Module Amenability of Restricted Semigroup Algebras Under Module Actions

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Abstract. In this article, we show that module amenability with the canonical action of restricted semigroup algebra $l^1_r(S)$ and semigroup algebra $l^1(E)$ are equivalent, where $S_r$ is the restricted semigroup of associated to the inverse semigroup $S$. We use this to give a characterization of module amenability of restricted semigroup algebra $l^1_r(S)$ with the canonical action, where $S$ is a Clifford semigroup.

1. Introduction

The notion of module amenability for a Banach algebra $A$ which is a Banach module over another Banach algebra $U$ is defined by Amini in [1]. He showed that for an inverse semigroup $S$, the semigroup algebra $l^1(S)$ is module amenable as a $l^1(E)$-module with the multiplication right action and the trivial left action, where $E$ is the set of idempotents of $S$ if and only if $S$ is amenable.

In this paper we show that module amenability of $l^1(S)$ as an $l^1(E)$-module with the canonical action implies its module amenability as an $l^1(E)$-module with the trivial left action. The main difference is that the corresponding equivalence relation leads a Clifford homomorphic image. We characterize module amenability of the restricted semigroup algebra $l^1_r(S)$ as an $l^1(E)$-module with the canonical action, for each Clifford semigroup $S$. Also we show that in the canonical action, the module amenability of the semigroup algebra $l^1_r(S)$ and the restricted semigroup algebra $l^1_r(S)$ are equivalent. This could be considered as the module version of a result of [6], [9], which asserts that the amenability of the semigroup algebra $l^1(S)$ and the restricted semigroup algebra $l^1_r(S)$ are equivalent.

Throughout this paper, $A$ and $U$ are Banach algebras such that $A$ is a Banach $U$-module with compatible actions

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in A, \alpha \in U),$$

and

$$\alpha \cdot (\beta \cdot a) = (a\beta) \cdot a, \quad (a \cdot \beta) \cdot \alpha = a \cdot (\beta \alpha) \quad (a \in A, \alpha, \beta \in U).$$

The Banach algebra $U$ acts trivially on $A$ from left (right) if for each $\alpha \in U$ and $a \in A$, $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), where $f$ is a continuous character on $U$.

Let $X$ be a Banach $A$-module and a Banach $U$-module with compatible actions

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

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when

\[(a \cdot x) \cdot a = a \cdot (x \cdot a) \ (a \in \mathcal{A}, a \in \mathcal{U}, x \in X);\]

and similarly for the right and two sided actions. We call \(X\) a \(\mathcal{A}\)-\(\mathcal{U}\)-module. If in addition,

\[a \cdot x = x \cdot a \quad (a \in \mathcal{U}, x \in X)\]

then \(X\) is called a commutative \(\mathcal{A}\)-\(\mathcal{U}\)-module. If \(X\) is a commutative \(\mathcal{A}\)-\(\mathcal{U}\)-module, then so is \(X'\), under the actions

\[(a \cdot f)(x) = f(x \cdot a), \quad (a \cdot f)(x) = f(x \cdot a) \quad (a \in \mathcal{A}, a \in \mathcal{U}, x \in X, f \in X')\]

and similarly for the right actions.

Let \(J\) be the closed ideal of \(\mathcal{A}\) generated by elements of the form \(a \cdot ab - ab \cdot a\) for \(a \in \mathcal{U}, a, b \in \mathcal{A}\).

Let \(\mathcal{A}, \mathcal{U}\) and \(X\) be as above. A bounded map \(D : \mathcal{A} \to X\) is called a module derivation if

\[D(a \pm b) = D(a) \pm D(b), \quad D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in \mathcal{A})\]

and

\[D(a \cdot a) = a \cdot D(a) \quad D(a \cdot a) = D(a) \cdot a \quad (a \in \mathcal{A}, a \in \mathcal{U}).\]

Note that \(D\) is not necessarily linear, but still its boundedness implies its norm continuity (since \(D\) preserves subtraction). When \(X\) is commutative, each \(x \in X\) defines a module derivation

\[\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).\]

These are called inner module derivations.

