A Fixed Point Theorem for JS-contraction Type Mappings with Applications to Polynomial Approximations

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Abstract. A fixed point theorem is established for a new class of JS-contraction type mappings. As applications, some Kelisky-Rivlin type results are obtained for linear and nonlinear \(q\)-Bernstein-Stancu operators.

1. Introduction

Let \(\Theta\) be the set of functions \(\theta : (0, \infty) \to (1, \infty)\) satisfying the following conditions:

\((\Theta_1)\) \(\theta\) is non-decreasing;

\((\Theta_2)\) For each sequence \(\{t_n\} \subset (0, \infty)\), we have

\[
\lim_{n \to \infty} \theta(t_n) = 1 \iff \lim_{n \to \infty} t_n = 0^+;
\]

\((\Theta_3)\) There exist \(r \in (0, 1)\) and \(\ell \in (0, \infty]\) such that \(\lim_{t \to 0^+} \frac{\theta(t)-1}{t} = \ell\).

Recently, Jleli and Samet [4] introduced the class of JS-contraction mappings as follows.

Definition 1.1. Let \((X, d)\) be a metric space, and let \(T : X \to X\) be a given mapping. The mapping \(T\) is said to be a JS-contraction if there exist \(\theta \in \Theta\) and \(k \in (0, 1)\) such that

\[(x, y) \in X \times X, d(Tx, Ty) > 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k.\]

In [4], the following generalization of Banach contraction principle was established.

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Theorem 1.2. Let \((X,d)\) be a complete metric space, and let \(T : X \to X\) be JS-contraction. Then \(T\) has a unique fixed point.

Observe that Banach contraction principle follows from Theorem 1.2 by taking \(\theta(t) = e^{\sqrt{t}}\). For other related results, we refer the reader to [13, 16].

In this paper, a fixed point theorem for a new class of JS-contraction type mappings is presented. Next, this theorem is used to study the iterates properties of some polynomial operators including \(q\)-Bernstein-Stancu operators and \(q\)-Bernstein-Stancu operators of nonlinear type.

2. A Fixed Point Theorem

In this section, a new fixed point theorem is established for a new class of JS-contraction type mappings. The obtained result is an extension of Theorem 1.2.

At first, let us introduce some notations. Let \(M\) be a nonempty set, and let \(T : M \to M\) be a given mapping. We denote by \(\text{Fix}(T)\) the set of all the fixed points of \(T\), that is,

\[
\text{Fix}(T) = \{x \in M : x = Tx\}.
\]

Suppose that \(M\) is a group with respect to a certain operation \(+\). For \(x \in M\) and \(N \subset M\), we denote by \(x + N\) the subset of \(M\) defined by

\[
x + N = \{x + y : y \in N\}.
\]

We denote by \(\mathbb{N}\) the set of positive integers, that is,

\[
\mathbb{N} = \{0, 1, 2, \cdots\}.
\]

We denote by \(\mathbb{N}^\ast\) the set defined by

\[
\mathbb{N}^\ast = \{1, 2, 3, \cdots\}.
\]

Our fixed point theorem can be stated as follows.

Theorem 2.1. Let \(E\) be a group with respect to a certain operation \(+\). Let \(X\) be a subset of \(E\) endowed with a certain metric \(d\) such that \((X,d)\) is complete. Let \(X_0 \subset X\) be a closed subset of \(X\) such that \(X_0\) is a subgroup of \(E\). Let \(T : X \to X\) be a given mapping satisfying

\[
(x,y) \in X \times X, \ x - y \in X_0, \ d(Tx,Ty) \neq 0 \implies \theta(d(Tx,Ty)) \leq [\theta(d(x, y))]^k,
\]

where \(k \in (0, 1)\) is a constant and \(\theta \in \Theta\). Suppose that the operation mapping \(\pm : X \times X \to X\) defined by

\[
\pm(x,y) = x \pm y, \quad (x,y) \in X \times X
\]

is continuous with respect to the metric \(d\). Moreover, suppose that

\[
x - Tx \in X_0, \quad x \in X.
\]

Then we have

(i) For every \(x \in X\), the Picard sequence \(\{T^n x\}\) converges to a fixed point of \(T\).

(ii) For every \(x \in X\),

\[
(x + X_0) \cap \text{Fix}(T) = \left\{ \lim_{n \to \infty} T^n x \right\}.
\]
Proof. Let \( x \in X \) be an arbitrary point in \( X \). If for some \( p \in \mathbb{N} \), we have \( T^p x = T^{p+1} x \), then \( T^p x \) will be a fixed point of \( T \). So, without restriction of the generality, we can suppose that \( d(T^n x, T^{n+1} x) > 0 \), for all \( n \in \mathbb{N} \).

