On the Existence of Bounded Solutions to a Class of Nonlinear Initial Value Problems with Delay

Muhammad Usman Ali*, Fairouz Tchierb, Calogero Vetroc

*Department of Mathematics, COMSATS Institute of Information Technology, Attock Pakistan
bMathematics Department College of Science (Malaz), King Saud University, PO Box 22452 Riyadh, King Saudi Arabia
cDepartment of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy

Abstract. We consider a class of nonlinear initial value problems with delay. Using an abstract fixed point theorem, we prove an existence result producing a unique bounded solution.

1. Introduction

In this paper, we study the existence and uniqueness of a bounded solution for the following nonlinear initial value problem with delay:

\[ u(t) = \begin{cases} \int_{t-\tau}^{t} g(s, u(s), u'(s)) \, ds, & t \in [0, t_1], \\ \phi(t), & t \in [-\tau, 0], \end{cases} \]

under assumption

\[ \begin{cases} \phi(0) = \int_{-\tau}^{0} g(s, \phi(s), \phi'(s)) \, ds, \\ \phi'(0) = g(0, \phi(0), \phi'(0)) - g(-\tau, \phi(-\tau), \phi'(-\tau)). \end{cases} \]

In this problem, \( u(t) \) is the proportion of infectious individuals (in population) at time \( t \), \( \tau > 0 \) is the length of time for which an individual remains infectious; \( u'(t) \) is the speed of infectivity and \( g(t, u(t), u'(t)) \) is the proportion of new infectious individuals per unit time. For a comprehensive study of integral equations with delay, the reader is referred to Precup [7].

We denote by \( X \) the product space \( X = C^1[-\tau, t_1] \times C[-\tau, t_1] \). Then, we consider the Bielecki metric \( d_B : X \times X \to \mathbb{R}_2 \), where \( \mathbb{R}_2 \) is the set of all \( 2 \times 1 \) matrices, given as

\[ d_B((u_1, v_1), (u_2, v_2)) = (\|u_1 - u_2\|_B, \|v_1 - v_2\|_B)^T, \quad \text{for all } (u_1, v_1), (u_2, v_2) \in X, \]

2010 Mathematics Subject Classification. Primary 34A12; Secondary 35F25, 47H10

Keywords. Nonlinear initial value problem with delay, Perov’s fixed point theorem, \( \Lambda \)-admissible mapping

Received: 21 November 2016; Accepted: 15 April 2017

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Email addresses: muh_usman_al@yahoo.com (Muhammad Usman Ali), ftchier@ksu.edu.sa (Fairouz Tchier), calogero.vetro@unipa.it (Calogero Vetro)
where
\[ \|z\|_B = \max\{|z(t)|e^{-\beta(t+\tau)} : t \in [-\tau, t_1]\}, \quad \text{for a chosen } \beta > 0 \text{ and any } z \in C[-\tau, t_1]. \]

We need the following functional space
\[ X_+ = \{ (u,v) \in X : u(t) \geq 0, t \in [-\tau, t_1] \}. \]

Of course, \( X \) is a complete metric space, \( X_+ \subseteq X \) is closed in \( X \) and so \( X_+ \) is a complete metric space too. Clearly, from (1) we have
\[
\begin{align*}
    u'(t) &= \begin{cases} 
            g(t, u(t), u'(t)) - g(t - \tau, u(t - \tau), u'(t - \tau)), & t \in [0, t_1], \\
            \phi'(t), & t \in [-\tau, 0].
        \end{cases}
\end{align*}
\]

This problem is largely investigated by Bica-Muresan [2], where some existence and uniqueness results of solution for problem (1) are obtained by using classical tools of fixed point theory (see Banach [1] and Perov [6]). In this paper, by using the same approach in Bica-Muresan [2] and a concept of admissibility for mappings (based on an idea of Samet-Vetro-Vetro [9]), we obtain the existence and uniqueness of a bounded solution for problem (1). First we prove an abstract result which is a generalization of Perov’s fixed point theorem [6], then we work with a suitable integral operator associated to a large class of nonlinear initial value problems.

2. Mathematical Background and Preliminaries

We fix notation as follows. Let \( X \) be a non-empty set. By \( \mathbb{R}_+ \) we denote the set of all non-negative numbers and by \( \mathbb{R}_m \) the set of all \( m \times 1 \) real matrices. Let \( \alpha, \beta \in \mathbb{R}_m \), that is \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_m)^T \), then by \( \alpha \preceq \beta \) (resp., \( \alpha \prec \beta \)) we mean \( \alpha_i \leq \beta_i \) (resp., \( \alpha_i < \beta_i \)) for each \( i \in \{1, 2, \ldots, m\} \).