**Definition 1.1.** \(\mathcal{A}\) is called module amenable (as an \(\mathcal{U}\)-module) if for any commutative \(\mathcal{A}\)-\(\mathcal{U}\)-module \(X\), each module derivation \(D : \mathcal{A} \to X\) is inner.

**Definition 1.2.** A discrete semigroup \(S\) is called an inverse semigroup if for each \(x \in S\) there is a unique element \(x' \in S\) such that \(xx'x = x\) and \(x'xx' = x'\). An element \(e \in S\) is called an idempotent if \(e = e^2\).

Throughout this paper \(S\) is an inverse semigroup with the set of idempotents \(E\). An inverse semigroup whose idempotents are in the center is called a Clifford semigroup [3]. A Clifford semigroup \(S\) is called a semilattice if each element of \(S\) is idempotent [7]. It is easy to see that \(E\) is a commutative subsemigroup of \(S\) and \(l^1(E)\) can be regarded as a subalgebra of \(l^1(S)\).

Let \(l^1(E)\) acts on \(l^1(S)\) by the multiplication from right and trivially from left, that is

\[\delta_s \ast \delta_e = \delta_{se}, \quad \delta_s \cdot \delta_e = \delta_{se} \ (s \in S, e \in E).\]

In this case, \(J\) is the closed ideal generated by

\[\{\delta_s - \delta_{se} : \ s \in S, e \in E\}.
\]

Consider an equivalence relation on \(S\) as follows

\[h \equiv k \iff \delta_h = \delta_k \in J \quad (h, k \in S).\]

It is shown in [8] that the quotient \(S/\approx\) is a discrete group.

### 2. Module amenability of restricted semigroup algebras

Here we consider \(l^1(E)\) acts on \(l^1(S)\) with canonical actions, that is

\[\delta_s \cdot \delta_e = \delta_{se}, \quad \delta_s \cdot \delta_e = \delta_{se} \ (s \in S, e \in E).\]
The closed ideal $I_c$ of $l^1(S)$ is generated by
\[ \{ \delta_{cs} - \delta_{ce} : s, e \in S \}. \]

We consider an equivalence relation on $S$ as follows
\[ s \sim t \iff \delta_s - \delta_t \in I_c \quad (s, t \in S). \]

An equivalence relation $R$ on a semigroup $S$ is called a congruence if
\[ (s, t) \in R \implies (as, at), (sa, ta) \in R \quad (s, t, a \in S). \]

Congruences on any semigroup provide some information about its homomorphic images\[2\].

Let $\rho$ be a congruence on $S$ and $P$ a property of homomorphic image $S/\rho$, we call $\rho$ a $P$ congruence. A least congruence $\rho$ such that $S/\rho$ is a $P$ congruence is called the least $P$ congruence.

**Lemma 2.1.** $\sim$ is the least Clifford congruence on $S$.

**Proof.** Since $I_c$ is an ideal of $l^1(S)$, $\sim$ is a congruence. From definition of $\sim$, it follows that $es \sim se$. Thus $S/\sim$ is a Clifford semigroup. Hence the least Clifford congruence $\xi \subseteq \sim$.

Let $I_\gamma$ be the closed ideal of $l^1(S)$ generated by
\[ \{ \delta_{cs} - \delta_{ce} : (s, t) \in \gamma \}, \]
for each Clifford congruence $\gamma$ on $S$. Clearly
\[ sy t \iff \delta_{cs} - \delta_{ce} \in I_\gamma \quad (s, t \in S). \]

Since $(es, se) \in \gamma$, it follows that $\delta_{cs} - \delta_{ce} \in I_\gamma$. Thus $I_\gamma \subseteq I_\gamma$ and so $\sim \subseteq \gamma$, for each Clifford congruence $\gamma$. Hence $\sim \subseteq \xi$. $\square$

Let $X$ be a commutative $l^1(S)$-$l^1(E)$-module. Throughout this paper we denote by $\bullet$ the left and right actions of $l^1(E)$ on $X$ and by $\cdot$ the left and right actions of $l^1(S)$ on $X$.

