C-Class Functions and Pair \((F, h)\) upper Class on Common Best Proximity Points Results for New Proximal C-Contraction mappings

A.H.Ansari\(^a\), Geno Kadwin Jacob\(^b\), D. Chellapillai\(^b\)

\(^a\)Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
\(^b\)Department of Mathematics, Bharathidasan University,Trichy- 620 024, Tamil Nadu, India.

Abstract. In this paper, using the concept of C−class and Upper class functions we prove the existence of unique common best proximity point. Our main result generalizes results of Kumam et al. \([17]\) and Parvaneh et al. \([21]\).

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Consider a pair \((A, B)\) of nonempty subsets of a metric space \((X, d)\). Assume that \(f\) is a function from \(A\) into \(B\). An element \(w \in A\) is said to be a best proximity point whenever 

\[ d(w, f\, w) = d(A, B), \]

where 

\[ d(A, B) = \inf\{d(s, t) : s \in A, t \in B\}. \]

Best proximity point theory of non-self functions was initiated by Fan \([1]\) and Kirk et al. \([16]\); see also \([17][15][11][13] [4][8][9][24][25][20][18]\).

Definition 1.1. Consider non-self functions \(f_1, f_2, \ldots, f_n : A \to B\). We say the a point \(s \in A\) is a common best proximity point of \(f_1, f_2, \ldots, f_n\) if

\[ d(s, f_1(s)) = d(s, f_2(s)) = \cdots = d(s, f_n(s)) = d(A, B). \]

Definition 1.2. \([17]\)Let \((X, d)\) be a metric space and \(\emptyset \neq A, B \subset X\). We say the pair \((A, B)\) has the V-property if for every sequence \(\{t_n\}\) of \(B\) satisfying 

\[ d(s, t_n) \to d(s, B) \]

for some \(s \in A\), there exists a \(t \in B\) such that 

\[ d(s, t) = d(s, B). \]

Definition 1.3. \([5]\) A continuous function \(F : [0, \infty)^2 \to \mathbb{R}\) is called C-class function if for any \(s, t \in [0, \infty)\), the following conditions hold:

1. \(F(s, t) \leq s;\)
2. \(F(s, t) = s\) implies that either \(s = 0\) or \(t = 0\).

2010 Mathematics Subject Classification. Primary 47H10; Secondary 54H25

Keywords. common best proximity point; triangular α-proximal admissible; proximal C-contraction, C-class functions, pair \((F, h)\) upper class

Received: 17 September 2016; Accepted: 30 January 2017
Communicated by Vladimir Rakočević
Email addresses: analysisamirmath2@gmail.com (A.H.Ansari), genomaths@gmail.com (Geno Kadwin Jacob), chellapillai2@gmail.com (D. Chellapillai)
An extra condition on $F$ that $F(0, 0) = 0$ could be imposed in some cases if required. The letter $C$ will denote the class of all $C$-functions.

**Example 1.4.** ([5]) Following examples show that the class $C$ is nonempty:

1. $F(s, t) = s - t$.
2. $F(s, t) = m + t$, for some $m \in (0, 1)$.
3. $F(s, t) = \frac{n^2}{(1 + t^2)}$ for some $r \in (0, \infty)$.
4. $F(s, t) = \frac{\log(t + a^2)}{(1 + t)}$, for some $a > 1$.
5. $F(s, t) = \frac{1}{r(1 + t)} \int_0^\infty \frac{e^{-x}}{\sqrt{x^2 + r}}$ where $\Gamma$ is the Euler Gamma function.

**Definition 1.5.** ([6, 7]) We say that the function $h: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function of subclass of type $I$, if $x \geq 1 \implies h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

**Example 1.6.** ([6, 7]) Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by:

- $(a)$ \( h(x, y) = (y + l)^r, l > 1 \);
- $(b)$ \( h(x, y) = (x + l)^r, l > 1 \);
- $(c)$ \( h(x, y) = x^m y, m \in \mathbb{N} \);
- $(d)$ \( h(x, y) = y \);
- $(e)$ \( h(x, y) = \frac{1}{n+1} \left( \sum_{i=1}^n y^i \right), n \in \mathbb{N} \);
- $(f)$ \( h(x, y) = \left[ \frac{1}{n+1} \left( \sum_{i=1}^n y^i \right) + l \right]^r, l > 1, n \in \mathbb{N} \)

for all $x, y \in \mathbb{R}^+$. Then $h$ is a function of subclass of type $I$.

**Definition 1.7.** ([6, 7]) Let $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^* \rightarrow \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class, if $h$ is a function of subclass of type $I$ and: (i) $0 \leq s \leq 1 \implies F[s, t] \leq F[1, t]$, (ii) $h(1, y) \leq F[1, t] \implies y \leq t$ for all $t, y \in \mathbb{R}^+$.

**Example 1.8.** ([6, 7]) Define $h, F : \mathbb{R}^+ \times \mathbb{R}^* \rightarrow \mathbb{R}$ by:

- $(a)$ \( h(x, y) = (y + l)^r, l > 1 \) and $F[s, t] = st + l$;
- $(b)$ \( h(x, y) = (x + l)^r, l > 1 \) and $F[s, t] = (1 + l)^s$;
- $(c)$ \( h(x, y) = x^m y, m \in \mathbb{N} \) and $F[s, t] = st$;
- $(d)$ \( h(x, y) = y \) and $F[s, t] = t$;
- $(e)$ \( h(x, y) = \frac{1}{n+1} \left( \sum_{i=1}^n x^i \right), n \in \mathbb{N} \) and $F[s, t] = st$;
- $(f)$ \( h(x, y) = \left[ \frac{1}{n+1} \left( \sum_{i=1}^n x^i \right) + l \right]^r, l > 1, n \in \mathbb{N} \) and $F[s, t] = (1 + l)^s$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(F, h)$ is an upper class of type $I$.

