Orbital Continuity and Fixed Points

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Abstract. The aim of the present paper is to show the significance of the concept of orbital continuity introduced by Ciric. We prove that orbital continuity of a pair of R-weak commuting self-mappings of type A_1 or of type A_2 of a complete metric space is equivalent to fixed point property under Jungck type contraction. We also establish a situation in which orbital continuity is a necessary and sufficient condition for the existence of a common fixed point of a pair of mappings yet the mappings are necessarily discontinuous at the fixed point.

1. Introduction

In 1971 Ciric [1] introduced the notion of orbital continuity. If \( f \) is a self-mapping of a metric space \((X,d)\) then the set \( O(x,f) = \{f^nx : n = 0, 1, 2, \ldots \} \) is called the orbit of \( f \) at \( x \) and \( f \) is called orbitally continuous if \( u = \lim_n f^nx \) implies \( fu = \lim_n f^{n+1}x \). Every continuous self-mapping is orbitally continuous but not conversely [1]. Shastri et al [11] defined the notion of orbital continuity for a pair of mappings. If \( f \) and \( g \) are self-mappings of a metric space \((X,d)\) and if \( \{x_n\} \) is a sequence in \( X \) such that \( f x_n = g x_{n+1}, n = 0, 1, 2, \ldots \), then the set \( O(x_0,f,g) = \{fx_n : n = 0, 1, 2, \ldots \} \) is called the \((f,g)\)-orbit at \( x_0 \) and \( g \) (or \( f \)) is called \((f,g)\)-orbitally continuous if \( \lim_n fx_n = u \) implies \( \lim_n gx_n = gu \) (or \( \lim_n fx_n = u \) implies \( \lim_n fx_{n+m} = fu \)). We now give some relevant definitions.

Definition 1.1 ([14]). Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called \( R \)-weakly commuting if there exists some real number \( R > 0 \) such that \( d(fgx, gfx) \leq Rd(fx, gx) \) for all \( x \) in \( X \). The mappings \( f \) and \( g \) are called point-wise \( R \)-weakly commuting on \( X \) if given \( x \) in \( X \) there exists \( R > 0 \) such that \( d(fgx, gfx) \leq Rd(fx, gx) \) (see [5]). The notion of point-wise \( R \)-weak commuting implies commutativity at coincidence points and is, therefore, equivalent to the notion of weak compatibility.

Definition 1.2 ([13]). Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called compatible if \( \lim_n d(fgx_n, gfx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \).

Definition 1.3 ([9]). Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called \( R \)-weakly commuting of type \( A_2 \) if there exists some real number \( R > 0 \) such that \( d(ffx, gfx) \leq Rd(fx, gx) \) for all \( x \) in \( X \). Similarly, the self-mappings \( f \) and \( g \) are called \( R \)-weakly commuting of type \( A_3 \) if there exists some real number \( R > 0 \) such that \( d(fgx, gfx) \leq Rd(fx, gx) \) for all \( x \) in \( X \).
Definition 1.4 ([10]). Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called \( g \)-compatible or \( f \)-compatible according as \( \lim_n d(fx_n, gx_n) = 0 \) or \( \lim_n d(gx_n, fx_n) = 0 \) whenever \([x_n]\) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \).

Definition 1.5 ([8]). Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called compatible of type \((P)\) if \( \lim_n d(fx_n, gx_n) = 0 \) whenever \([x_n]\) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \).

Example 1.11. Let \( X \) be a metric space and \( f \) and \( g \) be self-mappings of \( X \) that are \( f \)-compatible or \( g \)-compatible. Then \( f \) and \( g \) are compatible of type \((P)\) if \( \lim_n d(fx_n, gx_n) = 0 \) whenever \([x_n]\) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \).

In a recent work [7], the authors introduced the following definitions:

Definition 1.6. Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called quasi \( R \)-commuting provided there exists a positive real number \( R \) such that given \( x \) in \( X \) we have \( d(fx, gx) \leq Rd(fx, gx) \) or \( d(gx, fx) \leq Rd(fx, gx) \) or \( d(fgx, ggx) \leq Rd(fx, gx) \) or \( d(fgx, ggx) \leq Rd(fx, gx) \) for all \( x \) in \( X \) and \( f \) and \( g \) are compatible of type \((P)\).