**Proposition 2.2.** If $l^1(S)$ is module amenable as an $l^1(E)$-module with the canonical action then $l^1(S)$ is module amenable as an $l^1(E)$-module with the trivial left action.

**Proof.** Suppose that $l^1(E)$ acts on $l^1(S)$ with the trivial left action and the multiplication right action. Let $X$ be a commutative $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to X^*$ be a module derivation. We have
\[ \delta_{se} \cdot x = \delta_{s} \cdot (\delta_{e} \cdot x) = \delta_{s} \cdot (x \cdot \delta_{e}) = (\delta_{s} \cdot x) \cdot \delta_{e} = (\delta_{e} \cdot x) \cdot \delta_{s} = (\delta_{s} \cdot \delta_{e}) \cdot x = \delta_{(s,e)} \cdot x. \]

Thus $f \cdot X = 0$ and similarly $X \cdot f = 0$. Now since $S/\sim$ is a group, $es \approx se$ and so $\delta_{es} - \delta_{es} \in I_c$. It follows that $X \cdot I_c = I_c \cdot X = 0$ and even if $l^1(E)$ acts on $l^1(S)$ with the canonical action, $X$ is a commutative $l^1(S)$-$l^1(E)$-module. In additions, we have
\[ D(\delta_{se}) = D(\delta_{s}) \cdot \delta_{e} = \delta_{e} \cdot D(\delta_{s}) = D(\delta_{e}). \]

Therefore $Dl_I = 0$ and so $D(\delta_{cs} - \delta_{ce}) = 0$. Now if $l^1(E)$ acts on $l^1(S)$ with the canonical action, then we have
\[ D(\delta_{s} \cdot \delta_{e}) = D(\delta_{s} \cdot \delta_{e}) = D(\delta_{s}) \cdot \delta_{e} = \delta_{e} \cdot D(\delta_{s}) \quad (f \in E, s \in S). \]

Hence $D$ is a module derivation. So by assumption it is inner. $\square$

Similar to the Proposition 2.1.5 of [10] we have the following Lemma.
Lemma 2.3. Let \( l^1(S) \) has a bounded approximate identity. Then \( l^1(S) \) is module amenable as an \( l^1(E) \)-module with the canonical action if and only if each module derivation \( D : l^1(S) \to X^* \) is inner, for each pseudo-unital \( l^1(S) \)-\( l^1(E) \)-module \( X \).

Theorem 2.4. Let \( S \) be a semilattice. Then \( l^1(S) \) is module amenable as an \( l^1(E) \)-module with the canonical action if and only if \( l^1(S) \) admits a bounded approximate identity.

Proof. Suppose that \( l^1(S) \) admits a bounded approximate identity \((\lambda_t)\). Consider a module derivation \( D : l^1(S) \to X^* \). For each \( e \in S \) and \( \lambda \in \mathbb{C} \) we have

\[
D(\lambda \delta_e) = \lambda \delta_e \bullet D(\delta_e) = \lambda D(\delta_e).
\]

Thus \( D \) is a derivation. By Lemma 2.3, we may suppose that \( X \) is pseudo-unital \( l^1(S) \)-module. That is, for each \( x \in X \), there exist \( f, g \in l^1(S) \) and there is \( y \in X \) such that \( x = f \cdot y \cdot g \). It follows that

\[
D(\delta_e)(x) = D(\delta_e)(f \cdot y \cdot g)
= D(\delta_e)((\lim_i \lambda_i \cdot f) \cdot y \cdot g)
= D(\delta_e)((\lim_i \lambda_i \cdot (f \cdot y \cdot g))
= \lim_i D(\delta_e) \cdot \lambda_i(f \cdot y \cdot g)
= \lim_i D(\delta_e) \cdot \lambda_i(e).
\]