Let $\Phi_\alpha$ denote the class of the functions $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ which satisfy the following conditions:

- $(a)$ $\varphi$ continuous;
- $(b)$ $\varphi(u, v) > 0, (u, v) \neq (0, 0)$ and $\varphi(0, 0) \geq 0$.

Let $\Psi_\alpha$ be a set of all continuous functions $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- $(\psi_1)$ $\psi$ is continuous and strictly increasing.
- $(\psi_2)$ $\psi(t) = 0$ if and only if $t = 0$.

Also we denote by $\Psi$ the family of all continuous functions from $[0, +\infty) \times [0, +\infty)$ to $[0, +\infty)$ such that $\psi(u, v) = 0$ if and only if $u = v = 0$ where $\psi \in \Psi$.
Lemma 1.9. ([14]) Suppose \((X, d)\) is a metric space. Let \(\{x_n\}\) be a sequence in \(X\) such that \(d(x_n, x_{n+1}) \to 0\) as \(n \to \infty\). If \(\{x_n\}\) is not a Cauchy sequence then there exist an \(\varepsilon > 0\) and sequences of positive integers \(\{m(k)\}\) and \(\{n(k)\}\) with \(m(k) > n(k) > k\) such that \(d(x_{m(k)}, x_{n(k)}) \geq \varepsilon\), \(d(x_{m(k)}-1, x_{n(k)}) < \varepsilon\) and

(i) \(\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)+1}) = \varepsilon\);

(ii) \(\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon\);

(iii) \(\lim_{k \to \infty} d(x_{m(k)}-1, x_{n(k)}) = \varepsilon\).

We note that also can see \(\lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon\) and \(\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon\).

Definition 1.10. ([21]) Let \((X, d)\) be a metric space, \(\emptyset \neq A, B \subset X\), \(\alpha : A \times A \to [0, \infty)\) a function and \(f, g : A \to B\) non-self mappings. We say that \((f, g)\) is a triangular \(\alpha\)-proximal admissible pair, if for all \(p, q, r, t_1, t_2, s_1, s_2 \in A\),

\[
\begin{align*}
T_1 : & \quad \alpha(t_1, t_2) \geq 1 \\
& \quad d(s_1, f(t_1)) = d(A, B) \quad \Rightarrow \alpha(s_1, s_2) \geq 1 \\
& \quad d(s_2, g(t_2)) = d(A, B)
\end{align*}
\]

\[
\begin{align*}
T_2 : & \quad \alpha(p, r) \geq 1 \\
& \quad \alpha(r, q) \geq 1 \quad \Rightarrow \alpha(p, q) \geq 1.
\end{align*}
\]

Let \((X, d)\) be a metric space, \(\emptyset \neq A, B \subset X\). We define

\[
\begin{align*}
A_0 = & \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\} \\
B_0 = & \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}
\end{align*}
\]

(1)

Definition 1.11. ([21]) Let \((X, d)\) be a metric space, \(\emptyset \neq A, B \subset X\), and \(f, g : A \to B\) non-self mappings. We say that \((f, g)\) is a generalized proximal C-contraction pair if, for all \(s, t, p, q \in A\),

\[
d(s, f(p)) = d(A, B) \quad \Rightarrow \quad d(s, t) \leq \frac{1}{2} d(p, t) + d(q, s) - \psi(d(p, t), d(q, s)),
\]

in which \(\psi \in \Psi\).

Definition 1.12. ([21]) Let \((X, d)\) be a metric space, \(\emptyset \neq A, B \subset X\), \(\alpha : A \times A \to [0, \infty)\) a function and \(f, g : A \to B\) non-self functions. If, for all \(s, t, p, q \in A\),

\[
d(s, f(p)) = d(A, B) \quad \Rightarrow \quad d(s, t) \leq \frac{1}{2} d(p, t) + d(q, s) - \psi(d(p, t), d(q, s)),
\]

then \((f, g)\) is said to be an \(\alpha\)-proximal \(C_1\)-contraction pair.

If in the definition above, we replace (2) by

\[
(\alpha(p, q) + l\psi(d(s, t))) \leq (l + 1)^{1} \frac{1}{2} \left( d(p, t) + d(q, s) \right) - \psi(d(p, t), d(q, s)),
\]

where \(l > 0\), then \((f, g)\) is said to be an \(\alpha\)-proximal \(C_2\)-contraction pair.

In this paper, we generalize some results of Parvaneh et al. ([21]) to obtain some new common best proximity point theorems. Next, by an example and some fixed point results, we support our main result.
2. Main Results

Definition 2.1. Let \(A\) and \(B\) are two nonempty subsets of a metric space, \((X,d)\). Let \(\mu : A \times A \to [0,\infty)\) a function and \(f, g : A \to B\) non-self mappings. We say that \((f, g)\) is a triangular \(\mu - \text{subproximal admissible pair}\), if for all \(p, q, r, s, t_1, t_2, s_1, s_2 \in A\),

\[
\begin{align*}
T_1 : & \quad \mu(t_1, t_2) \leq 1, \\
& \quad d(s_1, f(t_1)) = d(A, B), \quad \implies \mu(s_1, s_2) \leq 1 \\
& \quad d(s_2, f(t_2)) = d(A, B) \\
T_2 : & \quad \mu(p, r) \leq 1, \\
& \quad \mu(r, q) \leq 1 \implies \mu(p, q) \leq 1
\end{align*}
\]