Also, we denote the set of all \( m \times m \) matrices with non-negative elements by \( M_{m,m}(\mathbb{R}_+) \), the zero \( m \times m \) matrix by \( 0 \) and the identity \( m \times m \) matrix by \( I \). Let \( A \in M_{m,m}(\mathbb{R}_+) \), then \( A \) is said to be convergent to zero if and only if \( A^n \to 0 \) as \( n \to \infty \) (see Varga [11]). Also note that \( A^0 = I \). From Filip-Petrusel [4], we have:

**Theorem 2.1.** Let \( A \in M_{m,m}(\mathbb{R}_+) \). The following conditions are equivalent:

(i) \( A \) is convergent to zero;

(ii) the eigenvalues of \( A \) are in the open unit disc, that is, \( |\lambda| < 1 \) for every \( \lambda \in \mathbb{C} \) with \( \det(A - \lambda I) = 0 \);

(iii) the matrix \( I - A \) is nonsingular (that is, its determinant is nonzero) and \( (I - A)^{-1} = I + A + \cdots + A^n + \cdots \).

Thus, it is easy to give some examples of matrices convergent to zero, from the literature (see Filip-Petrusel [4]). For example, we consider the following:

\[
A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \quad \text{where } a, b \in \mathbb{R}_+ \text{ and } a + b < 1;
\]

\[
B := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad \text{where } a, b, c \in \mathbb{R}_+ \text{ and } \max\{a, c\} < 1.
\]

Now, we work in the setting of generalized metric spaces. Precisely, a mapping \( d : X \times X \to \mathbb{R}_m \) is called a vector-valued metric on \( X \) if the following properties are satisfied:

\((d_1)\) \( d(x, y) \geq 0 \) for all \( x, y \in X \); if \( d(x, y) = 0 \) then \( x = y \), and vice versa;

\((d_2)\) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

\((d_3)\) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).
Thus, a non-empty set $X$ equipped with a vector-valued metric $d$ is called a generalized metric space, say $(X, d)$. Notice that the convergence and Cauchyness of a sequence in generalized metric spaces are defined in a similar manner as in usual metric spaces. So Perov [6] proved the following interesting generalization of Banach contraction principle in [1].

**Theorem 2.2.** Let $(X, d)$ be a complete generalized metric space and $f : X \to X$ be a mapping for which there exists a matrix $A \in M_{m,n}(\mathbb{R})$ such that $d(fx, fy) \leq Ad(x, y)$ for all $x, y \in X$. If $A$ is a matrix convergent to zero, then

(i) $\text{Fix}(f) = \{x^*\}$, where $\text{Fix}(f) = \{x \in X : x = fx\}$;

(ii) the sequence of successive approximations $\{x_n\}$ such that $x_n = f^nx_0$ is convergent and admits the limit $x^*$, for all $x_0 \in X$.

Some interesting contributions to the development of fixed point theory and its applications in this context are obtained by Bica-Muresan [2], Bucur-Guran-Petrusu [3], Filip-Petrusel [4], O’Regan-Shahzad-Agarwal [5], Rus [8], Turinici [10].

3. Fixed Point Theorem

In this section we prove a fixed point theorem useful to obtain the existence and uniqueness of solution of problem (1). The crucial key to establish our generalization of Theorem 2.2 (Perov [6]) is the following notion of admissibility (inspired by Samet-Vetro-Vetro [9]).

**Definition 3.1.** Let $X$ be a non-empty set, $\Lambda : X \times X \to M_{m,m}(\mathbb{R})$ and $f : X \to X$ be a mapping. The function $f$ is said to be $\Lambda$-admissible if

$$x, y \in X, \quad \Lambda(x, y) \geq I \implies \Lambda(fx, fy) \geq I,$$

where $I$ is the $m \times m$ identity matrix and the inequality between matrices means entrywise inequality.

Let $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R})$ such that $(I - A_3 - A_4)^{-1}$ exists. Let $f : X \to X$. The hypotheses are the following:

(i) the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;

(ii) there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0) \geq I$;

(iii) $f$ is $\Lambda$-admissible;

(iv) a. for each sequence $\{x_n\} \subseteq X$ such that $\lim_{n \to \infty} x_n = x$ and $\Lambda(x_n, x_{n+1}) \geq I$ for all $n \in \mathbb{N}$, we have $\Lambda(x_n, x) \geq I$

or

b. $f$ is continuous.

Now we can have the first theorem producing existence and uniqueness of fixed point for a given mapping $f$.

**Theorem 3.2.** Let $(X, d)$ be a complete generalized metric space and $f : X \to X$ be a mapping such that, for all $x, y \in X$, we have

$$\Lambda(x, y)d(fx, fy) \leq A_1d(x, y) + A_2d(x, fx) + A_3d(y, fy) + A_4d(x, fy) + Bd(y, fx)$$