Definition 1.7. Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are called quasi \( R \)-commuting provided every sequence \([x_n]\) in \( X \) satisfying \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \) splits up in at most four sub-sequences such that any of the four sub-sequences, say \([x_n]\)_i, satisfies at least one of the four conditions \( \lim_n d(fx_n, gx_n) = 0 \), \( \lim_n d(gx_n, fx_n) = 0 \), \( \lim_n d(fgx_n, ggx_n) = 0 \), or \( \lim_n d(fgx_n, ggx_n) = 0 \) for some \( t \) in \( X \)

We now introduce the following notions:

Definition 1.8. Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) will be called semi \( R \)-commuting provided there exists \( R > 0 \) such that \( d(fx, gx) \leq Rd(fx, gx) \) or \( d(gx, fx) \leq Rd(fx, gx) \) or \( d(fgx, ggx) \leq Rd(fx, gx) \) or \( d(fgx, ggx) \leq Rd(fx, gx) \) for all \( x \) in \( X \) and \( f \) and \( g \) are \( f \)-compatible or \( g \)-compatible of type \((P)\).

Definition 1.9. Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) will be called semi \( \alpha \)-compatible provided every sequence \([x_n]\) in \( X \) satisfying \( \lim_n fx_n = \lim_n gx_n = t \) for some \( t \) in \( X \) satisfies \( \lim_n d(fx_n, gx_n) = 0 \) or \( \lim_n d(gx_n, fx_n) = 0 \) or \( \lim_n d(fgx_n, ggx_n) = 0 \) or \( \lim_n d(fgx_n, ggx_n) = 0 \) for some \( t \) in \( X \).

It is easy to see that semi \( R \)-commuting implies semi \( \alpha \)-compatible. It is also obvious that mappings which are \( f \)-compatible or \( g \)-compatible or \( f \)-compatible or \( g \)-compatible of type \((P)\) are semi \( \alpha \)-compatible.

Example 1.10. Let \( X = [0, \infty) \) equipped with the Euclidean metric. Define \( f, g : X \to X \) by
\[
fx = x/2 \text{ for each } x \in X, \quad gx = x \text{ for each } x \in X.
\]
Then \( f \) and \( g \) are commuting, \( R \)-weakly commuting, \( R \)-weakly commuting of type \( A \), \( R \)-weakly commuting of type \( A \) as well as semi \( R \)-commuting. It can also be verified that \( f \) and \( g \) are semi \( \alpha \)-compatible.

Example 1.11. Let \( X = [2, 20] \) equipped with the Euclidean metric. Define \( f, g : X \to X \) by
\[
f2 = 2, \quad fx = 3 \text{ if } 2 < x \leq 5, \quad fx = 3 \text{ if } x > 5, \\
g2 = 2, \quad gx = 12 \text{ if } 2 < x \leq 5, \quad gx = (x + 1)/2 \text{ if } x > 5.
\]
Then \( d(fgx, ggx) \leq d(fx, gx) \) for all \( x \) satisfying \( fx, gx \in (X \cap g(X)) \), that is, \( f \) and \( g \) are semi \( R \)-commuting with \( R = 1 \). However, \( f \) and \( g \) are not \( R \)-commuting. For example if we take \( x_n = 5 + 1/n \) then \( \lim_n fx_n = \lim_n gx_n = 3 \), \( \lim_n d(fx_n, gx_n) = 0 \), \( \lim_n d(gx_n, fx_n) = 9 \), \( \lim_n d(fgx_n, ggx_n) = 9 \), \( \lim_n d(fgx_n, ggx_n) = 9 \). Thus \( f \) and \( g \) fail to be \( R \)-commuting. These computations also show that \( f \) and \( g \) are neither compatible, nor \( f \)-compatible, nor \( g \)-compatible, nor compatible of type \((P)\). The notion of semi \( R \)-commuting is thus a proper generalization of these four conditions.