Definition 2.2. Let \(\Psi : X \times X \to [0,\infty)\) be a triangular function. We say that \((f, g)\) is a generalized proximal \(\Psi\)-admissible pair of type \(C\)-class, if for all \(s, t_1, t_2 \in X\),

\[
\begin{align*}
\left\{ \begin{array}{l}
\mu(t_1, t_2) \leq 1, \\
\psi(\mu(t_1, t_2)) = \mu(s_1, s_2) \leq 1 \\
\mu(s_1, s_2) \leq 1 \\
\psi(\mu(s_1, s_2)) = d(A, B)
\end{array} \right. \\
\end{align*}
\]

Definition 2.3. Let \(\alpha : A \times A \to [0,\infty)\) a function and \(f, g : A \to B\) non-self functions. If, for all \(s, t, p, q \in A\),

\[
\begin{align*}
\left\{ \begin{array}{l}
d(s, f(p)) = d(A, B) \\
d(t, g(q)) = d(A, B)
\end{array} \right. \\
\end{align*}
\]

then \((f, g)\) is said to be \(\alpha, \mu\)-proximal \(\Psi\)-admissible pair of type \(C\)-class.

Theorem 2.4. Let \(A\) and \(B\) are two nonempty subsets of a metric space, \((X,d)\). Let \(A\) be complete and \(A_0\) be nonempty. Moreover, assume that the non-self functions \(f, g : A \to B\) satisfy;

(i). \(f, g\) are continuous,

(ii). \(f(A_0) \subset B_0\) and \(g(A_0) \subset B_0\),

(iii). \((f, g)\) is a generalised proximal \(C\)-contraction pair of type \(C\)-class.

Then, the functions \(f\) and \(g\) have a unique common best proximity point.

Proof. Choose, \(s_0 \in A_0\) be arbitrary. Since \(f(A_0) \subset B_0\), there exists \(s_1 \in A_0\) such that

\[
d(s_1, f(s_0)) = d(A, B).
\]

Since \(g(A_0) \subset B_0\), there exists \(s_2 \in A_0\) such that \(d(s_2, g(s_1)) = d(A, B)\). Now as \(f(A_0) \subset B_0\), there exists \(s_3 \in A_0\) such that \(d(s_3, f(s_2)) = d(A, B)\).

We continue this process and construct a sequence \([s_n]\) such that

\[
\begin{align*}
d(s_{2n+1}, f(s_{2n})) = d(A, B), \\
d(s_{2n+2}, g(s_{2n+1})) = d(A, B).
\end{align*}
\]

for each \(n \in \mathbb{N}\)

Claim(1).

\[
\lim_{n \to \infty} d(s_n, s_{n+1}) = 0
\]
From (5) we get,

\[
d(s_{2n+1}, s_{2n+2}) \leq F\left(\frac{1}{2}(d(s_{2n}, s_{2n+2}) + d(s_{2n+1}, s_{2n+1})), \Psi(d(s_{2n}, s_{2n+2}), d(s_{2n+1}, s_{2n+1}))\right)
\]

\[
= F\left(\frac{1}{2}d(s_{2n}, s_{2n+2}), \Psi(d(s_{2n}, s_{2n+2}), 0)\right)
\]

\[
\leq \frac{1}{2}d(s_{2n}, s_{2n+2})
\]

\[
\leq \frac{1}{2}[d(s_{2n}, s_{2n+1}) + d(s_{2n+1}, s_{2n+2})]
\]

(9)

which implies \(d(s_{2n+1}, s_{2n+2}) \leq d(s_{2n}, s_{2n+1})\). Therefore, \(\{d(s_{2n}, s_{2n+1})\}\) is a non-negative decreasing sequence and so converges to \(d > 0\). Now, as \(n \to \infty\) in (9), we get

\[
d \leq \frac{1}{2}d(s_{2n}, s_{2n+1}) \leq \frac{1}{2}(d + d) = d
\]

that is,

\[
\lim_{n \to \infty} d(s_{2n}, s_{2n+1}) = 2d.
\]

(10)

Again, taking \(n \to \infty\) in (9), and using (10) we get

\[
F(d, \Psi(2d, 0)) = d
\]

So, \(d = 0\), or \(\Psi(2d, 0) = 0\) and hence \(d = 0\).

**Claim 2.** \(\{s_n\}\) is cauchy.

By (8) it is enough to show that subsequence \(\{s_{2n}\}\) is cauchy. Suppose, to the contrary, that \(\{s_{2n}\}\) is not a Cauchy sequence. By lemma (1.9) there exists \(\varepsilon > 0\) for which we can find subsequences \(\{s_{2n}\}\) and \(\{s_{2m}\}\) of \(\{s_{2n}\}\) with \(2n_k > 2m_k > 2k\) such that

\[
\varepsilon = \lim_{k \to \infty} d(s_{2m(k)}, s_{2n(k)}) = \lim_{k \to \infty} d(s_{2m(k)}, s_{2n(k)+1})
\]

\[
= \lim_{k \to \infty} d(s_{2m(k)+1}, s_{2n(k)}) = \lim_{k \to \infty} d(s_{2m(k)+1}, s_{2n(k)+1})
\]

(11)

From (5) we have

\[
d(s_{2m_k}, s_{2m_k}) \leq F\left(\frac{1}{2}(d(s_{2m_k}, s_{2m_k}) + s_{2m_k}, s_{2m_k}), \Psi(d(s_{2m_k}, s_{2m_k}), s_{2m_k}, s_{2m_k})\right)
\]