Similarly \( D(\delta_e)(x) = \lim_i \lambda_i \cdot D(\delta_e)(x) \). From the equalities \( D(\delta_{ef}) = D(\delta_e) \bullet D(\delta_f) = D(\delta_f) \bullet D(\delta_e) \), it follows that

\[
D(\delta_e) \cdot \delta_{ef} = D(\delta_f) \bullet D(\delta_e) \cdot \delta_{ef} \tag{1}
\]

for each \( e, f \in S \). In addition, we have

\[
D(\delta_{ef}) = \delta_e \bullet D(\delta_{ef}) = \delta_e \bullet (\delta_e \cdot D(\delta_f) + D(\delta_e) \cdot \delta_f) = \delta_e \cdot D(\delta_f) + D(\delta_e) \cdot \delta_{ef}.
\]

Thus

\[
D(\delta_e) \cdot \delta_{ef} = D(\delta_e) \cdot D(\delta_{ef}) \tag{2}
\]

From (1), (2), it follows that \( D(\delta_e) \cdot \delta_{ef} = D(\delta_f) \cdot \delta_{ef} \) for each \( e, f \in S \). Thus we have for each \( \lambda_i, D(\lambda_i \cdot \delta_e) = D(\delta_e) \cdot \lambda_i \) and so \( D(\delta_e) = \lim_i D(\lambda_i \cdot \delta_e) = \lim_i (D(\delta_e) \cdot \lambda_i + \lambda_i \cdot D(\delta_e)) \). This implies that \( D(\delta_e)(x) = \lim_i D(\delta_e) \cdot \lambda_i(x) + \lim_i \lambda_i \cdot D(\delta_e)(x) = 2D(\delta_e)(x), \) for each \( x \in X \). Hence \( D(\delta_e)(x) = 0, \) for \( x \in X \) and so \( D(\delta_e) = 0 \). Conversely, since \( l^1(S) \) is a commutative \( l^1(S) \)-module, it has a bounded approximate identity by [1].

Note that by the above theorem, for semilattice \( S = (\mathbb{N}, \lor) \), \( l^1(S) \) is module amenable as an \( l^1(E) \)-module with the canonical action. This example shows that module amenability of a semilattice algebra does not imply finiteness of the semilattice.

Consider the multiplication \( \circ \) on the Banach space \( l^1(S) \) by

\[
\sum_{s \in S} f(s) \delta_s \circ \sum_{t \in S} g(t) \delta_t = \sum_{s \in r \in S} \sum_{s \in s, \cdot t} f(s) g(t) \delta_r,
\]

if there are no elements \( t, s \in S \) with \( st = r \) and \( s's = tt' \), the multiplication is taken as zero. Under the usual \( l^1 \)-norm, \( (l^1(S), \circ) \) is a Banach algebra. We denote this Banach algebra by \( l^1(S) \) as in [6]. In the particular case,

\[
\delta_e \circ \delta_t = \begin{cases} 
\delta_{et} & s's = tt' \\
0 & \text{otherwise}.
\end{cases}
\]
Let that if $\delta s \sim \delta t \in I_B(s, t \in S)$. Note that in general, $\sim_B$ is not a congruence.

Let $X$ be a commutative $l_1^1(S)$-module. Throughout the rest of this paper we denote left and right actions of $l_1^1(E)$ on $X$ by $\ast$ and left and right actions of $l_1^1(S)$ on $X$ by $\cdot$.

**Proposition 2.5.** $l_1^1(S)$ is module amenable as an $l_1^1(E)$-module with the canonical action if and only if $l_1^1(S)/I_B$ is module amenable as an $l_1^1(E)$-module.

**Proof.** Let $X$ be a commutative $l_1^1(S)$-module. Consider a module derivation $D : l_1^1(S) \rightarrow X^*$. For each $s \in S$ such that $ss' \neq s's$, we have

$$D(\delta_s) = D(\delta_s \circ \delta_{s'}) = D(\delta_s) \cdot \delta_{s'}$$

$$= \delta_{s'} \circ D(\delta_s) = D(\delta_{s'} \circ \delta_s)$$

$$= 0.$$  

Thus $D|_{I_B} = 0$ and so $D : l_1^1(S)/I_B \rightarrow X^*$ defined by $D(\delta_s + I_B) = D(\delta_s)$ is a module derivation. We conclude that if $l_1^1(S)/I_B$ is module amenable as $l_1^1(E)$-module with the canonical action, then $l_1^1(S)$ is module amenable as $l_1^1(E)$-module with the canonical action. The converse follows using the module homomorphism $\pi : l_1^1(S) \rightarrow l_1^1(S)/I_B$ and Proposition 2.5 of [1].