(2)

with $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R})$ satisfying hypotheses (i)-(iv). Then $f$ has a fixed point. Moreover, if for all $x^*, x \in \text{Fix}(f)$ we have $\Lambda(x^*, x) \geq I$ and $A_1 + A_4 + B$ converges to zero then the fixed point is unique.
Proof. Because of hypothesis (ii), we see that there exists \( x_0 \in X \) such that \( \Lambda(x_0, f x_0) \geq I \). By putting \( x_1 = f x_0 \) and \( x_2 = f x_1 \), from (2), we have
\[
d(x_1, x_2) = d(f x_0, f x_1) = Id(f x_0, f x_1) \leq \Lambda(x_0, x_1)d(f x_0, f x_1)
\]
\[
\leq A_1 d(x_0, x_1) + A_2 d(x_0, f x_0) + A_3 d(x_1, f x_1) + A_4 d(x_0, f x_1) + B d(x_1, f x_0)
\]
\[
= A_1 d(x_0, x_1) + A_2 d(x_0, x_1) + A_3 d(x_1, x_2) + A_4 d(x_0, x_2) + B d(x_1, x_1)
\]
\[
\leq A_1 d(x_0, x_1) + A_2 d(x_0, x_1) + A_3 d(x_1, x_2) + A_4 [d(x_0, x_1) + d(x_1, x_2)] + B_0.
\]
After routine calculations, we get
\[
d(x_1, x_2) = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_0, x_1) = Ad(x_0, x_1).
\]
By putting \( x_3 = fx_2 \), hypothesis (iii) and (2) imply that
\[
d(x_2, x_3) = d(f x_1, f x_2) = Id(f x_1, f x_2) \leq \Lambda(x_1, x_2)d(f x_1, f x_2)
\]
\[
\leq A_1 d(x_1, x_2) + A_2 d(x_1, f x_1) + A_3 d(x_2, f x_2) + A_4 d(x_1, f x_2) + B d(x_2, x_1)
\]
\[
= A_1 d(x_1, x_2) + A_2 d(x_1, x_2) + A_3 d(x_2, x_3) + A_4 d(x_1, x_3) + B d(x_2, x_2)
\]
\[
\leq A_1 d(x_1, x_2) + A_2 d(x_1, x_2) + A_3 d(x_2, x_3) + A_4 [d(x_1, x_2) + d(x_2, x_3)] + B_0.
\]
This yields
\[
d(x_2, x_3) \leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_1, x_2) = Ad(x_1, x_2).
\]
Combining (3) and (4), we deduce that
\[
d(x_2, x_3) \leq A^2 d(x_0, x_1).
\]
Iterating this process, we construct a sequence \( \{x_n\} \subseteq X \) such that \( x_n = f x_{n-1} \), \( \Lambda(x_{n-1}, x_n) \geq I \) and
\[
d(x_n, x_{n+1}) \leq A^n d(x_0, x_1), \quad \text{for all } n \in \mathbb{N}.
\]
Next we show that \( \{x_n\} \) is a Cauchy sequence. Let \( n, m \) be arbitrary natural numbers. By using the triangular inequality \((d_3)\), for all \( n, m \in \mathbb{N} \), we have
\[
d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_i, x_{i+1})
\]
\[
\leq \sum_{i=0}^{m-1} A^i d(x_0, x_1) \leq A^n \left( \sum_{i=0}^{\infty} A^i \right) d(x_0, x_1)
\]
\[
= A^n(I - A)^{-1} d(x_0, x_1) \quad \text{(by condition (iii) of Theorem 2.1)}.
\]
Letting \( n \to \infty \) in the above inequality, we get
\[
\lim_{n \to \infty} d(x_n, x_{n+m}) = 0, \quad \text{(see hypothesis (i))},
\]
\[
\Rightarrow \{x_n\} \text{ is a Cauchy sequence.}
\]

From completeness of \((X, d)\), we deduce that there exists \( x' \in X \) such that \( x_n \to x' \). Next, we distinguish two cases.

Case 1: If hypothesis (iv.a) holds then we have \( \Lambda(x_n, x') \geq I \) for all \( n \in \mathbb{N} \). Thus, from (2), we get
\[
d(f x_n, f x') = Id(f x_n, f x') \leq \Lambda(x_n, x')d(f x_n, f x')
\]
\[
\leq A_1 d(x_n, x') + A_2 d(x_n, f x_n) + A_3 d(x', f x') + A_4 d(x_n, f x') + B d(x', f x_n)
\]
\[
= A_1 d(x_n, x') + A_2 d(x_n, x_{n+1}) + A_3 d(x', f x') + A_4 [d(x_n, f x') + d(x', x_{n+1})].
\]
By passing to the limit as \( n \to \infty \) in the above inequality, we obtain
\[
\begin{align*}
  d(x', f x') &\leq (A_3 + A_4) d(x', f x') , \\
  \Rightarrow (I - (A_3 + A_4)) d(x', f x') &\leq 0 ,
\end{align*}
\]

Since the matrix \( I - (A_3 + A_4) \) is nonsingular, we deduce that \( d(x', f x') = 0 \), and hence \( x' = f x' \).

**Case 2:** If hypothesis (iv.b) holds then, for \( n \to \infty \), we have \( f x_n \to f x' \), that is \( x_{n+1} = f x' \) and so \( f x' = x' \).