Example 1.12. Let \( X = [2, 11] \) equipped with the Euclidean metric. Define \( f, g : X \to X \) by
\[
f2 = (6 - x)/2 \text{ if } x \leq 2, \quad fx = 3 \text{ if } 2 < x \leq 5, \quad fx = (11 - x)/3 \text{ if } x > 5, \\
gx = x \text{ if } x \leq 2, \quad gx = 10 \text{ if } 2 < x \leq 5, \quad gx = (x + 1)/3 \text{ if } x > 5.
\]
Then for each \( x \leq 2 \) satisfying \( fx, gx \in (X \cap g(X)) \) we have \( d(fgx, ggx) \leq d(fx, gx) \). On the other hand, for each \( x \) satisfying \( 5 < x \leq 8 \) and \( fx, gx \in (X \cap g(X)) \) we have \( d(fgx, ggx) \leq d(fx, gx) \). This shows that \( f \) and \( g \) are not semi \( R \)-commuting. However, \( f \) and \( g \) can be shown to be \( R \)-commuting.

Examples 1.11 and 1.12 demonstrate that quasi \( R \)-commuting and semi \( R \)-commuting are independent notions. However, the notion of semi \( R \)-commuting is much easier to employ when both the conditions hold.
2. Main Results

**Theorem 2.1.** Let \( f \) and \( g \) be \( R \)-weakly commuting self-mappings of type \( A_f \) or of type \( A_g \) of a complete metric space \((X,d)\) such that \( f(X) \subseteq g(X) \) and

(i) \( d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1 \).

Then \( f \) and \( g \) have a common fixed point if and only if \( f \) and \( g \) are \((f, g)\)-orbitally continuous.

**Proof.** Let \( x_0 \) be any point in \( X \). Define sequences \( \{y_n\} \) and \( \{x_n\} \) in \( X \) such that

\[
y_n = fx_n = gx_{n+1}, \quad n = 0, 1, 2, \ldots
\]  

(1)

This can be done since \( f(X) \subseteq g(X) \). Now using a standard argument and by virtue of (i) it follows easily that \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists a point \( t \) in \( X \) such that \( y_n \to t \) as \( n \to \infty \). Also, \( \lim_n fx_n = t \) and \( \lim_n gx_n = t \). Let us assume that \( f \) and \( g \) are orbitally continuous. Then

\[
\lim_n fgx_n = \lim_n ffx_n = ft, \quad \text{and}
\]

(2)

\[
\lim_n ggx_n = \lim_n gfx_n = gt.
\]

(3)

Suppose \( f \) and \( g \) are \( R \)-weakly commuting of type \( A_f \). Then \( d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n) \). This, in view of (2) and (3) implies that \( ft = gt \). Now if \( t \neq ft \), using (i) we get

\[
d(fx_n, ft) \leq hd(gx_n, gt).
\]

On letting \( n \to \infty \) this yields, \( d(t, ft) \leq hd(t, gt) = hd(t, ft) \), that is, \( t = ft = gt \). Hence \( t \) is a common fixed point of \( f \) and \( g \). The proof is similar if \( f \) and \( g \) are \( R \)-weakly commuting of type \( A_g \). Moreover, condition (i) implies uniqueness of the common fixed point.

Conversely let us assume that the mappings \( f \) and \( g \) satisfy (i) and possess a common fixed point, say \( z \). Then \( z = fz = gz \). Also, the \((f,g)\)-orbit of any point \( x_0 \) defined by (1) converges to \( z \), that is, \( \lim_n fx_n = \lim_n gx_n = z \). Suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \( A_g \). Then we have

\[
d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n).
\]

This implies

\[
\lim_n d(ffx_n, gfx_n) = 0.
\]  

(4)

Now by virtue of (i) we have

\[
d(ffx_n, fz) \leq hd(gfx_n, gz) \leq h[d(gfx_n, ffx_n) + d(ffx_n, gz)] = h[d(gfx_n, ffx_n) + d(ffx_n, fz)].
\]

This yields \((1 - h)d(ffx_n, fz) \leq hd(ffx_n, gfx_n) \) which, in view of (4), yields \( \lim_n ffx_n = fz = z \). Hence \( f \) is \((f, g)\)-orbitally continuous. Also \( \lim_n d(ffx_n, gfx_n) = 0 \) implies \( \lim_n gfx_n = fz = gz \), that is, \( g \) is \((f, g)\)-orbitally continuous. Similarly, \( f \) and \( g \) are orbitally continuous if \( f \) and \( g \) are assumed \( R \)-weakly commuting of type \( A_f \). This establishes the theorem. \( \square \)

The following examples illustrate the above theorem.

