\bigcup B$ is dense in $X|$. The weight, density and character of a space $X$ are denoted by $\text{w}(X)$, $d(X)$ and $\chi(X)$, respectively. The $i$-weight of a topological space $X$, is the least of cardinals $\text{w}(Y)$ of the Tychonoff spaces $Y$ which are continuous one-to-one images of $X$. The pseudocharacter of a space $X$ at a subset $A$, denoted by $\psi(A, X)$, is defined as the smallest cardinal number of the form $|U|$, where $U$ is a family of open subsets of $X$ such that $\bigcap U = A$. If $A = \{x\}$ is a singleton, then we write $\psi(x, X)$ instead of $\psi(\{x\}, X)$. The pseudocharacter of a space $X$ is defined to be $\psi(X) = \sup \{ \psi(x, X) : x \in X \}$. The diagonal number $\Delta(X)$ of a space $X$ is the pseudocharacter of its square $X \times X$ at its diagonal $\Delta_X = \{(x, x) : x \in X\}$.

The pseudocharacter of the space $C(X, Y)$ has been studied, and some remarkable equalities or inequalities was obtained between the pseudocharacter of the space $C(X, Y)$ for certain topologies and some cardinal
functions on the spaces $X$ and $Y$. For instance, in [3], the inequalities $\psi(Y) \leq \psi(C_p(X, Y)) \leq \psi(Y) \cdot d(X)$ and, in [1] and [2], the equalities $\psi(C_p(X, R)) = d(X) = \text{ie}(C_p(X, R))$, and in [6], $\psi(C_\alpha(X, R)) = \Delta(C_\alpha(X, R)) = \text{wac}(X)$ were obtained. In this paper, when the range space $Y$ is an arbitrary topological space instead of the space $R$, we obtained some inequalities between the pseudocharacter of the space $C_\alpha(X, Y)$ at a point $f$ and the weak $\alpha$-covering number of the domain space $X$ and some cardinal functions on the range space $Y$.

We assume that all cardinal invariants are at least the first infinite cardinal $\aleph_0$.

Notations and terminology not explained above can be found in [4] and [5].

2. Main Results

First, we give an inequality between the pseudocharacter of a point $f$ in the space $C_\alpha(X, Y)$ and some cardinal functions on spaces $X$ and $Y$.

**Theorem 2.1.** For each $f \in C_\alpha(X, Y)$, we have

$$\psi(f, C_\alpha(X, Y)) \leq \text{wac}(X) \cdot \sup_{A \in \mathcal{A}} \psi(f(A), Y) \cdot \sup_{A \in \mathcal{A}} \psi(f(A))$$

**Proof.** Let $\text{wac}(X) \cdot \sup \{\psi(f(A), Y) : A \in \mathcal{A}\} \cdot \sup \{\psi(f(A)) : A \in \mathcal{A}\} = \kappa$. The inequality $\text{wac}(X) \leq \kappa$ gives us a subfamily $\mathcal{B} = \{A_i : i \in I\}$ of $\mathcal{A}$ such that $|I| \leq \kappa$ and $X = \text{cl} \left( \bigcup \mathcal{B} \right) = \text{cl} \left( \bigcup \{A_i : i \in I\} \right)$. Since $\psi(f(A_i), Y) \leq \kappa$ for each $i \in I$, there exists a family $\mathcal{V}_i$ consisting of open subsets of the space $Y$ such that $|\mathcal{V}_i| \leq \kappa$ and $f(A_i) = \bigcap \{V : V \in \mathcal{V}_i\}$ for each $i \in I$. Since $\psi(f(A_i)) \leq \kappa$ for each $i \in I$, the subspace has a base $\mathcal{B}_i$ with $|\mathcal{B}_i| \leq \kappa$. For each $i \in I$, let

$$\mathcal{H}_i = \left\{ [A_i \cap f^{-1} (\text{cl}(G)), Y \setminus \text{cl}(U)) : G, U \in \mathcal{B}_i \text{ and } \text{cl}(G) \cap \text{cl}(U) = \emptyset \right\},$$

$$\mathcal{R}_i = [A_i, V] : V \in \mathcal{V}_i \text{ and } \mathcal{W} = (\bigcup_{i \in I} \mathcal{R}_i) \cup (\bigcup_{i \in I} \mathcal{H}_i).$$

