Fuzzy Hyper $p$-ideals of Hyper BCK-algebras

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Abstract. The paper is a reflection of “fuzzy sets” applied to “hyper $p$-ideals” and their comparison with simple “fuzzy hyper BCK-ideals”. The idea of “fuzzy (weak, strong) hyper $p$-ideals” is presented and characterization of these ideals is conferred using different concepts like that of “level subsets, hyper homomorphic pre-image” etc. The connections between “fuzzy (weak, strong) hyper $p$-ideals” are discussed and “the strongest fuzzy relation” on a “hyper BCK-algebra” is conferred.

1. Introduction

The “hyper structure theory” was presented by Marty [16], in 1934, at the “8th Congress of Scandinavian Mathematicians”. Now a days hyperstructures are widely used in both pure and applied mathematics. During the exploration of properties of set difference, Imai and Iseki in 1966 bring together a set of axioms commonly known as BCK-algebras. Komori [14] in 1983, introduced a new class of algebras called BCC-algebras or $BIC^+$-algebras. Dudek et al. [5, 8] discussed the properties of branches, ideals and atoms in weak BCC-algebras. Dudek [4] introduced the concept of solid weak BCC-algebras and further, he and Thomys [6] generalized the concept of BCC-algebras. Borzooei et al. [2] discussed the applications of hyperstructures in BCC-algebras. Later in 2000, this theory was applied to BCK-algebras by Jun et al. [13]. Jun et al. [12], deliberated the properties of “fuzzy strong hyper BCK-ideals”. The most apposite theory of “fuzzy sets” which is a tool for handling with uncertainties was presented by Zadeh [17] in 1965. Dudek et al. [7], “applied the fuzzy sets to BCC-algebras”. Moreover in 2001, “Jun and Xin [10] applied the fuzzy set
theory to hyper BCK-algebras”. This paper confers, “the concept of fuzzification of (weak, strong) hyper $p$-ideals in hyper BCK-algebras” and associated properties.

2. Preliminaries

“If $H$ is a non-empty set with the hyperoperation ‘$o$’ from $H \times H$ into $P^*(H)$ the collection of all non-empty subsets of $H$, then for any subsets $A$ and $B$ of $H$ by $A \circ B$ we denote the set $\bigcup \{a \circ b | a \in A, b \in B\}$”. “If $A = \{a\}$, then instead of $\{a\} \circ B$ we write $a \circ B$”.

**Definition 2.1.** [13] “Hyper BCK-algebra is a non-empty set $H$ equipped with a hyperoperation “$\circ$” and a constant $0$ fulfilling the following conditions:

(HK1) $(u \circ w) \circ (v \circ w) \ll u \circ v$

(HK2) $(u \circ v) \circ w = (u \circ w) \circ v$

(HK3) $u \circ H \ll \{u\}$

(HK4) $u \ll v$ and $v \ll u$ imply $u = v$

for any $u, v, w \in H$. Here $u \ll v$ is defined by $0 \in u \circ v$ and for any $G, I \subseteq H$, $G \ll I$ is defined as $\forall a \in G, \exists b \in I$ such that $a \ll b$. The relation “$\ll$” is called the hyper order in $H$”.

**Proposition 2.2.** [13] “For a hyper BCK-algebra $H$, the following properties are obvious:

(i) $u \circ 0 = \{u\}$

(ii) $u \circ v \ll u$

(iii) $0 \circ G = \{0\}$

(iv) $v \ll w$ implies $u \circ w \ll u \circ v$

(v) $G \subseteq I$ implies $G \ll I$

for any $u, v, w \in H$ and for non-empty subsets $G$ and $I$ of $H$”.

Moreover for the basic study relevant to “hyper BCK-subalgebras and (weak, strong, reflexive) hyper BCK-ideals”, please see [13]. From now onwards, $H$ will represent a “hyper BCK-algebra”.

**Lemma 2.3.** [12, 13] For any $H$,

(i) “any strong hyper BCK-ideal of $H$ is a hyper BCK-ideal of $H$”.

(ii) “any hyper BCK-ideal of $H$ is a weak hyper BCK-ideal of $H$”.

**Lemma 2.4.** [12] “For any reflexive hyper BCK-ideal $I$ of $H$, if $u \circ v \cap I \neq \emptyset$ then $u \circ v \ll I$, $\forall u, v \in H$”.

**Proposition 2.5.** [11] “If $G$ is a subset of $H$ and $I$ is any hyper BCK-ideal of $H$, such that, $G \ll I$ then $G \subseteq I$”.

**Definition 2.6.** For a “hyper BCK-algebra” $H$, a non-empty subset $I \subseteq H$, containing $0$ is known as

- a “weak hyper $p$-ideal” of $H$ if $(a \circ c) \circ (b \circ c) \subseteq I$ and $b \in I$ imply $a \in I$.

- a “hyper $p$-ideal” of $H$ if $(a \circ c) \circ (b \circ c) \ll I$ and $b \in I$ imply $a \in I$.

- a “strong hyper $p$-ideal” of $H$ if $(a \circ c) \circ (b \circ c) \cap I \neq \emptyset$ and $b \in I$ imply $a \in I$.
**Theorem 2.7.** Every “(strong, weak) hyper p-ideal” is a “(strong, weak) hyper BCK-ideal”.