**Proposition 2.6.** Let $S$ be a Clifford semigroup. Then $l_1^1(S)$ is module amenable as an $l_1^1(E)$-module with the canonical action if and only if $l^1(E)$ is amenable.

**Proof.** Suppose that $l_1^1(S)$ is module amenable as an $l_1^1(E)$-module with the canonical action. Since $S$ is a Clifford semigroup, $l_1^1(S)$ is a commutative $l_1^1(E)$-module with the canonical action. It follows from Proposition 2.2 of [1] that $l_1^1(S)$ has a bounded approximate identity. From [6], it follows that $E$ is finite. Let $I$ be the closed principal ideal of $S$ generated by $e \in E$. Thereby $l_1^1(I)$ is an $l_1^1(E)$-module with the following compatible actions

$$\delta_f \cdot \delta_i := \delta_f \circ \delta_i, \quad \delta_i \cdot \delta_f := \delta_i \circ \delta_f \quad (f, i \in I).$$

Consider the module homomorphism $\varphi : l_1^1(S) \rightarrow l_1^1(I)$ defined by $\varphi(\delta_s) = \delta_s \circ \delta_e$. Thus $l_1^1(I)$ is module amenable as an $l_1^1(E)$-module with the canonical action. Now put $I_e = \{b \in I : SB \subseteq I \}$. Similarly $I_e$ is an ideal of $I$ and $\psi : l_1^1(I) \rightarrow l_1^1(I/I_e)$ is a module homomorphism and so $l_1^1(I/I_e)$ is module amenable as an $l_1^1(E)$-module with the canonical action, by Proposition 2.5 of [1]. Similarly $I/I_e \approx |0 \cup G_e$ and $l_1^1(|0 \cup G_e)$ is module amenable as an $l_1^1(E)$-module with the canonical action. We claim that $l^1(G_e)$ is amenable. Let $X$ be a $l^1(G_e)$-module and $D : l^1(G_e) \rightarrow X^*$ be a derivation. Since $l^1(|0 \cup G_e) \subseteq l^1(|0 \cup G_e) \subseteq l^1(G_e) \oplus \mathbb{C} \delta_0$, with the following new definition, $X$ is a commutative $l^1(|0 \cup G_e)$-module with the compatible actions

$$x \cdot \delta_0 = \delta_0 \cdot x = 0.$$  

Consider $\Delta : l^1(|0 \cup G_e) \rightarrow X^*$ defined by $D(\delta_g) = D(\delta_g)(g \in G_e)$ and $D(\delta_0) = 0$. Clearly if $l^1(S)$ is an $l^1(E)$-module with the canonical action, then $D$ is a module derivation and so it is inner. Therefore $D$ is an inner derivation and this proves that $l^1(G_e)$ is amenable. It follows that $G_e$ is amenable and by [5], $l^1(S)$ is amenable. The converse is clear.

**Corollary 2.7.** Let $S$ be a semilattice. Then $l_1^1(S)$ is module amenable as an $l_1^1(E)$-module with the canonical action if and only if $S$ is finite.
Proof. It follows from the above proposition that \( l^2(S) \) is amenable. Since \( S \) is semilattice, \( S \) is finite. The converse is clear. \( \square \)

Proposition 2.6 means that if \( l^1(E) \) is module amenable as an \( l^1(I) \)-module with the canonical action then \( l^1(S) \) is module amenable with the canonical action for each Clifford semigroup. The converse fails in general. For example let \((\mathbb{N}, \vee)\) be the semigroup of positive integers with maximum operation, that is \( m \vee n = \max(m, n) \). By Theorem 2.4, \( l^1(S) \) is module amenable as an \( l^1(E) \)-module with the canonical action but \( l^2(S) \) is not module amenable as an \( l^1(E) \)-module with the canonical action by Corollary 2.7.