This concludes the existence part. The uniqueness part is obvious and is obtained by contradiction. Precisely, assume that there exist \( x', x \in \text{Fix}(f) \) with \( x' \neq x \). Clearly, we have \( \Lambda(x', x) \geq I \) and so (by (2))
\[
\begin{align*}
  d(f x', f x) &= Id(f x', f x) \leq \Lambda(x', x) d(f x', f x) \\
  &\leq A_1 d(x', x) + A_2 d(x', f x') + A_3 d(x, f x) + A_4 d(x', f x) + B d(x, f x') \\
  &= A_1 d(x', x) + A_2 d(x', x') + A_3 d(x, x) + A_4 d(x', x) + B d(x, x') \\
  &= (A_1 + A_4 + B) d(x', x) .
\end{align*}
\]

Consequently, by iterating this process, we obtain
\[
\begin{align*}
  d(x', x_n) &\leq (A_1 + A_4 + B)^n d(x', x), \quad \text{for all } n \in \mathbb{N}, \\
  \Rightarrow d(x', x) &= 0 \quad \text{(letting } n \to \infty \text{)}, \\
  \Rightarrow x' &= x , \quad \text{a contradiction.}
\end{align*}
\]

Thus, the fixed point of \( f \) is unique. \( \square \)

**Example 3.3.** The following mappings and matrices satisfy the hypotheses of Theorem 3.2. Let \( X = \mathbb{R}^2 \) be endowed with the generalized metric \( d \) defined by
\[
\begin{align*}
  d(x, y) &= (|x_1 - y_1|, |x_2 - y_2|)^T , \quad \text{for all } x = (x_1, x_2), y = (y_1, y_2) \in X .
\end{align*}
\]

Let \( f : X \to X \) be given by
\[
\begin{align*}
  f x &= \begin{cases} 
  \left( \frac{2x_1}{3} - \frac{x_2}{3} + 1, \frac{x_2}{3} + 1 \right) , & \text{for all } x = (x_1, x_2) \in X \text{ with } x_1 \leq 3 , \\
  \left( x_1 - \frac{x_2}{2} + 1, \frac{x_2}{2} + 1 \right) , & \text{for all } x = (x_1, x_2) \in X \text{ with } x_1 > 3 .
\end{cases}
\end{align*}
\]

For the sake of simplicity, we put \( f x = f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) , \) where
\[
\begin{align*}
  f_1(x_1, x_2) &= \begin{cases} 
  \frac{2x_1}{3} - \frac{x_2}{3} + 1 , & \text{if } x_1 \leq 3 , \\
  x_1 - \frac{x_2}{2} + 1 , & \text{if } x_1 > 3 ,
\end{cases}
\end{align*}
\]

and
\[
\begin{align*}
  f_2(x_1, x_2) &= \begin{cases} 
  \frac{x_2}{3} + 1 , & \text{if } x_1 \leq 3 , \\
  \frac{x_2}{2} + 1 , & \text{if } x_1 > 3 .
\end{cases}
\end{align*}
\]

Consider \( \Lambda : X \times X \to M_{2,2}(\mathbb{R}^+) \) defined by
\[
\Lambda(x, y) = \Lambda((x_1, x_2), (y_1, y_2)) = \begin{cases} 
  \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , & \text{if } 0 \leq x_1, x_2, y_1, y_2 \leq 3 , \\
  \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} , & \text{if } x_1, x_2, y_1, y_2 > 3 , \\
  \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , & \text{otherwise} .
\end{cases}
\]
We show only that the condition (2) holds for all $x, y \in X$, by distinguishing some cases; we leave to the reader to check the remaining hypotheses of Theorem 3.2. Let $A_1 = \left(\begin{array}{c} \frac{3}{5} \\ 0 \\ \frac{1}{3} \end{array}\right)$.

Case 1: If $0 \leq x_1, x_2, y_1, y_2 \leq 3$, then we have

$$A(x, y)d(fx, fy) = \left| \frac{f_1(x_1, x_2) - f_1(y_1, y_2)}{|f_2(x_1, x_2) - f_2(y_1, y_2)|} \right| \leq \left(\begin{array}{c} \frac{3}{5} \\ 0 \\ \frac{1}{3} \end{array}\right) \left(\begin{array}{c} |x_1 - y_1| \\ |x_2 - y_2| \end{array}\right) = A_1d(x, y).$$

Case 2: If $x_1, x_2, y_1, y_2 > 3$, then we have

$$A(x, y)d(fx, fy) = \left(\begin{array}{c} \frac{3}{5} \\ 0 \\ \frac{1}{3} \end{array}\right) \left(\begin{array}{c} |x_1 - y_1| \\ |x_2 - y_2| \end{array}\right) = A_1d(x, y).$$

Case 3: For other choices of $x_1, x_2, y_1$ and $y_2$, we have

$$A(x, y)d(fx, fy) = \left(\begin{array}{c} 0 \\ 0 \\ \frac{1}{2} \end{array}\right) \left(\begin{array}{c} |x_1 - y_1| \\ |x_2 - y_2| \end{array}\right) = A_1d(x, y).$$

Thus (2) holds for all $x, y \in X$ with $A_1 = \left(\begin{array}{c} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{array}\right)$ and $A_2 = A_3 = A_4 = B = \overline{0}$. Of course, we have $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4) = A_1$, which is convergent to zero.