**Proof.** Let \( I \) be a “hyper p-ideal of \( H \)”. Then, for any \( i, j, k \in H \),
\[
(i \circ k) \circ (j \circ k) \ll I \text{ and } j \in I \text{ imply } i \in I.
\]
Putting \( k = 0 \) we get
\[
(i \circ 0) \circ (j \circ 0) \ll I \text{ and } j \in I \text{ imply } i \in I.
\]
Therefore, \((i \circ j) \ll I \text{ and } j \in I \Rightarrow i \in I. \) Hence proved. \( \square \)

Generally, every “(strong, weak) hyper BCK-ideal” is not a “(strong, weak) hyper p-ideal”. It can be observed with the help of examples given below:

**Example 2.8.** “Let \( H = \{0, a, b\} \). We contemplate the following table:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0]</td>
<td>[0]</td>
<td>[0]</td>
</tr>
<tr>
<td>a</td>
<td>[0,a]</td>
<td>[0]</td>
<td>[0,a]</td>
</tr>
<tr>
<td>b</td>
<td>[b]</td>
<td>[b]</td>
<td>[0,a]</td>
</tr>
</tbody>
</table>

Then \( H \) is a hyper BCK-algebra. Take \( I = \{0, a\} \). Then \( I \) is a “weak hyper BCK-ideal”, however, not a “weak hyper p-ideal of \( H \)” as \((b \circ b) \circ (0 \circ b) = \{0, a\} \subseteq I \text{ and } 0 \in I \text{ but } b \notin I. \)

**Example 2.9.** “Let \( H = \{0, a, b\} \). We contemplate the following table:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0]</td>
<td>[0]</td>
<td>[0]</td>
</tr>
<tr>
<td>a</td>
<td>[0]</td>
<td>[0]</td>
<td>[0]</td>
</tr>
<tr>
<td>b</td>
<td>[b]</td>
<td>[b]</td>
<td>[0,b]</td>
</tr>
</tbody>
</table>

Then \( H \) is a hyper BCK-algebra”. Take \( I = \{0, b\} \). Then, \( I \) is a “hyper BCK-ideal” but not a “hyper p-ideal” as \((a \circ a) \circ (0 \circ a) = \{0\} \ll I, 0 \in I \text{ but } a \notin I. \)

Here \( I = \{0, b\} \) is also a “strong hyper BCK-ideal” however, it is not a “strong hyper p-ideal of \( H \)” as \((a \circ a) \circ (0 \circ a) = \{0\} \cap I \neq \emptyset \text{ and } 0 \in I \text{ but } a \notin I. \)

**Theorem 2.10.** For any “hyper BCK-algebra”,
(i) “any hyper p-ideal is also a weak hyper p-ideal”,
(ii) “any strong hyper p-ideal is also a hyper p-ideal”.

**Proof.** (i) Let \( I \) is a “hyper p-ideal of \( H \)”. Let, \((i \circ k) \circ (j \circ k) \subseteq I \text{ and } j \in I \). Then, \((i \circ k) \circ (j \circ k) \subseteq I \text{ implies } (i \circ k) \circ (j \circ k) \ll I \) (by Proposition 2.2(v)), which along with \( j \in I \) implies \( i \in I \), which is our required condition.

(ii) Let, \( I \) is a “strong hyper p-ideal of \( H \)”. Let, \((i \circ k) \circ (j \circ k) \ll I \text{ and } j \in I \). Then, \( \forall \alpha \in (i \circ k) \circ (j \circ k), \exists \beta \in I \) such that \( \alpha \ll \beta \). Thus \( 0 \in \alpha \circ \beta \) and \((\alpha \circ \beta) \cap I \neq \emptyset \), which along with \( \beta \in I \) implies \( \alpha \in I \), that is \((i \circ k) \circ (j \circ k) \subseteq I \). Thus \((i \circ k) \circ (j \circ k) \cap I \neq \emptyset \), which along with \( j \in I \) implies \( i \in I \), which is our required condition. \( \square \)
Generally, the converse of above theorem doesn’t hold. It can be observed by the following examples:

**Example 2.11.** “Let \( H = \{0, a, b\} \). We contemplate the following table:

<table>
<thead>
<tr>
<th>o</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>a</td>
<td>{a}</td>
<td>{0, a}</td>
<td>{0, a}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{b}</td>
<td>{0, a, b}</td>
</tr>
</tbody>
</table>

Then \( H \) is a hyper BCK-algebra”. Take \( I = \{0, b\} \). Clearly, \( I \) is a “weak hyper \( p \)-ideal of \( H \)”. But for \( (a \circ a) \circ (0 \circ a) = \{0, a\} \ll I \) and \( 0 \in I, a \notin I \), so \( I \) isn’t a “hyper \( p \)-ideal”.

**Example 2.12.** “We cogitate the table given below which explains the hyper BCK-algebra \( H = \{0, a, b\} \):

<table>
<thead>
<tr>
<th>o</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>a</td>
<td>{a}</td>
<td>{0, a}</td>
<td>{0, a}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{a, b}</td>
<td>{0, a, b}</td>
</tr>
</tbody>
</table>

Take \( I = \{0, a\} \)”. Clearly, \( I \) is a “hyper \( p \)-ideal” but not a “strong hyper \( p \)-ideal of \( H \)” as, \( (b \circ 0) \circ (a \circ 0) \cap I = \{a, b\} \cap I \neq \emptyset \) and \( a \in I \) but \( b \notin I \).

For detail study of “fuzzy (weak, strong) hyper BCK-ideals”, one must consult [10].

**Theorem 2.13.** [10] For any \( H \),
(i) “any fuzzy hyper BCK-ideal of \( H \) is a fuzzy weak hyper BCK-ideal of \( H \)”.
(ii) “any fuzzy strong hyper BCK-ideal of \( H \) is a fuzzy hyper BCK-ideal of \( H \)”.