(12)

Taking \(k \to \infty\) in the above inequality and using (11), and the continuity of \(F, \Psi\), we would obtain

\[
F\left(\frac{1}{2}(\varepsilon + \varepsilon), \Psi(\varepsilon, \varepsilon)\right) = \varepsilon
\]

and therefore, \(\varepsilon = 0\), or \(\Psi(\varepsilon, \varepsilon) = 0\), which would imply \(\varepsilon = 0\), a contradiction. Thus, \(\{s_n\}\) is a cauchy sequence. Since \(A\) is complete, there is a \(z \in A\) such that \(s_n \to z\). Now, from

\[
d(s_{2n+1}, f(s_{2n})) = d(A, B),
\]

\[
d(s_{2n+2}, g(s_{2n+1})) = d(A, B)
\]

By continuity of \(f\) and \(g\), taking \(n \to \infty\) we have \(d(z, f(z)) = d(z, g(z)) = d(A, B)\). So, \(z\) is a common best proximity point of the mappings \(f\) and \(g\). Let, \(w\) is also a common best proximity point of mappings \(f\) and \(g\). From (1) we have

\[
d(z, w) \leq F\left(\frac{1}{2}(d(z, w) + d(w, z)), -\Psi(d(z, w), d(w, z))\right)
\]

\[
= F(d(z, w), \Psi(d(z, w), d(w, z))
\]

(13)

So, \(d(z, w) = 0\), or \(\Psi(d(z, w), d(z, w) = 0\), Hence \(d(z, w) = 0\), and therefore \(z = w\). \(\Box\)
Theorem 2.5. Let \( A \) and \( B \) are two nonempty subsets of a metric space, \((X, d)\). Let \( A \) be complete and \( A_0 \) be nonempty. Moreover, assume that the non-self functions \( f, g : A \rightarrow B \) satisfy:

(i). \( f, g \) are continuous,

(ii). \( f(A_0) \subset B_0 \) and \( g(A_0) \subset B_0 \),

(iii). \( (f, g) \) is an \( \alpha, \mu \)-proximal C-contraction pair of type C-class ,

(iv). \( (f, g) \) is a triangular \( \alpha \)-proximal admissible pair and a triangular \( \mu \) – subproximal admissible pair,

(v). there exist \( s_0, s_1 \in A_0 \) such that \( d(s_1, f(s_0)) = d(A, B), \alpha(s_1, s_0) \geq 1, \mu(s_1, s_0) \leq 1 \). Then, the functions \( f \) and \( g \) have a common best proximity point.Furthermore, if \( z, w \in X \) are common best proximity points and \( \alpha(z, w) \geq 1, \mu(z, w) \leq 1 \), then common best proximity point is unique.

Proof. By (iv), we can find \( s_0, s_1 \in A_0 \) such that

\[
d(s_1, f(s_0)) = d(A, B), \quad \alpha(s_1, s_0) \geq 1, \mu(s_1, s_0) \leq 1.
\]

Define the sequence \( \{s_n\} \) as in (7) of the theorem(2.4). Since, \( (f, g) \) is triangular \( \alpha \)-proximal admissible and triangular \( \mu \) – subproximal admissible, we have \( \alpha(s_n, s_{n+1}) \geq 1, \mu(s_n, s_{n+1}) \leq 1 \). Then

\[
\begin{align*}
&\alpha(s_n, s_{n+1}) \geq 1, \\
&d(s_{2n+1}, f(s_{2n})) = d(A, B) \\
&d(s_{2n+2}, g(s_{2n+1})) = d(A, B)
\end{align*}
\]

and

\[
\begin{align*}
&\mu(s_n, s_{n+1}) \leq 1, \\
&d(s_{2n+1}, f(s_{2n})) = d(A, B) \\
&d(s_{2n+2}, g(s_{2n+1})) = d(A, B)
\end{align*}
\]

If \( s = s_{2n+1}, t = s_{2n+2}, p = s_{2n}, q = s_{2n+1} \), and \( (f, g) \) is an \( \alpha, \mu \)-proximal C-contraction pair of type C-class. Then,

\[
\begin{align*}
h(1, d(s_{2n+1}, s_{2n+2})) &\leq h(\alpha(s_{2n}, s_{2n+1}), d(s_{2n+1}, s_{2n+2})) \\
&\leq F\left[\mu(s_{2n}, s_{2n+1}), F\left(\frac{1}{2}(d(s_{2n}, s_{2n+2}) + d(s_{2n+1}, s_{2n+1}))\right)\right] \\
&\leq F\left[1, F\left(\frac{1}{2}(d(s_{2n}, s_{2n+2}) + d(s_{2n+1}, s_{2n+1}))\right)\right].
\end{align*}
\]

so,

\[
\begin{align*}
d(s_{2n+1}, s_{2n+2}) &\leq F\left(\frac{1}{2}(d(s_{2n}, s_{2n+2}) + d(s_{2n+1}, s_{2n+1}))\right) \\
&= F\left(\frac{1}{2}d(s_{2n}, s_{2n+2}), \psi(d(s_{2n}, s_{2n+2}), d(s_{2n+1}, s_{2n+1}))\right) \\
&\leq \frac{1}{2}d(s_{2n}, s_{2n+2}) \\
&\leq \frac{1}{2}[d(s_{2n}, s_{2n+1}) + d(s_{2n+1}, s_{2n+2})]
\end{align*}
\]