Using two generalized metrics, one can have the following variant of Theorem 3.2.

**Theorem 3.4.** Let $(X, d)$ be a complete generalized metric space, $\rho$ a second generalized metric and $f : (X, \rho) \to (X, \rho)$ be a mapping such that, for all $x, y \in X$, we have

$$A(x, y)\rho(fx, fy) \leq A_1\rho(x, y) + A_2\rho(x, fx) + A_3\rho(y, fy) + A_4\rho(x, f(y, f) + B\rho(y, fx)$$

with $A, A_1, A_2, A_3, A_4, B \in M_{m,n}(\mathbb{R}^*)$ satisfying hypotheses (i)-(iii). Further, assume that

(v) there exists $C \in M_{m,n}(\mathbb{R}^*)$ such that $d(fx, fy) \leq C\rho(x, y)$, whenever there exists a sequence $\{x_n\}_{n=0}^\infty$ with $A(x_i, x_{i+1}) \geq 1$, where $x_0 = x$ and $x_n = y$;

(vi) $f : (X, d) \to (X, d)$ is $\Lambda$-continuous, that is, if $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} d(x_n, x) = 0$ and $A(x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have $\lim_{n \to \infty} d(fx_n, fx_n) = 0$.

Then $f$ has a fixed point. Moreover, if for all $x, y \in \text{Fix}(f)$ we have $A(x, y) \geq I$ and $A_1 + A_4 + B$ converges to zero then the fixed point is unique.

**Remark 3.5.** The proof of Theorem 3.4 essentially follows step by step the proof of Theorem 3.2, by replacing the generalized metric $d$ with $\rho$. The difference between the two proofs is relative to the fact that here we have to establish the Cauchyness of the sequence $\{x_n\}$ in respect both of $(X, \rho)$ and $(X, d)$, because only $(X, d)$ is complete by hypothesis. To this aim, by construction of $\{x_n\}$, for all $n, m \in \mathbb{N}$, we have $A(x_i, x_{i+1}) \geq 1$ for each $i \in \{n, n + 1, \ldots, n + m - 1\}$. So, by using hypothesis (v), we get

$$d(x_{n+1}, x_{n+m+1}) = d(fx_n, fx_{n+m}) \leq C\rho(x_n, x_{n+m}) \leq C[A^n(I - A)^{-1}\rho(x_0, x_1)].$$

(by triangular inequality for $\rho$ and (iii) of Theorem 2.1). So, passing to the limit as $n \to \infty$, we deduce easily that $\{x_n\}$ is Cauchy in $(X, d)$. Finally, by using hypothesis (vi), we deduce that $f$ has a fixed point.

**Example 3.6.** The following mappings and matrices satisfy the hypotheses of Theorem 3.4. Let $X = \mathbb{R}_+ \setminus \{0\}$ be endowed with the generalized metrics $\rho$ and $d$ defined by

$$\rho(x, y) = (|x - y|, |x - y|)^T, \quad \text{for all } x, y \in X,$
The hypotheses are the following:

\[ d(x, y) = \begin{cases} 
  (|x - y| + 1, |x - y| + 1)^T, & \text{if } x \in (0, 1) \text{ or } y \in (0, 1) \text{ or } x, y \in (0, 1) \text{ with } x \neq y, \\
  (0, 0)^T, & \text{if } x = y \in (0, 1), \\
  (|x - y|, |x - y|)^T, & \text{otherwise}.
\]

Let \( f : X \to X \) be given by

\[ f(x) = \begin{cases} 
  x^2, & \text{if } x \in (0, 1), \\
  \frac{x+20}{2}, & \text{otherwise}.
\]

Consider \( \Lambda : X \times X \to M_{2,2}(\mathbb{R}_+ \otimes \mathbb{R}_+) \) defined by

\[ \Lambda(x, y) = \begin{cases} 
  1 \quad \text{if } x, y \geq 1, \\
  0 \quad \text{if } x \leq 1, \\
  0 \quad \text{otherwise}.
\]

In particular, the condition (5) holds true with \( A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( A_2 = A_3 = A_4 = B = \tilde{0} \).

4. Solution of an Initial Value Problem with Delay

In this section we prove a theorem producing the existence of a unique bounded solution of problem (1). We follow the presentation in Bica-Muresan [2] and in Samet-Vetro-Vetro [9]. First, we consider a more general integral operator than the one in Bica-Muresan [2].