3. Fuzzy Hyper \( p \)-ideals

Now we present the idea of “fuzzy (weak, strong) hyper \( p \)-ideals” and confer associated properties.

**Definition 3.1.** For a “hyper BCK-algebra” \( H \), a “fuzzy set” \( \omega \) in \( H \) is called a

- “fuzzy weak hyper \( p \)-ideal of \( H \)” if, for any \( a, b, c \in H \)
  \[ \omega(0) \geq \omega(a) \geq \min \{ \inf_{x \in \{0\}} \omega(x), \omega(b) \} \]

- “fuzzy hyper \( p \)-ideal of \( H \)” if, for any \( a, b, c \in H \)
  \[ \omega(a) \geq \min \{ \inf_{x \in \{0\}} \omega(x), \omega(b) \} \]

- “fuzzy strong hyper \( p \)-ideal of \( H \)” if, for any \( a, b, c \in H \)
  \[ \inf_{x \in \{0\}} \omega(x) \geq \omega(a) \geq \min \{ \sup_{x \in \{0\}} \omega(x), \omega(b) \} \]

**Theorem 3.2.** Any “fuzzy (weak, strong) hyper \( p \)-ideal” is a “fuzzy (weak, strong) hyper BCK-ideal”.
Proof. Let, $\tilde{\omega}$ is a “fuzzy hyper $p$-ideal of $H$”. Then, $\forall \ i, j, k \in H$ we get,

$$\tilde{\omega}(i) \geq \min \{ \inf_{a \in \mathcal{P}(i \lor (j \land k))} \tilde{\omega}(a), \tilde{\omega}(j) \}$$

Putting $k = 0$ we get,

$$\tilde{\omega}(i) \geq \min \{ \inf_{a \in \mathcal{P}(i \lor (j \lor 0))} \tilde{\omega}(a), \tilde{\omega}(j) \}$$

which gives,

$$\tilde{\omega}(i) \geq \min \{ \inf_{a \in \mathcal{P}(i \lor j)} \tilde{\omega}(a), \tilde{\omega}(j) \}$$

Hence proved. □

Generally, the converse of above theorem doesn’t hold. Consider the “hyper BCK-algebra $H = [0, a, b]$” defined by the table, given in Example (2.9). Define a “fuzzy set $\tilde{\omega}$ in $H$” by:

$$\tilde{\omega}(0) = 1, \tilde{\omega}(a) = 0.6, \tilde{\omega}(b) = 0$$

It is easy to substantiate that $\tilde{\omega}$ is a “fuzzy weak hyper BCK-ideal” but not a “fuzzy weak hyper $p$-ideal of $H$” as

$$\tilde{\omega}(a) = 0.6 < 1 = \min \{ \inf_{a \in \mathcal{P}(a \lor 0)} \tilde{\omega}(a), \tilde{\omega}(0) \}$$

Now, again consider the “hyper BCK-algebra $H = [0, a, b]$” defined by the table given in Example (2.9) and define a “fuzzy set $\tilde{\omega}$ in $H$” by:

$$\tilde{\omega}(0) = 0.8, \tilde{\omega}(a) = 0.5, \tilde{\omega}(b) = 0.3$$

Clearly $\tilde{\omega}$ is a “fuzzy hyper BCK-ideal” but not a “fuzzy hyper $p$-ideal” of $H$ since

$$\tilde{\omega}(a) = 0.5 < 0.8 = \min \{ \inf_{a \in \mathcal{P}(a \lor 0)} \tilde{\omega}(a), \tilde{\omega}(0) \}$$

Example 3.3. “Let $H = [0, a, b, c]$ be a hyper BCK-algebra defined by the table given below:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>[a]</td>
<td>[0, a]</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>[b]</td>
<td>[b]</td>
<td>[0, a]</td>
<td>[0, a]</td>
</tr>
<tr>
<td>c</td>
<td>[c]</td>
<td>[c]</td>
<td>[c]</td>
<td>[0, a]</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\tilde{\omega}$ in $H$ by”:

$$\tilde{\omega}(0) = \tilde{\omega}(a) = 1, \tilde{\omega}(b) = \frac{1}{2}, \tilde{\omega}(c) = \frac{1}{2}$$

Clearly, $\tilde{\omega}$ is a “fuzzy strong hyper BCK-ideal” of but not a “fuzzy strong hyper $p$-ideal” of $H$ since

$$\tilde{\omega}(b) = \frac{1}{2} < 1 = \min \{ \sup_{b \in \mathcal{P}(b \lor 0)} \tilde{\omega}(b), \tilde{\omega}(a) \}$$

Theorem 3.4. For any “hyper BCK-algebra”,

(i) “Any fuzzy hyper $p$-ideal is a fuzzy weak hyper $p$-ideal”.

(ii) “Any fuzzy strong hyper $p$-ideal is a fuzzy hyper $p$-ideal”.