It is clear that $|\mathcal{W}| \leq \kappa$ and $f \in \mathcal{W}$ for each $W \in \mathcal{W}$. Now, we shall prove that $\bigcap \mathcal{W} = \{f\}$. Take a $g \in \bigcap \mathcal{W}$. We claim that $g_{|A_i} = f_{|A_i}$ for each $i \in I$. Assume the contrary. Suppose $g_{|A_i} \neq f_{|A_i}$ for some $i \in I$ that is, we have an $x \in A_i$ such that $g(x) \neq f(x)$. Since $g \in \bigcap \mathcal{W}$ and $f(A_i) = \bigcap \{V : V \in \mathcal{V}_i\}$, we have $g(A_i) \subseteq f(A_i)$. Therefore $g(x) \in f(A_i)$ and $f(x) \in f(A_i)$. Since $f(x) \neq g(x)$ and the space $Y$ is regular, there exist $G$ and $U$ in $\mathcal{B}_i$ such that $f(x) \in \text{cl}(G)$, $g(x) \in \text{cl}(U)$ and $\text{cl}(G) \cap \text{cl}(U) = \emptyset$. On the other hand, since $[A_i \cap f^{-1} (\text{cl}(G)), Y \setminus \text{cl}(U)] \in \mathcal{H}_i$ and $g \in \bigcap \mathcal{W}$, we have $g \in [A_i \cap f^{-1} (\text{cl}(G)), Y \setminus \text{cl}(U)]$. But this contradicts the fact that $g(x) \in \text{cl}(U)$. Hence, $g_{|A_i} = f_{|A_i}$ for each $i \in I$, or in other words $g_{|\bigcup_{i \in I} A_i} = f_{|\bigcup_{i \in I} A_i}$. Hausdorffness of the space $Y$ and the equality $X = \text{cl} \left( \bigcup \mathcal{B} \right) = \text{cl} \left( \bigcup \{A_i : i \in I\} \right)$ lead us to the fact that $g = f$. Therefore $\bigcap \mathcal{W} = \{f\}$, that is $\psi(f, C_\alpha(X, Y)) \leq \kappa$. □

Recall that a cover $\mathcal{A}$ of a set $X$ is called a separating cover if

$\bigcap \{A \in \mathcal{A} : x \in A\} = \{x\}$, for each $x \in X$. Also recall that the point separating weight $psw(X)$ of a topological space $X$ is the smallest infinite cardinal $\kappa$ such that the space $X$ has a separating open cover $\mathcal{V}$ with $\text{ord}(\mathcal{V}) \leq \kappa$ for each $x \in X$.

**Definition 2.2.** Let $A$ be a subset of a topological space $(X, \tau)$. We say that the point separating exterior weight $psw_e(A, X) \leq \kappa$, if there exists a subfamily $\mathcal{V} \subseteq \tau$ satisfying $\text{ord}(a, \mathcal{V}) \leq \kappa$ and $\bigcap \{V \in \mathcal{V} : a \in V\} = \{a\}$, for each $a \in A$.

The following lemmas are needed for the second main theorem, and in order to prove them, let us recall the Miščenko’s lemma.

**Lemma 2.3 (Miščenko’s lemma [5]).** Let $\kappa$ be an infinite cardinal, let $X$ be a set, and let $\mathcal{A}$ be a collection of subsets of $X$ such that $\text{ord}(x, \mathcal{A}) \leq \kappa$ for all $x \in X$. Then the number of finite minimal covers of $X$ by elements of $\mathcal{A}$ is at most $\kappa$. 

Lemma 2.4. Let $Z$ be subspace of the space $X$ such that $psw_e(Z,X) \leq \kappa$. Then $\psi(K,X) \leq \kappa$ for each compact subset $K$ of $Z$.

Proof. Let $\mathcal{V}$ be a family of open subsets of $X$ satisfying $ord(z,\mathcal{V}) \leq \kappa$ and $\bigcap \{V \in \mathcal{V} : z \in V\} = \{z\}$, for each $z \in Z$. Let $K$ be any compact subspace of $Z$ and let $\mu = \{W : W \subseteq \mathcal{V} \text{ and } W \text{ is a minimal finite open cover for } K\}$. By Miščenko's lemma, we have $|\mu| \leq \kappa$. Define the family $O = \bigcup_{W \in \mu} W$. It is clear that $|O| \leq \kappa$ and it can be easily seen that $\bigcap O = K$. Hence $\psi(K,X) \leq \kappa$. □

Lemma 2.5. Let $Z$ be subspace of the space $X$ such that $psw_e(Z,X) \leq \kappa$. Then we have $w(K) \leq \kappa$ for each compact subset $K$ of $Z$.

Proof. Let $K$ be a compact subset of $Z$. Clearly, $psw(K) \leq psw(Z) \leq psw_e(Z, X) \leq \kappa$. The compactness of $K$ leads us to the fact that $w(K) = psw(K)$. [in [5], Ch. 1, Theorem 7.4]. Hence the claim. □

Now, we are ready to give another bound for the pseudocharacter of the space $C_{\alpha}(X,Y)$ at a point $f$.

Theorem 2.6. For each $f \in C_{\alpha}(X,Y)$, we have

$$\psi(f,C_{\alpha}(X,Y)) \leq wac(X) \cdot psw_e(f(X), Y).$$

Proof. Let $wac(X) \cdot psw_e(f(X), Y) = \kappa$, and let $\beta = \{A_i : i \in I\}$ be a subfamily of $\alpha$ such that $cl(\bigcup \beta) = X$ and $|I| \leq \kappa$. The compactness of $A_i$ for each $i \in I$ and the inequality $psw_e(f(X), Y) \leq \kappa$ lead us to the facts that $\psi(f(A_i), Y) \leq \kappa$ and $w(A_i) \leq \kappa$ for each $i \in I$, by lemmas 2.4 and 2.5. Therefore, by Theorem 2.1, we have $\psi(f,C_{\alpha}(X,Y)) \leq \kappa$. □

Acknowledgement: The author acknowledge the help of Mr. Hasan Gül in the preparation of the manuscript.

References