Let \( f : X_+ \to X_+ \) be the integral operator defined for all \((u, v) \in X_+\) by

\[ f(u, v)(t) = \begin{cases} 
  \int_{t-\tau}^t g(s, u(s), v(s))ds, & \text{if } t \in [0, t_1], \\
  h(t, u(t), v(t)), & \text{if } t \in [-\tau, 0].
\]

The hypotheses are the following:

**H1:** \( \zeta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is a function such that

(i) there exists \((u_0, v_0) \in X_+ \) such that \( \zeta((u_0(t), v_0(t)), (f(u_0, v_0)(t))) \geq 0 \) for all \( t \in [-\tau, t_1] \);

(ii) for all \( t \in [-\tau, t_1] \), \((u_1, v_1), (u_2, v_2) \in X_+\), we have

\[ \zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0 \quad \Rightarrow \quad \zeta(f(u_1, v_1)(t), f(u_2, v_2)(t))) \geq 0; \]

(iii) for each sequence \( \{(u_n, v_n)\} \subseteq X_+ \) such that \((u_n, v_n) \to (u, v)\) as \( n \to \infty \) and \( \zeta((u_n, v_n), (u_{n+1}, v_{n+1})) \geq 0 \) for all \( n \in \mathbb{N} \), we have \( \zeta((u_n, v_n), (u, v)) \geq 0 \) for all \( n \in \mathbb{N} \).

**H2:** \( g, h \in C([-\tau, t_1] \times X_+ \times \mathbb{R}_+) \) are functions such that

(i) there exist \( g, \bar{g}, \phi, \bar{\phi} \in \mathbb{R}_+ \) such that

\[ \begin{cases} 
  g \leq g(t, u, v) \leq \bar{g}, & t \in [-\tau, t_1], \quad u \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\
  \phi \leq \phi(t) \leq \bar{\phi}, & t \in [-\tau, 0];
\]
(ii) there exist \(a_1, a_2, \rho, \delta > 0\) with \(a_1 \delta^{-1} + \rho a_2 < 1\) such that, for all \(u, u' \in \mathbb{R}_+\), \(v, v' \in \mathbb{R}\) with \(\zeta((u, v), (u', v')) \geq 0\), we have

\[
|g(t, u, v) - g(t, u', v')| \leq a_1|u - u'| + a_2|v - v'|, \quad \text{for all } t \in [-\tau, t_1],
\]

and

\[
|h(t, u, v) - h(t, u', v')| \leq \rho(a_1|u - u'| + a_2|v - v'|), \quad \text{for all } t \in [-\tau, t_1].
\]

(iii) we have

\[
\begin{align*}
\phi(0) &= \int_0^t g(s, \phi(s), \phi'(s))ds, \\
\phi'(0) &= g(0, \phi(0), \phi'(0)) - g(-\tau, \phi(-\tau), \phi'(-\tau)) = h(0, \phi(0), \phi'(0)).
\end{align*}
\]

Now, we can have the theorem producing a unique fixed point of \(f\).

**Theorem 4.1.** If hypotheses \(H_1\) and \(H_2\) hold, then the integral operator (6) has a unique fixed point in \(X_+\).

**Proof.** Note that hypothesis \(H_2(i)\) implies that \(f(X_+) \subseteq X_+\) and so \(f\) is well-defined. Since \(f(u_1, v_1)(t) = f(u_2, v_2)(t)\) for all \(t \in [-\tau, 0]\), then we have

\[
d_B(f(u_1, v_1), f(u_2, v_2)) = \left(\max_{t \in [\tau, 1]} \left|\int_{t-\tau}^t g(s, u_1(s), v_1(s))ds - \int_{t-\tau}^t g(s, u_2(s), v_2(s))ds\right| e^{-\delta(t+\tau)} ,\right)_{T_t},
\]

for all \((u_1, v_1), (u_2, v_2) \in X_+.\) If \(\zeta((u_1(t), v_1(t))(u_2(t), v_2(t))) \geq 0\) for all \(t \in [-\tau, t_1]\), then we have

\[
\left|\int_{t-\tau}^t g(s, u_1(s), v_1(s))ds - \int_{t-\tau}^t g(s, u_2(s), v_2(s))ds\right| \\
\leq \int_{t-\tau}^t \left|g(s, u_1(s), v_1(s)) - g(s, u_2(s), v_2(s))\right| ds \\
\leq \int_{t-\tau}^t (a_1|u_1(s) - u_2(s)| + a_2|v_1(s) - v_2(s)|)ds \\
= \int_{t-\tau}^t (a_1|u_1(s) - u_2(s)|e^{-\delta(t+\tau)} + a_2|v_1(s) - v_2(s)|e^{-\delta(t+\tau)}) e^{\delta(t+\tau)} ds \\
\leq \left(\frac{a_1}{\delta}\|u_1 - u_2\|_B + \frac{a_2}{\delta}\|v_1 - v_2\|_B\right) \int_{t-\tau}^t e^{\delta(t+\tau)} ds \\
= \left(\frac{a_1}{\delta}\|u_1 - u_2\|_B + \frac{a_2}{\delta}\|v_1 - v_2\|_B\right) \left(e^{\delta(t+\tau)} - e^{\delta\tau}\right).
\]