Proof. (i) Let, $\tilde{\omega}$ be a “fuzzy hyper $p$-ideal of $H$”. Since, “every fuzzy hyper $p$-ideal is a fuzzy hyper BCK-ideal” (by Theorem 3.2) and “every fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal” (by Theorem 2.13(i)), therefore $\tilde{\omega}$ is also a “fuzzy weak hyper BCK-ideal of $H$”. Hence $\tilde{\omega}$ satisfies $\tilde{\omega}(i) \geq \tilde{\omega}(i)$, for all $i \in H$. Also being a “fuzzy hyper $p$-ideal” $\tilde{\omega}$ satisfies:

$$\tilde{\omega}(i) \geq \min \{ \inf_{a \in \mathcal{P}(i \lor (j \lor k))} \tilde{\omega}(a), \tilde{\omega}(j) \}$$

$\forall \ i, j, k \in H$. Hence $\tilde{\omega}$ is a “fuzzy weak hyper $p$-ideal of $H$".
(ii) Let $\omega$ is a “fuzzy strong hyper $p$-ideal of $H$”. Since, “every fuzzy strong hyper $p$-ideal is a fuzzy strong hyper BCK-ideal” (by Theorem 3.2) and “every fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal” (by Theorem 2.13(ii)), therefore $\omega$ is also a “fuzzy hyper BCK-ideal” of $H$. Hence for any $i, j \in H$, if $i \ll j$ then $\omega(i) \geq \omega(j)$.

Also being a “fuzzy strong hyper $p$-ideal”, $\omega$ satisfies for any $i, j, k \in H$

$$\omega(i) \geq \min \{\sup_{x \in (i \circ k) \cap (j \circ k)} \omega(x), \omega(j)\}$$

Since $\sup_{x \in (i \circ k) \cap (j \circ k)} \omega(x) \geq \omega(y), \forall y \in (i \circ k) \circ (j \circ k)$, therefore we get,

$$\omega(i) \geq \min \{\sup_{x \in (i \circ k) \cap (j \circ k)} \omega(x), \omega(j)\} \geq \min \{\omega(y), \omega(j)\},$$

for all $y \in (i \circ k) \circ (j \circ k)$

Since $\omega(y) \geq \inf_{x \in (i \circ k) \cap (j \circ k)} \omega(k), \forall y \in (i \circ k) \circ (j \circ k)$, therefore we have,

$$\omega(i) \geq \min \{\omega(y), \omega(j)\} \geq \min \{\inf_{x \in (i \circ k) \cap (j \circ k)} \omega(z), \omega(j)\},$$

that is

$$\omega(i) \geq \min \{\inf_{x \in (i \circ k) \cap (j \circ k)} \omega(z), \omega(j)\}$$

Hence proved. $\square$

Generally, the converse of above theorem doesn’t hold. Consider the “hyper BCK-algebra $H = [0, a, b]$” defined by the table given in Example (2.11). Define a “fuzzy set $\omega$ in $H$” by:

$$\omega(0) = 1, \omega(a) = 0.6, \omega(b) = 0.9$$

Then $\omega$ is a “fuzzy weak hyper $p$-ideal” but not a “fuzzy hyper $p$-ideal of $H$” as:

$$a \leq b \text{ but } \omega(a) = 0.6 < 0.9 = \omega(b)$$

**Example 3.5.** “Consider a hyper BCK-algebra $H = [0, a, b]$ defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$[0]$</th>
<th>$[a]$</th>
<th>$[b]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
</tr>
<tr>
<td>$a$</td>
<td>$[0, a]$</td>
<td>$[a]$</td>
<td>$[a]$</td>
</tr>
<tr>
<td>$b$</td>
<td>$[b]$</td>
<td>$[b]$</td>
<td>$[0, b]$</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\omega$ in $H$ by”:

$$\omega(0) = \omega(a) = 1, \omega(b) = \frac{1}{2}$$

Then $\omega$ is a “fuzzy hyper $p$-ideal” but it is not a “fuzzy strong hyper $p$-ideal of $H$” as:

$$\omega(b) = \frac{1}{2} < 1 = \min \{\sup_{x \in [0, b]} \omega(x), \omega(a)\}$$

**Theorem 3.6.** A “fuzzy set $\omega$ in $H$”, $\omega$ is a “fuzzy (weak, strong) hyper $p$-ideal of $H$” iff $\forall t \in [0, 1], \omega_t \neq \emptyset$ is a “(weak, strong) hyper $p$-ideal of $H$”.

*Proof.* Let, $\omega$ is a “fuzzy hyper $p$-ideal of $H$”. Since $\omega_t \neq \emptyset$, so for any $i \in \omega_t$, $\omega(i) \geq t$. “Since every fuzzy hyper $p$-ideal is also a fuzzy weak hyper $p$-ideal” (by Theorem 3.4(ii)), so $\omega$ is also a “fuzzy weak hyper $p$-ideal of $H$”. Thus $\omega(0) \geq \omega(i) \geq t$, for all $i \in H$, which implies $0 \in \omega_t$.

Let $(i \circ k) \circ (j \circ k) \ll \omega_t$ and $j \in \omega_t$, then $\forall x \in (i \circ k) \circ (j \circ k), \exists y \in \omega_t$ such that $x \ll y$. So $\omega(x) \geq \omega(y) \geq t, \forall x \in (i \circ k) \circ (j \circ k)$. Thus $\inf_{x \in (i \circ k) \cap (j \circ k)} \omega(x) \geq t$. Also $\omega(j) \geq t$, as $j \in \omega_t$. Therefore

$$\omega(i) \geq \min \{\inf_{x \in (i \circ k) \cap (j \circ k)} \omega(x), \omega(j)\} \geq \min \{t, t\} = t$$

$\Rightarrow i \in \omega_t$. Hence $\omega_t$ is “hyper $p$-ideal” of $H$.

Conversely, Let, “$\omega_t \neq \emptyset$ is a “hyper $p$-ideal of $H$”, $\forall t \in [0, 1]$”. Let $i \ll j$ for some $i, j \in H$ and put $\omega(j) = t$.