For an arbitrary inverse semigroup \( S \) with the set of idempotents \( E \), the restricted product of elements \( x \) and \( y \) of \( S \) is \( xy \) if \( x'y = y'y \) and undefined, otherwise. The set \( S \) with this restricted product forms a discrete groupoid [4]. If we adjoin a zero element 0 to this groupoid and put \( 0' = 0 \), then we get an inverse semigroup \( S_r \) with the multiplication rule

\[
x \circ y = \begin{cases} xy & \text{if } x'y = y'y \\ 0 & \text{otherwise}, \end{cases}
\]

for each \( x, y \in S \cup \{0\} \). The inverse semigroup \( S_r \) is called the restricted semigroup of \( S \) (see[6]). Note that \((\mathbb{N}, \vee)\) could be regarded as a subalgebra of \( l^1(S_r) \) and we denote this Banach algebra by \( l^1(E_r) \). Thereby \( l^1(S_r) \) is an \( l^1(E_r) \)-module with the canonical action. The closed ideal \( J_r \) of \( l^1(S_r) \) is generated by

\[
[\delta_r - \delta_0] s \in S, s's \neq ss'.
\]

We consider an equivalence relation \( \sim \) on \( S_r \) as follows

\[
s \sim t \iff \delta_s - \delta_t \in J_r \quad (s, t \in S_r).
\]

**Proposition 2.8.** \( \sim \) is the least Clifford congruence on \( S_r \).

**Proof.** From definition of \( J_r \), \( s \sim t \) for each \( s, t \in S \) such that \( ss' \neq s's \) and \( tt' \neq t't \). Since each element \( s \) such that \( ss' = s's \) is contained in a maximal subgroup of \( S \), \( S \) is a semilattice of groups. Thus \( S/\sim \) is a Clifford semigroup by Theorem 4.2.1 of [3]. Thus \( \sim \) is a Clifford equivalence. Suppose that \( s \sim t \) and \( l \in S \). Since \( \delta_s - \delta_t \in J_r \), it follows that \( \delta_l \cdot (\delta_s - \delta_t) \in J_r \) and \( (\delta_s - \delta_t) \cdot \delta_l \in J_r \). Thus \( \delta_{s, \text{tot}} - \delta_{t, \text{tot}} \in J_r \) and \( \delta_{s, \text{tot}} - \delta_{t, \text{tot}} \in J_r \) and so \( \sim \) is a congruence on \( S_r \). Finally suppose that \( \rho \) is a Clifford congruence on \( S_r \). Let \( l_\rho \) be the closed ideal of \( l^1(S_r) \) generated by \( [\delta_s - \delta_t : (s, t) \in \rho] \). Clearly

\[
sp t \leftrightarrow \delta_s - \delta_t \in l_\rho.
\]

Since for each \( s \in S \) such that \( ss' \neq s's \) we have \((s, 0) \in \rho \), it follows that \( \delta_s - \delta_0 \in l_\rho \), for each Clifford congruence \( \rho \). Thus \( J_r \subseteq l_\rho \) and so for each Clifford congruence \( \rho, \sim \subseteq \rho \). Hence \( \sim \) is the least Clifford congruence on \( S_r \). \( \square \)

Note that \( \delta_s \circ \delta_t = 0 \) in \( l^1(S) \) but \( \delta_s \cdot \delta_t = \delta_0 \) in \( l^1(S_r) \), for each \( s, t \in S \) such that \( s's \neq tt' \). Thus \( l^2(S) \) is not a subalgebra of \( l^1(S_r) \).

**Proposition 2.9.** Let \( S \) be an inverse semigroup. Then the following statements are equivalent:

(i): \( l^1(S) \) is module amenable as an \( l^1(E) \)-module with the canonical action.

(ii): \( l^1(S_r) \) is module amenable as an \( l^1(E_r) \)-module with the canonical action.