which implies \( d(s_{2n+1}, s_{2n+2}) \leq d(s_{2n}, s_{2n+1}) \). Therefore, \( \{d(s_{2n}, s_{2n+1})\} \) is an non-negative decreasing sequence and so converges to \( d > 0 \). Now, as \( n \to \infty \) in (16), we get

\[
d \leq \frac{1}{2}d(s_{2n}, s_{2n+1}) \leq \frac{1}{2}(d + d) = d.
\]
Again, taking $n \to \infty$ in (9), and using (17) we get

$$F(d, \Psi(2d, 0)) = d$$

So, $d = 0$, or, $\Psi(2d, 0) = 0$ and hence $d = 0$. Therefore,

$$\lim_{n \to \infty} d(s_n, s_{n+1}) = 0 \quad (18)$$

Now we prove that

$$\alpha(s_{2m_i-1}, s_{2n_i}) \geq 1, \mu(s_{2m_i-1}, s_{2n_i}) \leq 1, \quad n_k > m_k > k. \quad (19)$$

Since $(f, g)$ is triangular $\alpha$–proximal admissible and triangular $\mu$ – subproximal admissible and

$$\begin{aligned}
\alpha(s_{2m_i-1}, s_{2m_i}) \geq 1 \\
\alpha(s_{2m_i}, s_{2m_i+1}) \geq 1 \\
\mu(s_{2m_i-1}, s_{2m_i}) \leq 1 \\
\mu(s_{2m_i}, s_{2m_i+1}) \leq 1
\end{aligned}$$

From $(T_2)$ of definition(1.10) and definition(2.1) we have

$$\begin{aligned}
\alpha(s_{2m_i-1}, s_{2m_i+1}) &\geq 1 \\
\mu(s_{2m_i-1}, s_{2m_i+1}) &\leq 1.
\end{aligned}$$

Again, since $(f, g)$ is triangular $\alpha$–proximal admissible and triangular $\mu$ – subproximal admissible and

$$\begin{aligned}
\alpha(s_{2m_i-1}, s_{2m_i+2}) \geq 1 \\
\alpha(s_{2m_i+1}, s_{2m_i+2}) \geq 1 \\
\mu(s_{2m_i-1}, s_{2m_i+2}) \leq 1 \\
\mu(s_{2m_i+1}, s_{2m_i+2}) \leq 1
\end{aligned}$$

From $(T_2)$ of definition(1.10) and definition(2.1) again, we have

$$\begin{aligned}
\alpha(s_{2m_i-1}, s_{2m_i+2}) &\geq 1 \\
\mu(s_{2m_i-1}, s_{2m_i+2}) &\leq 1.
\end{aligned}$$

By continuing this process, we get (19).

Now, we prove that $\{s_n\}$ is cauchy.

By, (18) it is enough to show that subsequence $\{s_{2n}\}$ is cauchy. Suppose, to the contrary, that $\{s_{2n}\}$ is not a Cauchy sequence. By lemma (1.9) there exists $\epsilon > 0$ for which we can find subsequences $\{s_{2n_i}\}$ and $\{s_{2m_i}\}$ of $\{s_{2n}\}$with $2n_i > 2m_i > 2k$ such that

$$\epsilon = \lim_{k \to \infty} d(s_{2m(k)}, s_{2n(k)}) = \lim_{k \to \infty} d(s_{2m(k)}, s_{2n(k)+1}) = \lim_{k \to \infty} d(s_{2m(k)+1}, s_{2n(k)}) = \lim_{k \to \infty} d(s_{2m(k)+1}, s_{2n(k)+1}) \quad (20)$$
Now if \( s = s_{2m+1} \), \( t = s_{2m} \), \( p = s_{2m} \), \( q = s_{2m} - 1 \), then
\[
\begin{align*}
\phi(1, d(s_{2m+1}, s_{2m})) & \leq \phi(a(s_{2m}, s_{2m+1}), d(s_{2m+1}, s_{2m})) \\
& \leq \mathcal{F} \left[ \mu(s_{2m}, s_{2m+1}), \mathcal{F} \left( \frac{1}{2} \left( d(s_{2m}, s_{2m}) + d(s_{2m+1}, s_{2m+1}) \right) \right) \right] \\
& \leq \mathcal{F} \left[ 1, \mathcal{F} \left( \frac{1}{2} \left( d(s_{2m}, s_{2m}) + d(s_{2m+1}, s_{2m+1}) \right) \right) \right]
\end{align*}
\]

Therefore,
\[
d(s_{2m+1}, s_{2m}) \leq \mathcal{F} \left( \frac{1}{2} \left( d(s_{2m}, s_{2m}) + d(s_{2m+1}, s_{2m+1}) \right) \right)
\]

Taking \( k \to \infty \) in the above inequality and using (20), and the continuity of \( F, \Psi \), we would obtain
\[
\mathcal{F} \left( \frac{1}{2} (\epsilon + \epsilon), \Psi(\epsilon, \epsilon) \right) = \epsilon
\]

and therefore, \( \epsilon = 0 \), or \( \Psi(\epsilon, \epsilon) = 0 \), which would imply \( \epsilon = 0 \), a contradiction. Thus, \( \{s_n\} \) is a cauchy sequence. Since \( A \) is complete, there is a \( z \in A \) such that \( s_n \to z \). Now, from
\[
d(s_{2m+1}, f(s_{2m})) = d(A, B), \quad d(s_{2m+2}, g(s_{2m+1})) = d(A, B)
\]