Then $j \in \omega_t$. So $i \ll j \in \omega_t \Rightarrow i \ll \omega_t$. “Being a hyper $p$-ideal, $\omega_t$ is also a hyper BCK-ideal of $H$” (By Theorem
(2.7)) therefore by Proposition 2.5, \( i \in \omega_i \). Hence \( \bar{\omega}(i) \geq t = \omega(j) \). That is \( i \ll j \Rightarrow \bar{\omega}(i) \geq \bar{\omega}(j) \), for all \( i, j \in H \).

Moreover, for any \( i, j, k \in H \), let \( d = \min \{ \inf_{x \in (i \circ k) \cap (j \circ k)} \bar{\omega}(x), \ \bar{\omega}(j) \} \). Then \( \bar{\omega}(j) \geq d \Rightarrow j \in \omega_d \) and for all \( e \in (i \circ k) \cap (j \circ k), \bar{\omega}(e) \geq \inf_{x \in (i \circ k) \cap (j \circ k)} \bar{\omega}(z) \geq d \), which implies \( e \in \omega_d \). Thus \((i \circ k) \cap (j \circ k) \subseteq \omega_d \). By Proposition 2.2(v), \((i \circ k) \cap (j \circ k) \subseteq \omega_d \Rightarrow (i \circ k) \circ (j \circ k) \ll \omega_d \), which along with \( j \in \omega_d \) implies \( i \in \omega_d \). Hence we get

\[ \bar{\omega}(i) \geq d = \min \{ \inf_{x \in (i \circ k) \circ (j \circ k)} \bar{\omega}(x), \ \bar{\omega}(j) \}. \]

Hence proved. \( \square \)

**Theorem 3.7.** If \( \bar{\omega} \) is a “fuzzy (weak, strong) hyper p-ideal of \( H \)" then, \( A = \{ i \in H | \omega(i) = \omega(0) \} \) is a “(weak, strong) hyper p-ideal of \( H \)".

**Proof.** Let, \( \bar{\omega} \) is a “fuzzy strong hyper p-ideal of \( H \)”. Clearly, \( 0 \in A \). Let \((i \circ k) \cap (j \circ k) \neq \emptyset \) and \( j \in A \) for some \( i, j, k \in H \). Then \( \exists i_0, \in (i \circ k) \cap (j \circ k) \cap A \) such that \( \bar{\omega}(i_0) = \omega(0) \). Also \( \bar{\omega}(j) = \omega(0) \). Then

\[
\bar{\omega}(i) \geq \min \{ \sup_{x \in (i \circ k) \cap (j \circ k)} \bar{\omega}(x), \ \bar{\omega}(j) \} \geq \min \{ \bar{\omega}(i_0), \ \bar{\omega}(j) \} = \min \{ \bar{\omega}(0), \ \bar{\omega}(0) \} = \bar{\omega}(0) \Rightarrow \bar{\omega}(i) \geq \bar{\omega}(0)
\]

“Being a fuzzy strong hyper p-ideal, \( \bar{\omega} \) is also a fuzzy weak hyper p-ideal of \( H \)” (by Theorem 3.4), so it satisfies \( \bar{\omega}(0) \geq \bar{\omega}(i), \ \forall i \in H \). Therefore \( \bar{\omega}(0) = \bar{\omega}(i) \) and so \( i \in A \). Hence proved. \( \square \)

Likewise, as done above, we can Corroborate the result for the other two cases. The “transfer principle” for “fuzzy sets” described in [15] suggest the following result.

**Theorem 3.8.** Let \( \bar{\omega} \) be a “fuzzy set in \( H \)” defined by:

\[
\bar{\omega}(a) = \begin{cases} 
  t & \text{if } a \in A \\
  0 & \text{if } a \notin A 
\end{cases}
\]

\( \forall a \in H, \text{ where, } A \subseteq H \text{ and } t \in (0, 1] \). Then, “\( A \) is a (weak, strong) hyper p-ideal iff \( \bar{\omega} \) is a fuzzy (weak, strong) hyper p-ideal”.

**Proof.** Let, \( A \) is a “strong hyper p-ideal of \( H \)”. Then for any \( i, j, k \in H \) if \( (i \circ k) \cap (j \circ k) \neq \emptyset \) and \( j \in A \) then \( i \in A \). Thus we have

\[
\bar{\omega}(i) = t = \min \{ \sup_{x \in (i \circ k) \cap (j \circ k)} \bar{\omega}(x), \ \bar{\omega}(j) \}
\]

If \((i \circ k) \cap (j \circ k) \cap A = \emptyset \) and \( j \notin A \) then \( \bar{\omega}(y) = 0 \), \( \forall y \in (i \circ k) \cap (j \circ k) \) and \( \bar{\omega}(j) = 0 \), therefore

\[
\min \{ \sup_{x \in (i \circ k) \cap (j \circ k)} \bar{\omega}(x), \ \bar{\omega}(j) \} = 0 \
\]

Now by Proposition 2.2(ii), “we have \( i \circ i \leq i, \ \forall i \in H \)”. Then, \( \forall z \in i \circ i, z \ll i \).

“Being a strong hyper p-ideal of \( H, A = \omega_i \) is a hyper p-ideal of \( H \)” (by Theorem 2.10(ii)) and hence \( \bar{\omega} \) is a “fuzzy hyper p-ideal” of \( H \) (by Theorem 3.6). Therefore

\[
z \ll i \Rightarrow \bar{\omega}(z) \geq \bar{\omega}(i), \ \forall z \in i \circ i
\]

Hence \( \bar{\omega} \) is a “fuzzy strong hyper p-ideal” of \( H \).