It follows that

\[
\left|\int_{t-\tau}^t g(s, u_1(s), v_1(s))ds - \int_{t-\tau}^t g(s, u_2(s), v_2(s))ds\right| e^{-\delta(t+\tau)} \\
\leq \left(\frac{a_1}{\delta}\|u_1 - u_2\|_B + \frac{a_2}{\delta}\|v_1 - v_2\|_B\right) (1 - e^{-\delta\tau}) \\
\leq \frac{a_1}{\delta}\|u_1 - u_2\|_B + \frac{a_2}{\delta}\|v_1 - v_2\|_B, \quad t \in [0, t_1].
\]
Therefore
\[
\max_{t \in [0,1]} \left| \int_{t-\varepsilon}^{t} g(s, u_1(s), v_1(s)) ds - \int_{t-\varepsilon}^{t} g(s, u_2(s), v_2(s)) ds \right| e^{-\varepsilon(t+\varepsilon)} \\
\leq \frac{a_1}{\varepsilon} ||u_1 - u_2||_{\infty} + \frac{a_2}{\varepsilon} ||v_1 - v_2||_{\infty},
\]
for all \((u_1, v_1), (u_2, v_2) \in X_+\) such that \(\zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0\) for all \(t \in [-\tau, \tau]\). Similarly, we get
\[
|h(t, u_1(t), v_1(t)) - h(t, u_2(t), v_2(t))| \\
\leq \rho(\alpha_1 ||u_1 - u_2||_{\infty} + a_2 ||v_1 - v_2||_{\infty}) + a_2 ||v_1(t) - v_2(t)|| e^{-\varepsilon(t+\varepsilon)} e^{\varepsilon(t+\varepsilon)} \\
\leq \rho(\alpha_1 ||u_1 - u_2||_{\infty} + a_2 ||v_1 - v_2||_{\infty}),
\]
for all \((u_1, v_1), (u_2, v_2) \in X_+\) such that \(\zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0\) for all \(t \in [-\tau, \tau]\). Consequently, we obtain
\[
\max_{t \in [0,1]} |h(t, u_1(t), v_1(t)) - h(t, u_2(t), v_2(t))| e^{-\varepsilon(t+\varepsilon)} \\
\leq \rho(\alpha_1 ||u_1 - u_2||_{\infty} + a_2 ||v_1 - v_2||_{\infty}),
\]
for all \((u_1, v_1), (u_2, v_2) \in X_+\) such that \(\zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0\) for all \(t \in [-\tau, \tau]\).

Then, for all \((u_1, v_1), (u_2, v_2) \in X_+\) such that \(\zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0\) for all \(t \in [-\tau, \tau]\), we have
\[
d_h(f(u_1, v_1), f(u_2, v_2)) \leq A \delta((u_1, v_1), (u_2, v_2)),
\]
where
\[
A = \begin{pmatrix} a_1 \delta^{-1} & a_2 \delta^{-1} \\ \rho a_1 & \rho a_2 \end{pmatrix}. \tag{7}
\]

Consider \(\Lambda : X \times X \to M_{2,2}(\mathbb{R}_+\), defined by
\[
\Lambda((u_1, v_1), (u_2, v_2)) = \begin{cases} 1 & 0 \\ 0 & 1 \end{cases}, \quad \text{if } \zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0 \text{ for all } t \in [-\tau, \tau], \\
0 & 0 \end{cases}, \quad \text{otherwise.}
\]

Finally, for all \((u_1, v_1), (u_2, v_2) \in X_+\), we have
\[
\Lambda((u_1, v_1), (u_2, v_2)) d_h(f(u_1, v_1), f(u_2, v_2)) \leq A \delta((u_1, v_1), (u_2, v_2)).
\]

Next, the eigenvalues of \(A\) are
\[
\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = a_1 \delta^{-1} + \rho a_2. \tag{8}
\]

Therefore, from (8) and hypothesis \(H_2(ii)\) (i.e., \(a_1 \delta^{-1} + \rho a_2 < 1\)) we infer that \(\lambda_1, \lambda_2\) are in the open unit disc and so \(A\) is convergent to zero (see conditions (i) and (ii) of Theorem 2.1). Also, the above calculations and the relation between the matrix \(\Lambda\) and function \(\zeta\) (by using hypotheses \(H_1\)) show that the hypotheses of Theorem 3.2 hold with \(A_1 = A\) given by (7) and \(A_2 = A_3 = A_4 = B = \bar{B}.\) For instance, by \(H_1(ii)\), we have
\[
\Lambda((u_1, v_1), (u_2, v_2)) \geq I \\
\Rightarrow \zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) \geq 0 \\
\Rightarrow \zeta(f(u_1, v_1(t)), f(u_2, v_2(t))) \geq 0 \\
\Rightarrow \Lambda(f(u_1, v_1(t)), f(u_2, v_2(t))) \geq I,
\]
so \(f\) is \(\Lambda\)-admissible. Thus, the existence and uniqueness of a fixed point of \(f\) in \(X_+\) is an immediate consequence of Theorem 3.2. \(\square\)
By particularizing the choice of \( h \in C([-\tau, t_1] \times \mathbb{R}_+ \times \mathbb{R}) \), we can have the theorem producing a unique bounded solution of problem (1). This theorem is more general than the analogous of Bica-Muresan [2, Theorem 5]. Let \( h(t, u(t), v(t)) = g(t, u(t), v(t)) - g(t - \tau, u(t - \tau), v(t - \tau)) \) for all \( t \in [0, t_1] \) and consider the integral operator