By continuity of \( f \) and \( g \), taking \( n \to \infty \) we have \( d(z, f(z)) = d(z, g(z)) = d(A, B) \). So, \( z \) is a common best proximity point of the mappings \( f \) and \( g \). Let, \( w \) is also a common best proximity point of mappings \( f \) and \( g \). Since \( \alpha(w, z) \geq 1, \mu(z, w) \leq 1 \)

from (6) we have
\[
\begin{align*}
\phi(1, d(z, w)) & \leq \phi(a(z, w), d(z, w)) \\
& \leq \mathcal{F} \left[ \mu(z, w), \mathcal{F} \left( \frac{1}{2} \left( d(z, w) + d(w, z) \right) \right) \right] \\
& \leq \mathcal{F} \left[ 1, \mathcal{F} \left( \frac{1}{2} \left( d(z, w) + d(w, z) \right) \right) \right]
\end{align*}
\]

therefore,
\[
d(z, w) \leq \mathcal{F} \left( \frac{1}{2} \left( d(z, w) + d(w, z) \right) \right) - \Phi \left( \mathcal{F} \left( \frac{1}{2} \left( d(z, w) + d(w, z) \right) \right) \right)
\]

So, \( d(z, w) = 0 \), or \( \Phi(d(z, w), d(z, w)) = 0 \), hence \( d(z, w) = 0 \), and therefore \( z = w \). \( \square \)

**Definition 2.6.** ([21]) Let \( \alpha : X \times X \to \mathbb{R} \) be a function and \( f, g : X \to X \) self-mappings and \( p, q, r \in X \) be any three elements. We say that \( (f, g) \) is a triangular \( \alpha \)-admissible pair if

\( \text{(i)} \) \( \alpha(p, q) \geq 1 \quad \implies \quad \alpha(f(p), g(q)) \geq 1 \) or \( \alpha(g(p), f(q)) \geq 1 \),

\( \text{(ii)} \) \( \begin{cases} \alpha(p, r) \geq 1 \\
\alpha(r, q) \geq 1 \end{cases} \quad \implies \quad \alpha(p, q) \geq 1 \)

**Definition 2.7.** Let \( \mu : X \times X \to \mathbb{R} \) be a function and \( f, g : X \to X \) self-mappings and \( p, q, r \in X \) be any three elements. We say that \( (f, g) \) is a triangular \( \mu \)-subadmissible pair if

\( \text{(i)} \) \( \mu(p, q) \leq 1 \quad \implies \quad \mu(f(p), g(q)) \leq 1 \) or \( \mu(g(p), f(q)) \leq 1 \),

\( \text{(ii)} \) \( \begin{cases} \mu(p, r) \leq 1 \\
\mu(r, q) \leq 1 \end{cases} \quad \implies \quad \mu(p, q) \leq 1 \)
The corollary is an consequence of the last theorem.

**Corollary 2.8.** Let \((X, d)\) be a complete metric space and \(f, g : X \to X\). Moreover, let the self functions \(f\) and \(g\) satisfy:

(i) \(f\) and \(g\) are continuous,
(ii) there exists \(s_0 \in X\) such that \(a(s_0, f(s_0)) \geq 1\),
(iii) \((f, g)\) is a triangular \(\alpha\)--admissible pair and triangular \(\mu\) --subadmissible pair ,
(iv) for all \(p, q \in X\),

\[
\alpha(p, q)d(f(p), g(q)) \leq \frac{1}{2}\mu(p, q)\left(d(p, g(q)) + d(q, f(p))\right) - \Psi(d(p, g(q)), d(q, f(p)))
\]

(or)

\[
\alpha(p, q) + \frac{1}{2}\mu(p, q)\left(d(p, g(q)) + d(q, f(p))\right) - \Psi(d(p, g(q)), d(q, f(p)))
\]

Then \(f\) and \(g\) have common fixed point. Moreover, if \(x, y \in X\) are common fixed points and \(a(x, y) \geq 1\), \(\mu(x, y) \leq 1\), then the common fixed point of \(f\) and \(g\) is unique, that is \(x = y\).

Now, we remove the continuity hypothesis of \(f\) and \(g\) and get the following theorem.

**Theorem 2.9.** Let \(A\) and \(B\) be two nonempty subsets of a metric space, \((X, d)\). Let \(A\) be complete, the pair \((A, B)\) have the \(V\)--property, and \(A_0\) be nonempty. Moreover, assume that the non-self functions \(f, g : A \to B\) satisfy:

(i) \(f(A_0) \subseteq B_0\) and \(g(A_0) \subseteq B_0\),
(ii) \((f, g)\) is a generalised proximal \(C\)--contraction pair of type \(C\)-class,

Then, the functions \(f\) and \(g\) have a unique common best proximity point.

Proof. By Theorem(2.4), there is a cauchy sequence \(\{s_n\} \subseteq A\) and \(z \in A_0\) such that (7) holds and \(s_n \to z\). Moreover, we have

\[
d(z, B) \leq d(z, f(s_{2n}))
\]

\[
\leq d(z, s_{2n+1}) + d(s_{2n+1}, f(s_{2n}))
\]

\[
\leq d(z, s_{2n+1}) + d(A, B).
\]

we take \(n \to \infty\) in the above inequality, and we get

\[
\lim_{n \to \infty} d(z, f(s_n)) = d(z, B) = d(A, B).
\]  

(22)

Since the pair \((A, B)\) has the \(V\)--property, there is a \(p \in B\) such that \(d(z, p) = d(A, B)\) and so \(z \in A_0\). Moreover, Since \(f(A_0) \subseteq B_0\), there is a \(q \in A\) such that

\[
d(q, f(z)) = d(A, B).
\]  

(23)

Furthermore \(d(s_{2n+2}, g(s_{2n+1})) = d(A, B)\) for every \(n \in \mathbb{N}\).