Conversely, Let \( \bar{\omega} \) is a “fuzzy strong hyper p-ideal of \( H \)”. Then, by Theorem 3.6, “\( \forall t \in (0, 1], \omega_t = A \) is a strong hyper p-ideal of \( H \)”. Correspondingly, we can verify the result for the other two types of ideals. \( \square \)
Theorem 3.9. The family of “fuzzy strong hyper p-ideals” is a “completely distributive lattice with respect to join and meet”.

Proof. Let \( \{ \omega_i \mid i \in I \} \) be a family of “fuzzy strong hyper p-ideals of \( H \)”. “Since \([0,1]\) is a completely distributive lattice with respect to the usual ordering in \([0,1]\), it is sufficient to corroborate that, \( \bigvee_{i \in I} \omega_i \) and \( \bigwedge_{i \in I} \omega_i \) are “fuzzy strong hyper p-ideals of \( H \)”.

For any \( a \in H \) we have,
\[
\inf_{x \in H} (\bigvee_{i \in I} \omega_i)(x) = \inf_{x \in H} (\sup_{i \in I} \omega_i(x))
\]
\[
= \sup_{i \in I} (\inf_{x \in H} \omega_i(x)) \geq \sup_{i \in I} \omega_i(a) = (\bigvee_{i \in I} \omega_i)(a)
\]
\[
\Rightarrow \inf_{x \in H} (\bigvee_{i \in I} \omega_i)(x) \geq (\bigvee_{i \in I} \omega_i)(a)
\]
Moreover, for any \( a, b, c \in H \), we have
\[
(\bigvee_{i \in I} \omega_i)(a) = \sup_{i \in I} \omega_i(a) \geq \sup_{i \in I} [\min \{\sup_{y \in [0,1]} \omega_i(y), \omega_i(b)\}]
\]
\[
= \min \{\sup_{i \in I} (\sup_{y \in [0,1]} \omega_i(y)), \sup_{i \in I} \omega_i(b)\}
\]
\[
= \min \{\sup_{y \in [0,1]} (\bigvee_{i \in I} \omega_i)(y), (\bigvee_{i \in I} \omega_i)(b)\}
\]
\[
\Rightarrow (\bigvee_{i \in I} \omega_i)(a) \geq \min \{\sup_{y \in [0,1]} (\bigvee_{i \in I} \omega_i)(y), (\bigvee_{i \in I} \omega_i)(b)\}
\]
Hence \( \bigvee_{i \in I} \omega_i \) is a “fuzzy strong hyper p-ideal” of \( H \).

Now, we prove that \( \bigwedge_{i \in I} \omega_i \) is a “fuzzy strong hyper p-ideal of \( H \)”.

For any \( a \in H \) we have,
\[
\inf_{x \in H} (\bigwedge_{i \in I} \omega_i)(x) = \inf_{x \in H} (\inf_{i \in I} \omega_i(x))
\]
\[
= \inf_{i \in I} (\inf_{x \in H} \omega_i(x)) \geq \inf_{i \in I} \omega_i(a) = (\bigwedge_{i \in I} \omega_i)(a)
\]
\[
\Rightarrow \inf_{x \in H} (\bigwedge_{i \in I} \omega_i)(x) \geq (\bigwedge_{i \in I} \omega_i)(a)
\]
Moreover, for any \( a, b, c \in H \), we have
\[
(\bigwedge_{i \in I} \omega_i)(a) = \inf_{i \in I} \omega_i(a) \geq \inf_{i \in I} [\min \{\sup_{y \in [0,1]} \omega_i(y), \omega_i(b)\}]
\]
\[
= \min \{\inf_{i \in I} (\sup_{y \in [0,1]} \omega_i(y)), \inf_{i \in I} \omega_i(b)\}
\]
\[
= \min \{\inf_{y \in [0,1]} (\bigwedge_{i \in I} \omega_i)(y), (\bigwedge_{i \in I} \omega_i)(b)\}
\]
\[
\Rightarrow (\bigwedge_{i \in I} \omega_i)(a) \geq \min \{\inf_{y \in [0,1]} (\bigwedge_{i \in I} \omega_i)(y), (\bigwedge_{i \in I} \omega_i)(b)\}
\]
Hence \( \bigwedge_{i \in I} \omega_i \) is a “fuzzy strong hyper p-ideal of \( H \)”.

Hence proved. \( \square \)

Correspondingly, as done above, we can Corroborate the result for the other two cases. For the definition of “the stronges fuzzy relation on \( H \)”, one must see [1].

Theorem 3.10. Let \( \omega \) be a “fuzzy set” and let \( \lambda_{ab} \) be “the strongest fuzzy relation on \( H \)”. \( \omega \) is a “fuzzy strong hyper p-ideal iff \( \lambda_{ab} \) is a fuzzy strong hyper p-ideal of \( H \times H \)”.