(iii): \( l^1(S_r/\sim \sim) \) is amenable.

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( X \) is a commutative \( l^1(S_r) \)-\( l^1(E_r) \)-module and \( D : l^1(S_r) \to X^* \) is a module derivation. Then the following module actions are well-defined

\[
\delta_s \ast_B x = \begin{cases} 0 & \delta_s \cdot x = \delta_0 \cdot y (\text{for some } y \in X) \\ \delta_s \cdot x & \text{otherwise}, \end{cases}
\]

**References**


for each $s \in S$ and similarly for the right action. Also $l^1_r(E)$ acts on $X$ by the following action

$$\delta_c \cdot_R x = \begin{cases} 0 & \delta_c \cdot x = \delta_0 \cdot y \text{ (for some } y \in X) \\ \delta_c \cdot x & \text{otherwise.} \end{cases}$$

Therefore $X$ is a commutative $l^1(S)$-$l^1_r(E)$-module. Consider $\tilde{D} : l^1(S) \to X^*$ defined by

$$\tilde{D}(\delta_s) = \begin{cases} D(\delta_s) & \delta_s \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\tilde{D}$ extends to a module derivation and so it is inner. Therefore $D$ is inner.

(ii) $\Rightarrow$ (i) Suppose that $X$ is a commutative $l^1(S)$-$l^1_r(E)$-module. It is enough to define $\delta_0 \cdot x = \delta_0 \cdot y = 0$, then $X$ is a commutative $l^1(S_r)$-$l^1_r(E_r)$-module. Let $D : l^1_r(S_r) \to X^*$ be a module derivation. Consider $D : l^1(S_r) \to X^*$ defined by

$$D(\delta_s) = \begin{cases} D(\delta_s) & s \in S \\ 0 & s = 0. \end{cases}$$

It is easy to see that $D$ extends to a module derivation and so it is inner. Therefore $D$ is inner.

(iii) $\Rightarrow$ (ii) Since $l^1_r(S_r)$ is module amenable as an $l^1_r(E_r)$-module with the canonical action, it follows from Proposition 2.5 [1] that $l^1(S_r/\sim_r)$ is module amenable as an $l^1_r(E_r)$-module with the canonical action. Now by Propositions 2.6, 2.8, $l^1(S_r/\sim_r)$ is amenable.

Let $X$ be a commutative $l^1(S_r)$-$l^1_r(E_r)$-module. Since $J_r \cdot X = X \cdot J_r = 0$, the following module actions are well-defined

$$(\delta_s + J_r) \cdot x := \delta_s + J_r \cdot (\delta_s + J_r) := x \cdot \delta_s (x \in X, \delta_s \in l^1(S_r)),$$

therefore $X$ is an $l^1(S_r)/J_r$-module. Suppose that $D : l^1(S_r) \to X^*$ is a module derivation, and consider $\tilde{D} : l^1(S_r)/J_r \to X^*$ defined by $\tilde{D}(\delta_s + J_r) = D(\delta_s)$ ($s \in S_r$). We have

$$D(\delta_s) = D(\delta_s \cdot s \cdot \delta_s) = D(\delta_s) \cdot \delta_s \cdot \delta_s = D(\delta_s) \cdot \delta_s = D(\delta_s \cdot \delta_s) = 0.$$

By the above observation, $\tilde{D}$ is also well-defined. Moreover,

$$D(\lambda \delta_s) = \lambda \delta_s \cdot s \cdot \delta_s = \lambda D(\delta_s) \quad (\lambda \in \mathbb{C}).$$

Thus $D$ is linear and so $\tilde{D}$ is linear. Hence $\tilde{D}$ is an inner module derivation. So $l^1(S_r)/J_r$ is module amenable as an $l^1_r(E_r)$-module with the canonical action and it follows from Proposition 2.5 of [1] and $l^1(S_r/\sim_r) \cong l^1(S_r)/J_r$, that $l^1(S_r/\sim_r)$ is module amenable as an $l^1_r(E_r)$-module with the canonical action. \qed

References