**Example 2.2.** Let \( X = [0, \infty) \) and \( d \) be the usual metric. Define \( f, g : X \to X \) by

\[
fx = x/2 \text{ for each } x \text{ in } X, \quad gx = x \text{ for each } x \text{ in } X.
\]

Then it is easily seen that \( f \) and \( g \) satisfy all the conditions of the above theorem and have a unique common fixed point \( x = 0 \).
Example 2.3. Let \( X = [2, 20] \) and \( d \) be the usual metric. Define \( f, g : X \to X \) by

\[
\begin{align*}
  f(x) &= 2 \text{ if } x = 2 \text{ or } x > 5, \quad f(x) = 6 \text{ if } 2 < x \leq 5, \\
  g(x) &= 2, \quad g(x) = 12 \text{ if } 2 < x \leq 5, \quad g(x) = (x + 1)/3 \text{ if } x > 5.
\end{align*}
\]

Then the mappings \( f \) and \( g \) are \( R \)-weakly commuting mappings of type \( A_r \), \( f(X) \subseteq g(X) \), \( d(f(x), f(y)) \leq (4/d) d(g(x), g(y)) \), and \( x = 2 \) is the unique common fixed point of \( f \) and \( g \). It is also easy to see that \( f \) and \( g \) are \((f, g)\)-orbitally continuous.

Remark 2.4. The mappings \( f \) and \( g \) in Example 2.3 are non-compatible. If we consider the sequence \( \{x_n\} = \{5 + 1/n : n \geq 1\} \) then \( \lim_n f x_n = 2, \lim_n g x_n = 2, \lim_n g x_n = 6 \) and \( \lim_n d(x_n, f x_n) = 2 \). Hence \( f \) and \( g \) are non-compatible. In view of non-compatibility of \( f \) and \( g \) and following the proof of Theorem 2 in Pant [6] it follows that both \( f \) and \( g \) are discontinuous at the common fixed point \( x = 2 \), though both the mappings are orbitally continuous. The contraction condition (i) pertaining to a pair of mappings employed in the above theorem was introduced by Jungck [2] and is often referred to as Jungck contraction condition.

Theorem 2.5. Let \( f \) and \( g \) be orbitally continuous self-mappings of a complete metric space \( (X, d) \) such that \( f(X) \subseteq g(X) \) and

\[
\text{(ii) } d(f(x), f(y)) \leq h d(g(x), g(y)), 0 \leq h < 1.
\]

If \( f \) and \( g \) are semi \( R \)-commuting then \( f \) and \( g \) have a coincidence point which is their unique common fixed point.

Proof. Let \( x_0 \) be any point in \( X \). Define sequences \( \{y_n\} \) and \( \{x_n\} \) in \( X \) as in (1) above. Then \( \{y_n\} \) is a Cauchy sequence and there exists a point \( t \) in \( X \) such that \( y_n \to t \) as \( n \to \infty \) and \( \lim_n f x_n = \lim_n g x_n = t \). Orbital continuity of \( f \) and \( g \) implies that (2) and (3) hold. Since the sequence \( \{x_n\} \) satisfies \( f x_n, g x_n \in f(X) \cap g(X) \) and \( \lim_n f x_n = \lim_n g x_n = t \), semi \( R \)-commutativity of \( f \) and \( g \) implies that \( d(f f x_n, g f x_n) \leq R d(f x_n, g x_n) \) or \( d(f g x_n, g f x_n) \leq R d(f x_n, g x_n) \) or \( d(f g x_n, g g x_n) \leq R d(f x_n, g x_n) \) or \( d(f f x_n, f g x_n) \leq R d(f x_n, g x_n) \). Suppose \( d(f f x_n, f g x_n) \leq R d(f x_n, g x_n) \) is satisfied. This implies (4), that is, \( \lim_n d(f f x_n, f g x_n) = 0 \). This, in view of (2) and (3) implies that \( f t = g t \). Thus, semi \( R \)-commutativity in combination with orbital continuity implies that \( t \) is a coincidence point of \( f \) and \( g \). It may be observed here that weak compatibility will not imply \( f t = g t \) since weak compatibility does not imply (4). Now if \( f t \neq g t \), using (ii) we get

\[
d(f x_n, f t) \leq h d(g x_n, g t).
\]