Proof. Let, \( \omega \) is a “fuzzy strong hyper p-ideal of \( H \)”. Consider
\[
\inf_{(x,y) \in (i_1,i_2) \times (j_1,j_2)} \lambda_{ab}(x,y) = \inf_{(x,y) \in (i_1,i_2) \times (j_1,j_2)} [\min \{\omega(x), \omega(y)\}]
\]
\[
= \min \{\inf_{(x,y) \in (i_1,i_2)} \omega(x), \inf_{(x,y) \in (i_1,i_2)} \omega(y)\} \geq \min \{\omega(i_1), \omega(i_2)\} = \lambda_{ab}(i_1,i_2)
\]
\[
\Rightarrow \inf_{(x,y) \in (i_1,i_2) \times (j_1,j_2)} \lambda_{ab}(x,y) \geq \lambda_{ab}(i_1,i_2), \forall (i_1,i_2) \in H \times H
\]
Now, for any \((i_1,i_2), (j_1,j_2), (k_1,k_2) \in H \times H\), consider
\[ \lambda_\omega(i_1, i_2) = \min \{ \omega(i_1), \omega(i_2) \} \geq \min \{ \sup_{z \in (i_1 \circ k_1) \circ (j_1 \circ k_1)} \omega(z), \sup_{d \in (i_2 \circ k_2) \circ (j_2 \circ k_2)} \omega(d) \}, \omega(j_2) \]  

where  

\[ z \in (i_1 \circ k_1) \circ (j_1 \circ k_1) \quad \text{and} \quad d \in (i_2 \circ k_2) \circ (j_2 \circ k_2) \]

\[ \Rightarrow \lambda_\omega(i_1, i_2) \geq \min \{ \sup \{ \min \{ \omega(z), \omega(d) \}, \lambda_\omega(j_1, j_2) \} \} \]

Hence, \( \lambda_\omega \) is a “fuzzy strong hyper \( p \)-ideal of \( H \times H \).

Conversely, let \( \lambda_\omega \) is a “fuzzy strong hyper \( p \)-ideal of \( H \times H \). Then, we have

\[ \inf_{(x,y) \in \{(i_1,i_2)\circ(k_1,k_2)\}} \lambda_\omega(x,y) \geq \lambda_\omega(i_1, i_2), \forall (i_1,i_2) \in H \times H \]

\[ \Rightarrow \inf_{(x,y) \in \{(i_1,i_2)\circ(k_1,k_2)\}} \min \{ \omega(x), \omega(y) \} \geq \min \{ \omega(i_1), \omega(i_2) \} \]

\[ \Rightarrow \min \{ \inf_{x \in i_1} \omega(x), \inf_{x \in i_2} \omega(x) \} \geq \min \{ \omega(i_1), \omega(i_2) \} \]

\[ \Rightarrow \inf_{x \in i_1} \omega(x) \geq \sup \{ \omega(i), \omega(i) \}, \forall i \in H \]

Hence the first condition for \( \omega \) to be a “fuzzy strong hyper \( p \)-ideal” is satisfied.

Note that “being a fuzzy strong hyper \( p \)-ideal of \( H \times H \), \( \lambda_\omega \) is also a fuzzy weak hyper \( p \)-ideal of \( H \times H \)” (by Theorem 3.4), thus \( \lambda_\omega \) satisfies

\[ \lambda_\omega(0, 0) \geq \lambda_\omega(i, i), \forall (0, 0), (i, i) \in H \times H \]

\[ \Rightarrow \min \{ \omega(0), \omega(0) \} \geq \min \{ \omega(i), \omega(i) \} \]

\[ \Rightarrow \omega(0) \geq \omega(i), \forall i \in H \]

Now, for any, \( (i_1, i_2), (j_1, j_2), (k_1, k_2) \) in \( H \times H \), \( \lambda_\omega \) satisfies

\[ \Rightarrow \lambda_\omega(i_1, i_2) \geq \min \{ \sup \lambda_\omega(e, f), \lambda_\omega(j_1, j_2) \} \]

where

\[ (e, f) \in ((i_1, i_2) \circ (k_1, k_2)) \circ ((j_1, j_2) \circ (k_1, k_2)) \]

\[ \Rightarrow \min \{ \omega(i_1), \omega(i_2) \} \geq \min \{ \sup \{ \min \{ \omega(e), \omega(f) \}, \min \{ \omega(j_1), \omega(j_2) \} \} \}

Putting \( i_1 = j_1 = k_1 = 0 \) we get

\[ \Rightarrow \min \{ \omega(0), \omega(0) \} \geq \min \{ \sup \{ \min \{ \omega(0), \omega(f) \}, \min \{ \omega(0), \omega(j_2) \} \} \}

Where

\[ (e, f) \in (0, (i_2 \circ k_2) \circ (j_2 \circ k_2)) \]

\[ \Rightarrow \omega(i_2) \geq \min \{ \sup_{e \circ k_2} \omega(f), \omega(j_2) \} \], \text{since} \ \omega(0) \geq \omega(i), \ \forall i \in H \]

Similarly by putting \( i_2 = j_2 = k_2 = 0 \), we get,
Hence proved. 

Hence we confer the product of two fuzzy hyper $p$-ideals. 

\[ \Rightarrow \omega(i_1) \geq \min \{\sup_{x \in (i_1 \times k_1) \cap (j_1 \times k_1)} \omega(e), \ \omega(j_1)\} \]

\[ \text{Hence } \omega \text{ is a “fuzzy strong hyper } p \text{-ideal of } H'. \]

Identically, as done above, we can corroborate the statement for the other two cases.

**Theorem 3.11.** Let, \( f : X \to Y \) be an onto hyper BCK-algebras from a hyper BCK-algebra \( X \) to a hyper BCK-algebra \( Y \). If, \( \nu \) is a “fuzzy strong hyper \( p \)-ideal of \( Y \) then the hyper homomorphic pre-image \( \omega \) of \( \nu \) under \( f \) is a fuzzy strong hyper \( p \)-ideal of \( X \).