Since \((f, g)\) is a generalised proximal \(C\)--contraction pair, we have

\[
d(q, s_{2n+2}) \leq F\left(\frac{1}{2}\left(d(z, s_{2n+2}) + d(s_{2n+1}, q)\right), \Psi\left(d(z, s_{2n+2}), d(s_{2n+1}, q)\right)\right)
\]

Letting \(n \to \infty\) in the above inequality, we have

\[
d(q, z) \leq F\left(\frac{1}{2}(d(z, q)), \Psi(d(z, q), 0)\right)
\]

So, \(d(q, z) = 0\), or \(\Psi(d(q, z), 0) = 0\), Thus \(d(z, q) = 0\), which implies that \(z = q\). Then, by (23), \(z\) is a best proximity point of \(f\).

Similarly, it is easy to prove that \(z\) is a best proximity point of \(g\). Then, \(z\) is a common best proximity point of \(f\) and \(g\). By the proof of Theorem(2.4), we conclude that \(f\) and \(g\) have unique common best proximity point.
**Theorem 2.10.** Let \( A \) and \( B \) be two nonempty subsets of complete metric space \((X,d)\). Let \( A \) be complete, the pair \((A,B)\) have \( V \)-property and \( A_0 \) is non-empty. Moreover, suppose that the non-self functions \( f, g : A \to B \) satisfy:

(i) \( f(A_0) \subset B_0 \) and \( g(A_0) \subset B_0 \\
(ii) (f, g) \) is an \( \alpha, \mu \)-proximal \( C \)-contraction pair of type \( C \)-class,

(iii)there exist \( s \) for which \( d(f(s)), f(s) \)

(iv)there exist \( s \) for which \( d(f(s)), f(s) \)

Proof. As similar to the proof of Theorem (2.5) that there exist a sequence \( \{s_n\} \) in \( A \) such that \( s_n \to z \) and \( \alpha(s_n, s_n + 1) \geq 1, \mu(s_n, s_n + 1) \leq 1. \) (\( n \in \mathbb{N} \))

Therefore, we have

\[
d(z, B) \leq d(z, f(s_n)) \\
= d(z, s_{2n+1}) + d(s_{2n+1}, f(s_n)) \\
\leq d(z, s_{2n+1}) + d(A, B).
\]

we take \( n \to \infty \) in the above inequality, and we get

\[
\lim_{n \to \infty} d(z, f(s_n)) = d(z, B) = d(A, B).
\] (24)

Since the pair \((A, B)\) has the \( V \)-property, there is an \( p \) in \( B \) such that \( d(z, p) = d(A, B) \) and so \( z \) in \( A_0 \). Moreover, Since \( f(A_0) \subset B_0 \), there is a \( q \) in \( A \) such that

\[
d(q, f(z)) = d(A, B).
\] (25)

Furthermore \( d(s_{2n+2}, f(s_{2n+1})) = d(A, B) \) for every \( n \in \mathbb{N} \). Also, by (v), \( \alpha(s_n, z) \geq 1, \mu(s_n, z) \leq 1 \) for every \( n \in \mathbb{N} \). By \((f, g)\) is an \( \alpha, \mu \)-proximal \( C \)-contraction pair of type \( C \)-class, we have

\[
h(1, d(q, s_{2n+2})) \\
\leq h(\alpha(z, s_{2n+1}), d(q, s_{2n+2})) \\
\leq F\left[\mu(z, s_{2n+1}), F(\frac{1}{2} d(z, s_{2n+2}) + d(s_{2n+1}, q)) - \Psi(d(z, s_{2n+2}), d(s_{2n+1}, q))\right] \\
\leq F\left[1, F\left(\frac{1}{2} d(z, s_{2n+2}) + d(s_{2n+1}, q), \psi(d(z, s_{2n+2}), d(s_{2n+1}, q))\right)\right]
\]

Therefore

\[
d(q, s_{2n+2}) \leq F\left(\frac{1}{2} d(z, s_{2n+2}) + d(s_{2n+1}, q), \psi(d(z, s_{2n+2}), d(s_{2n+1}, q))\right)
\]

Letting \( n \to \infty \) in the above inequality, we have

\[
d(q, z) \leq F\left(\frac{1}{2} d(z, q), \psi(0, d(z, q))\right)
\]

So, \( d(q, z) = 0 \), or \( \Psi(0, d(q, z)) = 0 \), which implies that \( z = q \). Then, by (25), \( z \) is a best proximity point of \( f \). Similarly, we can prove \( z \) is a best proximity point of \( g \). Therefore, \( z \) is an common
If $z, w \in X$ are common best proximity points and $\alpha(z, w) \geq 1$, $\mu(z, w) \leq 1$, then we get
\[
d(z, w) \leq F\left(\frac{1}{2}(d(z, w) + d(w, z)), \Phi(d(z, w), d(w, z))\right)
= F(d(z, w), \Phi(d(z, w), d(w, z)))
\leq d(z, w)
\]
So, $d(z, w) = 0$, or $\Psi(d(z, w), d(z, w)) = 0$, Therefore, $d(z, w) = 0$ and hence $z = w$. □

The following corollary is an immediate consequence of the main theorem of this section.

