On the Local Uniqueness of the Fixed Point of the Probabilistic $q$-Contraction in Fuzzy Metric Spaces

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Abstract. In this paper we prove the local uniqueness of the fixed point of the probabilistic $q$-contraction in fuzzy metric space.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [21] for mappings $f : S \rightarrow S$, on Menger space $(S, \mathcal{F}, T_M)$, where $T_M = \min$. The real operation of triangular norms was introduced in the theory of probabilistic metric spaces by K. Menger [15], see [7–9, 11, 20, 22]. It turns out that t-norms are crucial operations in several fields, e.g., in statistics by copulas ([13, 14]), fuzzy sets, fuzzy logics (see [11]) and their applications, but also, among other fields, in the theory of generalized measures [11, 17, 23] and in nonlinear differential and difference equations [17]. Further investigations of the fixed point theory in a more general Menger space $(S, \mathcal{F}, T)$ was connected with investigations of the structure of the t-norm $T$, see [1, 4, 7]. Further development of the fixed point theory was obtained in a more general space - fuzzy metric spaces, see [2, 3, 6, 7, 16, 24].

We present in this paper a result on the local uniqueness of fixed point in fuzzy metric space. In Section 2 we give some results related t-norms. In Section 3 we give the definition of fuzzy metric space and Section 4 is devoted to the main result of the paper, the local uniqueness of the fixed point of the probabilistic $q$-contraction in fuzzy metric space.

2. Triangular Norms

A triangular norm (t-norm for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$ which is commutative, associative, monotone and $T(x,1) = x$ for every $x \in [0,1]$. A method of construction a new t-norm from a system of given t-norms is given in the following theorem, see [7, 11].
is investigated for some classes of t-norms
\( T \) holds also for

The equivalence

In the classical case

An arbitrary t-norm \( T \) can be extended (by associativity) in a unique way to an \( n \)-ary operation taking for \( (x_1, \ldots, x_n) \in [0, 1]^n \), \( n \in \mathbb{N} \), the values \( T(x_1, \ldots, x_n) \) which is defined by

Specially, we have

We can extend \( T \) to a countable infinitary operation taking for any sequence \( (x_n)_{n \in \mathbb{N}} \) from \([0, 1]\) the values

The limit on the right side of (1) exists since the sequence \( \left( \prod_{i=1}^{n} x_i \right)_{n \in \mathbb{N}} \) is non-increasing and bounded from below.

In the fixed point theory it is of interest to investigate the classes of t-norms \( T \) and sequences \( (x_n)_{n \in \mathbb{N}} \) from the interval \([0, 1]\) such that \( \lim_{n \to \infty} x_n = 1 \), and

In the classical case \( T = T_P \) we have \( (T_P)_{i=1}^{n} = \prod_{i=1}^{n} x_i \) and for every sequence \( (x_n)_{n \in \mathbb{N}} \) from the interval \([0, 1]\)

with

it follows that

The equivalence

holds also for \( T \geq T_L \).

In the paper [4] the condition

is investigated for some classes of t-norms \( T \) and sequences \( (x_i)_{i \in \mathbb{N}} \) from \([0, 1]\).
3. Fuzzy Metric Spaces

By [12] we have the following definition.

**Definition 2** A fuzzy metric space in the sense of Kramosil and Michálek is a triple \((X, M, T)\), where \(X\) is a nonempty set, \(T\) is a t-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty[\) satisfying the following conditions for all \(x, y, z \in X\) and \(s, t > 0\)

\begin{align*}
(FM-1) & \ M(x, y, 0) = 0; \\
(FM-2) & \ M(x, y, t) = 1, \text{ for all } t > 0 \text{ if and only if } x = y; \\
(FM-3) & \ M(x, y, t) = M(y, x, t); \\
(FM-4) & \ M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s)); \\
(FM-5) & \ M(x, y, \cdot) : \mathbb{R}^+ \to [0, 1] \text{ is left continuous.}
\end{align*}

We additionally suppose that \(M(x, y, t) > 0\) for \(t > 0\).

A sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) is a Cauchy sequence if for every \(\varepsilon > 0\) and \(\lambda \in ]0, 1[\) there exists \(n_0(\varepsilon, \lambda) \in \mathbb{N}\) such that \(M(x_n, x_m, \varepsilon) > 1 - \lambda\), for every \(n, m \geq n_0(\varepsilon, \lambda)\). A fuzzy metric space is complete if every Cauchy sequence converges.

4. A Fixed Point Theorem in Fuzzy Metric Spaces

It is well known that the uniqueness of a fixed point of probabilistic \(q\)-contraction does not follow immediately, as in the case of a Menger space, since \(\lim_{t \to \infty} M(x, y, t) = 1\) does not hold generally. One of the solution of this problem is to assume that on \((X, M, T)\) the following condition holds

\begin{equation}
M(x, y, t) \equiv C, \text{ for every } t > 0 \text{ implies } C = 1. \quad (2)
\end{equation}

In this paper we shall prove that a kind of the local uniqueness can be obtained without condition (2).

Let \(\text{Fix}(f)\) denote the set of fixed points of a function \(f : X \to X\).

**Definition 3** Let \((X, M, T)\) be a fuzzy metric space. A mapping \(f : X \to X\) is a probabilistic \(q\)-contraction \((q \in ]0, 1[)\) if

\[ M(f p_1, f p_2, x) \geq M(p_1, p_2, \frac{x}{q}) \]

for every \(p_1, p_2 \in X\) and every \(x \in \mathbb{R}^+\).