This yields \( t = f t = g t \). Hence \( t \) is a common fixed point of \( f \) and \( g \). The proof follows on similar lines when \( d(g x_n, g y_n) \leq R d(f x_n, g x_n) \) or \( d(g x_n, g g x_n) \leq R d(f x_n, g x_n) \) or \( d(f x_n, g g x_n) \leq R d(f x_n, g x_n) \). Uniqueness of the coincidence point or the common fixed point is a consequence of (ii). \( \square \)

We now give an example to illustrate the above theorem:

Example 2.6. Let \( X = [0, 11] \) and \( d \) be the Euclidean metric. Define \( f, g : X \to X \) by

\[
\begin{align*}
  f(x) &= (6 - x)/2 \text{ if } x \leq 2, \quad f(x) = 3 \text{ if } 2 < x \leq 5, \quad f(x) = 2 \text{ if } x > 5, \\
  g(x) &= x \text{ if } x \leq 2, \quad g(x) = 10 \text{ if } 2 < x \leq 5, \quad g(x) = (x + 1)/3 \text{ if } x > 5.
\end{align*}
\]

Then \( f \) and \( g \) satisfy all the conditions of Theorem 2.5 and have a unique common fixed point \( x = 2 \). It can be seen in this example that \( d(f x, g f x) \leq d(f x, g x) \) whenever \( f x, g x \in f(X) \cap g(X) \). Therefore the mappings \( f \) and \( g \) are semi \( R \)-commuting with \( R = 1 \). It can also be verified that \( f \) and \( g \) satisfy the contractive condition \( d(f x, f y) \leq 1/2 d(g x, g y) \) for all \( x, y \) in \( X \). Moreover, it is also easy to see that \( f \) and \( g \) are orbitally continuous mappings. It may be seen in this example that \( f \) and \( g \) are neither compatible, nor \( f \)-compatible, nor \( g \)-compatible nor compatible of type (P).

Remark 2.7. It is worth noting that in Theorem 2.5 we cannot replace semi \( R \)-commuting by pointwise \( R \)-weak commuting (equivalently weak compatibility). This can be seen from the following example.
Example 2.8. Let $X = [2, 20]$ and $d$ be the Euclidean metric. Define $f, g : X \to X$ by

\[
\begin{align*}
fx &= 6 \text{ if } 2 \leq x \leq 5, \quad fx = (x + 7)/6 \text{ if } x > 5, \\
gx &= 15 \text{ if } 2 \leq x \leq 5, \quad gx = (x + 1)/3 \text{ if } x > 5.
\end{align*}
\]

Then $f$ and $g$ satisfy the following conditions but do not have a common fixed point or a coincidence point:

\begin{enumerate}[(a)]
\item $f$ and $g$ are coincident maps.
\item $f$ and $g$ satisfy the contraction condition $d(fx, fy) \leq \frac{1}{2}d(gx, gy)$.
\item $f$ and $g$ are pointwise $R$-weakly commuting and vacuously weak compatible.
\item $f$ and $g$ are orbitally continuous. To see this, let $(fx_n = gx_{n+1}, n = 0, 1, 2, \ldots)$ be the $(f, g)$-orbit of some point $x_0$ in $X$. Then $x_n \to 5$ with $x_n > 5$, $\lim_n fx_n = \lim_n gx_n = 2$, $\lim_n ffx_n = \lim_n fgx_n = 6 = f2$, $\lim_n gfx_n = \lim_n ggx_n = 15 = g2$. Therefore $f$ and $g$ are orbitally continuous mappings.
\end{enumerate}

It may be observed that the mappings $f$ and $g$ in the above example are not semi $R$-commuting. This example and Theorem 2.5 very well demonstrate that while semi $R$-commuting condition is useful in establishing the existence of coincidence points and also implies commutativity at coincidence points, weak compatibility or pointwise $R$-weak commutativity may not ensure the existence of coincidence points. Proceeding on similar lines as in Theorem 2.5 we can prove the following:

Theorem 2.9. Let $f$ and $g$ be $(f, g)$-orbitally continuous self-mappings of a complete metric space $(X, d)$ such that $f(X) \subseteq g(X)$ and

(iii) $d(fx, fy) \leq hd(gx, gy)$, $0 \leq h < 1$.

If $f$ and $g$ are semi $\alpha$-compatible then $f$ and $g$ have a coincidence point which is their unique common fixed point.

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References