**Proof.** Let, \( \nu \) is a “fuzzy strong hyper \( p \)-ideal of \( Y \)”. Since, \( \omega \) is a “hyper homomorphic pre-image” of \( \nu \) under \( f \), so \( \omega \) is defined by \( \omega = \nu \circ f \) that is \( \omega(i) = \nu(f(i)), \forall i \in X \). Since \( \nu \) satisfies \( \inf_{f(i) = f(j), f(k) = f(l)} (\nu(f(i)), \nu(f(j))) \geq (\nu(f(k)), \nu(f(l))) \), \( \forall i, j, k, l \in X \) and \( f(i), f(j), f(k), f(l) \in Y \)

\[ \Rightarrow \inf_{x \in \omega} \omega(x) \geq \omega(i), \forall i \in X \]

Now for any \( i, j, k \in X \) consider

\[ \omega(i) = \nu(f(i)) \geq \min \{\sup_{y \in (i_1 \times k_1) \cap (j_1 \times k_1)} \nu(f(y)), \ \nu(f')\} \]

where \( f', k' \in Y \). Since \( f : X \to Y \) is an onto “hyper BCK-algebras”, so for \( f', k' \in Y, \exists j, k \in X \) such that \( f(j) = f', f(k) = k' \). Hence we get \( \omega(i) \geq \min \{\sup_{y \in (i_1 \times k_1) \cap (j_1 \times k_1)} \nu(f(y)), \ \nu(f(j))\} \)

\[ \Rightarrow \omega(i) \geq \min \{\sup_{y \in (i_1 \times k_1) \cap (j_1 \times k_1)} \omega(y), \ \omega(j)\}, \forall i, j, k \in X \]

Hence proved. 

Correspondingly, as done above, we can corroborate the statement for “fuzzy (weak) hyper \( p \)-ideals”. Lastly, we confer the product of two fuzzy \( p \)-ideals. One may consult [3], for basic material on the “product of fuzzy hyper BCK-ideals”.

**Theorem 3.12.** A fuzzy set \( \omega = \omega_1 \times \omega_2 \) is a “fuzzy (weak, strong) hyper \( p \)-ideal” of \( H = H_1 \times H_2 \) iff \( \omega_1 \) and \( \omega_2 \) are “fuzzy (weak, strong) hyper \( p \)-ideals” of \( H_1 \) and \( H_2 \) respectively.

**Proof.** Let \( \omega = \omega_1 \times \omega_2 \) be a “fuzzy hyper \( p \)-ideal” of \( H = H_1 \times H_2 \) and let \( i_1 \ll i_2 \) for some \( i_1, i_2 \in H_1 \). Then \( (i_1, 0) \ll (i_2, 0) \) which implies \( \omega((i_1, 0)) = \omega_1(i_1) \geq \omega((i_2, 0)) = \omega_1(i_2) \), that is, \( \omega_1(i_1) \geq \omega_1(i_2) \)

Moreover for any \( i_1, j_1, k_1 \in H_1 \), let \( t = \min\{\inf_{i \in (i_1 \times k_1) \cap (j_1 \times k_1)} \omega_1(a), \omega_1(j_1)\} \)

Then, \( \forall b \in (i_1 \circ k_1) \circ (j_1 \circ k_1), \omega_1(b) \geq \inf_{i \in (i_1 \times k_1) \cap (j_1 \times k_1)} \omega_1(a) \geq t \) and \( \omega_1(j_1) \geq t \)

\[ \Rightarrow \omega((b, 0)) \geq t \text{ and } \omega((j_1, 0)) \geq t, \forall (b, 0) \in ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0)) \]

\[ \Rightarrow \omega((b, 0)) \in \omega_1 \text{ and } (j_1, 0) \in \omega_1 \]

\[ \Rightarrow ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0)) \subseteq \omega_1 \text{ and } (j_1, 0) \in \omega_1 \]

\[ \Rightarrow ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0)) \ll \omega_1 \text{ and } (j_1, 0) \in \omega_1 \]

\[ \Rightarrow (i_1, 0) \in \omega_1, “\text{since } \omega_1 \text{ is a hyper } p \text{-ideal}” \text{ (by Theorem 3.6).} \]

Therefore, \( \omega((i_1, 0)) \geq t \). Thus \( \omega_1(i_1) \geq t = \min\{\inf_{i \in (i_1 \times k_1) \cap (j_1 \times k_1)} \omega_1(a), \omega_1(j_1)\} \), which is our required condition. Likewise, it can be proved that \( \omega_2 \) is a “fuzzy hyper \( p \)-ideal” of \( H_2 \). Conversely suppose that \( \omega_1 \) and \( \omega_2 \) are “fuzzy hyper \( p \)-ideals of \( H_1 \) and \( H_2 \)” respectively. For any \( (i, l), (j, m) \in H = H_1 \times H_2 \), where \( i, j \in H_1 \) and \( l, m \in H_2 \), let \( (i, l) \ll (j, m) \).
Hence proved. □

Correspondingly, as done above, we can corroborate the statement for the other two cases.

4. Conclusion

From our above discussion we can conclude that:

• a “(fuzzy) strong hyper p-ideal” is a “(fuzzy) hyper p-ideal” and a “(fuzzy) hyper p-ideal” is a “(fuzzy) weak hyper p-ideal”.

• $\lambda_\omega$, “the strongest fuzzy relation” on a “hyper BCK-algebra”, is a “fuzzy (weak, strong) hyper p-ideal” in case, $\omega$ is a “fuzzy (weak, strong) hyper p-ideal”.

• “Hyper homomorphic pre-image”, defined on an “onto hyper homomorphism”, of a “fuzzy (weak, strong) hyper p-ideal” is also a “fuzzy (weak, strong) hyper p-ideal”.

• The product of two “fuzzy (weak, strong) hyper p-ideal” is again a “fuzzy (weak, strong) hyper p-ideal”.

References