**Theorem 4** Let \((X, M, T)\) be a complete fuzzy metric space, \(T\) a continuous t-norm at the point \((1, 1)\), \(f : X \to X\) a probabilistic \(q\)-contraction and there exists \(x_0 \in X\) such that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M(x_0, f x_0, \frac{1}{q}) = 1. \quad (3) \]

If \(x = \lim_{n \to \infty} f^* x_0\) and

\[ A = \{y \mid y \in X, \lim_{t \to \infty} M(x_0, y, t) = 1\}, \]

then \(A \cap \text{Fix}(f) = \{x\}\).
Proof. Condition (3) implies the existence of $\lim_{n \to \infty} f^n x_0$, as in the case of Menger spaces, and the continuity of $f$ implies that $x \in \text{Fix}(f)$, see [7].

Firstly, we shall prove that $x \in A$, i.e., that

$$\lim_{t \to \infty} M(x_0, x, t) = 1. \quad (4)$$

In order to prove (4) we shall prove that for every $\lambda \in ]0, 1[\] there exists $t' > 0$ such that $M(x_0, x, t') > 1 - \lambda$. Let $n, m \in \mathbb{N}$. Then

$$M(x_0, f^n x_0, \frac{1}{q^n}) = T(M(x_0, f^n x_0, \frac{1}{q^n})) = T(T(\cdots(T(1, M(x_0, f^n x_0, \frac{1}{q^n}))\cdots))$$

$$(m)\text{-times}$$

$$\geq \prod_{i=n}^{\infty} M(x_0, f^n x_0, \frac{1}{q^n}).$$

Therefore by (3) we obtain

$$\lim_{n \to \infty} M(x_0, f^n x_0, \frac{1}{q^n}) = 1. \quad (5)$$

Since $M(x_0, f^n x_0, \cdot)$ is nondecreasing we obtain that

$$\lim_{t \to \infty} M(x_0, f^n x_0, t) = 1. \quad (6)$$

Since for every $m \in \mathbb{N}$ and $t > 0$ we have

$$M(f^m x_0, f^{m+1} x_0, t) \geq M(f^{m-1} x_0, f^m x_0, \frac{t}{q}) \geq \cdots \geq M(x_0, f^m x_0, \frac{t}{q^m}).$$

(5) implies that for every fixed $m \in \mathbb{N}$ we obtain

$$\lim_{t \to \infty} M(f^m x_0, f^{m+1} x_0, t) = 1. \quad (6)$$

Let $n$ be an arbitrary but fixed natural number. Then for every $t > 0$ we have

$$M(x_0, f^n x_0, t) \geq T(M(x_0, f^n x_0, \frac{t}{2}), M(f^{n-1} x_0, f^n x_0, \frac{t}{2}))$$

$$\geq \cdots$$

$$\geq T(T(\cdots(T(M(x_0, f^n x_0, \frac{t}{2^n-1}), M(f^{n-1} x_0, f^n x_0, \frac{t}{2^n-1})), \cdots, M(f^{n-1} x_0, f^n x_0, \frac{t}{2^n-1})))).$$

Since the t-norm $T$ is continuous at the point $(1, 1)$ then (6) implies that

$$\lim_{t \to \infty} M(x_0, f^n x_0, t) = 1 \quad (7)$$
for a fixed \( n \in \mathbb{N} \). Let \( \lambda \in ]0,1[, \; t > 0 \), and \( \delta(\lambda) \in ]0,1[ \) such that
\[
T(1 - \delta, 1 - \delta) > 1 - \lambda.
\]

Since \( \lim_{n \to \infty} f^n x_0 = x \) there exists \( n_0(t, \delta) \in \mathbb{N} \) such that
\[
M(x, f^{n_0} x_0, \frac{t}{2}) > 1 - \delta.
\]

By (7) we obtain that there exists \( t(\delta) > 0 \) such that
\[
M(x_0, f^{n_0} x_0, \frac{t(\delta)}{2}) > 1 - \delta.
\]

Let \( t' = \max\{t, t(\delta)\} \). Then we obtain
\[
M(x, x_0, t') \geq T(M(x, f^{n_0} x_0, \frac{t}{2}), M(f^{n_0} x_0, x_0, \frac{t(\delta)}{2}))
\]
\[
> T(1 - \delta, 1 - \delta)
\]
\[
> 1 - \lambda.
\]

Therefore \( x \in A \cap \text{Fix}(f) \).

If \( y \in A \cap \text{Fix}(f) \) then \( y = fy \) and \( \lim_{n \to \infty} M(x_0, y, t) = 1 \). Then
\[
M(x, y, t) = M(fx, fy, t)
\]
\[
\geq M(x, y, \frac{t}{q})
\]
\[
\ldots
\]
\[
\geq M(x, y, \frac{t}{q^n})
\]
\[
\geq T(M(x, y, \frac{t}{2q^n}), M(x, y, \frac{t}{2q^n})).
\]

Therefore
\[
M(x, y, t) \geq T\left( \lim_{n \to \infty} M(x, y, \frac{t}{2q^n}), \lim_{n \to \infty} M(x, y, \frac{t}{2q^n}) \right) = T(1, 1) = 1.
\]

Hence \( x = y \) and so \( A \cap \text{Fix}(f) = \{x\} \). \( \square \)

**Remark 5** For a class of \( \varphi \)-probabilistic contraction and \( t \)-norm of H-type Fang [2] proved a similar result (Theorem 4.1) about the local uniqueness of the fixed point in fuzzy metric spaces.

**References**