\[
\tilde{f}(u, v)(t) = \begin{cases}
\int_{-\tau}^{t} g(s, u(s), v(s))ds, & g(t, u(t), v(t)) - g(t - \tau, u(t - \tau), v(t - \tau)) \\
(\phi(t), \phi'(t)), & t \in [0, t_1],
\end{cases}
\]

Theorem 4.2. If hypotheses \( H_1 \) and \( H_2 \) hold, then problem (1) has a unique bounded solution in \( X_+ \).

Proof. The similar reasoning as in the proof of Theorem 4.1 shows that \( \tilde{f} \) has a unique fixed point in \( X_+ \), say \( ((u_1^*, v_1^*)) = \text{Fix}(\tilde{f}) \). To avoid repetition, we leave the details and point out just the difference. Precisely, here we obtain the matrix

\[
A = \begin{pmatrix}
a_1 \theta_{-1} & a_2 \theta_{-1} \\
a_1(1 + e^{-\theta}) & a_2(1 + e^{-\theta})
\end{pmatrix}
\]

with eigenvalues \( \lambda_1 = 0 \) and \( \lambda_2 = a_1 \theta_{-1} + a_2(1 + e^{-\theta}) \), that is, we have \( \rho = (1 + e^{-\theta}) \).

Next, we show that \( u_1^* \) is a unique bounded solution of (1). In fact, from hypothesis \( H_2(i) \), we get

\[
\begin{align*}
\frac{\theta_{-1}}{\theta} \leq u_1^*(t) = & \int_{-\tau}^{t} g(s, u_1^*(s), v_1^*(s))ds \leq \overline{\rho}, & t \in [0, t_1], \\
\underline{\varphi} \leq & u_1^*(t) \leq \overline{\varphi}, & t \in [-\tau, 0],
\end{align*}
\]

and hence the boundedness is proved. It remain to prove that

\[
(u_1^*, v_1^*) = \tilde{f}(u_1^*, v_1^*), \quad t \in [-\tau, t_1],
\]

(see Bica-Muresan [2], p. 25). We distinguish the following two cases:

Case 1: If \( t \in [0, t_1] \) then, from (6), we have

\[
(u_1^*, v_1^*) = \tilde{f}(u_1^*, v_1^*), \quad t \in [0, t_1].
\]

So

\[
\begin{align*}
u_1^*(t) = & \int_{-\tau}^{t} g(s, u_1^*(s), v_1^*(s))ds, \\
v_1^*(t) = & g(t, u_1^*(t), v_1^*(t)) - g(t - \tau, u_1^*(t - \tau), v_1^*(t - \tau)).
\end{align*}
\]

It follows easily that

\[
(u_1^*)'(t) = g(t, u_1^*(t), v_1^*(t)) - g(t - \tau, u_1^*(t - \tau), v_1^*(t - \tau))
\]

and so \( (u_1^*)'(t) = v_1^*(t) \) for all \( t \in [0, t_1] \).

Case 2: If \( t \in [-\tau, 0] \), again from (6), we have

\[
\tilde{f}(u_1^*, v_1^*) = (\phi(t), \phi'(t)) = (u_1^*(t), v_1^*(t))
\]

and so \( u_1^*(t) = \phi(t) \) and \( v_1^*(t) = \phi'(t) \). \( \square \)

Remark 4.3. Every non-negative constant function \( \zeta \) reduces Theorem 4.2 to Theorem 5 of Bica-Muresan [2], where (for the sake of exactness) the authors assume \( a_2 \in (0, 2^{-1}] \). On the other hand, other choices of function \( \zeta \) are possible. So Theorem 4.1 covers a large class of situations than those of the original version in [2]. For example, by assuming

\[
\zeta(u_1(t), v_1(t), u_2(t), v_2(t)) = u_1(t) - u_2(t), \quad \text{for all } t \in [-\tau, t_1], \quad (u_1, v_1), (u_2, v_2) \in X_+,
\]

the ordered approach to the study of initial value problems with delay arises naturally. Technically, this means to consider the product space \( X \) endowed with the partial order \( \leq \) defined by

\[
(u_1, v_1), (u_2, v_2) \in X, \quad (u_1, v_1) \leq (u_2, v_2) \iff u_1(t) \leq u_2(t), \quad t \in [-\tau, t_1].
\]

Thus, we have to check the contractive condition in Theorem 4.1 only for couples of points satisfying the partial order. Also, the hypothesis \( H_1(i) \) reduces to the existence of an upper solution for problem (1) and so on.
References