**Corollary 2.11.** Let $(X, d)$ be a complete metric space and $f, g : X \to X$. Moreover, let the self functions $f$ and $g$ satisfy:
(i) there exists $s_0 \in X$ such that $\alpha(s_0, f(s_0)) \geq 1$,
(ii) $(f, g)$ is a triangular $\alpha$–admissible pair,
(iii) for all $p, q \in X$,
\[
\alpha(p, q)d(f(p), g(q)) \leq \frac{1}{2}\mu(p, q)(d(p, g(q)) + d(q, f(p))) - \Psi(d(p, g(q)), d(q, f(p))
\]
(or)
\[
\left(\alpha(p, q) + 1\right)^{\frac{1}{d(f(p), g(q))}} \leq (l + 1)^{\frac{1}{d(p, g(q)) + d(q, f(p))}} - \Psi\left(d(p, g(q)), d(q, f(p))\right)
\]
(iv) if $\{s_n\}$ is a sequence in $A$ such that $\alpha(s_n, s_{n+1}) \geq 1$ and $s_n \to s_0$ as $n \to \infty$, then $\alpha(s_n, s_0) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then $f$ and $g$ have common fixed point. Moreover, if $x, y \in X$ are common fixed points and $\alpha(x, y) \geq 1$, then the common fixed point of $f$ and $g$ is unique, that is $x = y$.

**Example 2.12.** Consider $X = \mathbb{R}$ with the usual metric, $A = [-8, 0, 8]$ and $B = [-4, -2, 4]$. Then, $A$ and $B$ are nonempty closed subsets of $X$ with $d(A, B) = 2$, $A_0 = [0]$ and $B_0 = [-2]$. Define $f, g : A \to A$ by
\[
f(0) = -2, \quad f(8) = 4, \quad f(-8) = -4 \text{ and } g(x) = -2 \text{ for all } x \in A.
\]
and $\Psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ by $\Psi(s, t) = \sqrt{s^2 + t^2}$ also $F(s, t) = s - t$. If,
\[
\begin{align*}
d(u, f(p)) &= d(A, B) = 2 \\
d(v, f(q)) &= d(A, B) = 2
\end{align*}
\]
then, $u = v = p = 0$ and $q \in A$. Hence all the conditions of Theorem 2.4 hold for this example and clearly 0 is the unique best proximity point of $f$ and $g$.

**Example 2.13.** Let $X = [0, 2] \times [0, 2]$ and $d$ be the Euclidean metric. Let
\[
A = \{(0, m) : 0 \leq m \leq 2\} \quad B = \{(2, m) : 0 \leq m \leq 2\}
\]
Then, $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Define $f, g : A \to B$ by $f(0, m) = (2, m)$ and $g(0, m) = (2, 2)$. Also define $\alpha, \mu : A \times A \to [0, \infty)$ by $\mu(p, q) = 1$ and
\[
\alpha(p, q) = \begin{cases} 
0 & \text{if } p, q \in (0, 2) \times ((0, 0), (0, 2)), \\
1 & \text{otherwise}
\end{cases}
\]
and $\Psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ by
\[
\Psi(s, t) = 2 \quad \text{for all } s, t \in X
\]
also $F(s, t) = \frac{1}{2}h(x, y) = xy$, and $\mathcal{F}(s, t) = st$. Assume that
\[
\begin{align*}
    d(u, f(p)) &= d(A, B) = 2 \\
    d(v, f(q)) &= d(A, B) = 2
\end{align*}
\]
Hence, $u = p$ and $v = (0, 2)$, then $u = v$ and (2) holds. If $p \neq (0, 2)$, then $\alpha(p, q) = 0$ and (2) holds, which implies that $(f, g)$ is an $\alpha$–proximal $C$-contraction pair of type $C$-class. Hence, all the hypothesis of the Theorem(*) are satisfied. Moreover, if $s_n$ is a sequence such that $\alpha(s_n, s_{n+1}) \geq 1$ for every $n \in \mathbb{N} \cup \{0\}$ and $s_n \to s_0$, then $s_n = (0, 2)$ for all $n \in \mathbb{N} \cup \{0\}$ and hence $s_0 = (0, 2)$. Then $\alpha(s_n, s_0) \geq 1$ for every $n \in \mathbb{N} \cup \{0\}$. Clearly, $(A, B)$ has the $V$– property and then all the conclusions of Theorem(2.10) hold. Clearly, $(0, 2)$ is the unique common best proximity point of $f$ and $g$.

Example 2.14. Let $X = [0, 3] \times [0, 3]$ and $d$ be the Euclidean metric. Let $A = \{(0, m) : 0 \leq m \leq 3\}$ and $B = \{(3, m) : 0 \leq m \leq 3\}$

Then, $d(A, B) = 3$, $A_0 = A$ and $B_0 = B$. Define $f, g : A \to B$ by
\[
f(0, m) = \begin{cases} 
    (3, 3) & m = \frac{3}{2} \\
    (3, \frac{3}{2}) & m \neq \frac{3}{2}
\end{cases}
\]
and $g(0, m) = (3, 3)$. Also define $\alpha, \mu : A \times A \to [0, \infty)$ by $\mu(p, q) = 1$ and
\[
\alpha(p, q) = \begin{cases} 
    3 & \text{if } p, q \in (0, \frac{3}{2}) \times A, \\
    0 & \text{otherwise}
\end{cases}
\]
and $\Psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ by
\[
\Psi(s, t) = \frac{1}{2}(s + t) \quad \text{for all } s, t \in X
\]
also $F(s, t) = s - t, h(x, y) = xy$, and $\mathcal{F}(s, t) = st$.

It is easy to see that all required hypothesis of Theorem(2.10) are satisfied unless (iii). Clearly $f$ and $g$ have no common best proximity point. It is worth noting that pair $(f, g)$ does not have the triangular $\alpha$–proximal admissible property.

References