From (2), we have
\[
x - T x \in X_0.
\]
Using (1), we obtain
\[
\theta(d(T x, T^2 x)) = [\theta(d(x, T x))]^k.
\]
Again, using (2), we obtain
\[
T x - T^2 x = T x - T(T x) \in X_0,
\]
which implies from (1) that
\[
\theta(d(T^2 x, T^3 x)) \leq [\theta(d(T x, T^2 x))]^k \leq [\theta(d(x, T x))]^{k^2}.
\]
Therefore, by induction we obtain
\[
T^n x - T^{n+1} x \in X_0, \quad n \in \mathbb{N},
\]
and
\[
\theta(d(T^n x, T^{n+1} x)) \leq [\theta(d(x, T x))]^{k^n}, \quad n \in \mathbb{N}.
\]
Thus, we have
\[
1 \leq \theta(d(T^n x, T^{n+1} x)) \leq [\theta(d(x, T x))]^{k^n}, \quad n \in \mathbb{N}.
\]
Passing to the limit as \( n \to \infty \) in (4), we obtain
\[
\lim_{n \to \infty} \theta(d(T^n x, T^{n+1} x)) = 1,
\]
which implies from \((\Theta_2)\) that
\[
\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0.
\]
From condition \((\Theta_3)\), there exist \( r \in (0, 1) \) and \( \ell \in (0, \infty) \) such that
\[
\lim_{n \to \infty} \frac{\theta(d(T^n x, T^{n+1} x)) - 1}{[d(T^n x, T^{n+1} x)]^r} = \ell.
\]
Suppose that \( \ell < \infty \). In this case, let \( B = \ell/2 > 0 \). From the definition of the limit, there exists \( n_0 \in \mathbb{N} \) such that
\[
\left| \frac{\theta(d(T^n x, T^{n+1} x)) - 1}{[d(T^n x, T^{n+1} x)]^r} - \ell \right| \leq B, \quad n \geq n_0.
\]
This implies that
\[
\frac{\theta(d(T^n x, T^{n+1} x)) - 1}{[d(T^n x, T^{n+1} x)]^r} \geq \ell - B = B, \quad n \geq n_0.
\]
Then,
\[
n[d(T^n x, T^{n+1} x)]^r \leq An[\theta(d(T^n x, T^{n+1} x)) - 1], \quad n \geq n_0,
\]
where \( A = 1/B \).

Suppose now that \( \ell = \infty \). Let \( B > 0 \) be an arbitrary positive number. From the definition of the limit, there exists \( n_0 \in \mathbb{N} \) such that
\[
\frac{\theta(d(T^n x, T^{n+1} x)) - 1}{[d(T^n x, T^{n+1} x)]^r} \geq B, \quad n \geq n_0.
\]
This implies that
\[ n[d(T^n, T^{n+1})] \leq An[\theta(d(T^n, T^{n+1})) - 1], \quad n \geq n_0, \]
where \( A = 1/B. \)

Thus, in all cases, there exists \( A > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[ n[d(T^n, T^{n+1})] \leq An[\theta(d(T^n, T^{n+1})) - 1], \quad n \geq n_0. \]
Using (4), we obtain
\[ n[d(T^n, T^{n+1})] \leq An(\theta(d(T^n, T^{n+1}))^k - 1), \quad n \geq n_0. \]
Letting \( n \to \infty \) in the above inequality, we obtain
\[ \lim_{n \to \infty} n[d(T^n, T^{n+1})] = 0. \]
Therefore, by (3) and using the fact that (4), (8) and (1), we have
\[ T^n \to \infty, \quad n, p \geq 1. \]
Passing to the limit as \( n \to \infty \), using (7), the continuity of the operation mapping \( + \), and the closure of \( X_0 \), we obtain that
\[ T^n x - T^{n+p} x \in X_0, \quad n, p \geq 1. \]
Without restriction of the generality, we may suppose that \( d(T^n, T\omega) > 0 \), for all \( n \in \mathbb{N} \). Therefore, using (8) and (1), we have
\[ 1 \leq \theta(d(T^n+1, T\omega)) \leq \theta(T^n, \omega)^k, \quad n \in \mathbb{N}. \]
Passing to the limit as \( n \to \infty \), using (7) and (\( \Theta_2 \)), we deduce that
\[ \lim_{n \to \infty} d(T^n+1, T\omega) = 0. \]
Next, (7), (9) and the uniqueness of the limit yield $\omega = T\omega$, that is, $\omega$ is a fixed point of $T$. Then (i) is proved. In order to prove (ii), let $x \in X$ be fixed. We know that the Picard sequence $\{T^n x\}$ converges to $\omega \in X$, a fixed point of $T$. Moreover, from (8), we have $\omega - x \in X_0$, that is, $\omega \in x + X_0$. Therefore, we have

$$\left\{ \lim_{n \to \infty} T^n x \right\} \subset (x + X_0) \cap \text{Fix}(T).$$

Now, let $z \in (x + X_0) \cap \text{Fix}(T)$ be fixed. Then

$$Tz = z \quad \text{and} \quad z - x \in X_0.$$

Therefore, we have

$$z - Tx = Tz - Tx = (Tz - z) + (x - Tx) + (z - x) \in X_0.$$

Again,

$$z - T^2 x = T^2 z - T^2 x = (T^2 z - Tz) + (Tx - T^2 x) + (z - Tz) \in X_0.$$

Hence, by induction we obtain

$$z - T^n x \in X_0, \quad n \in \mathbb{N}.$$

Without restriction of the generality, we may suppose that $z \neq T^n x$, for all $n \in \mathbb{N}$. Therefore, by (1) we have

$$1 \leq \theta(d(z, T^{n+1} x)) = \theta(d(Tz, T^{n+1} x)) \leq [\theta(d(z, Tx))]^k \leq \cdots \leq [\theta(d(z, x))]^{k^{n+1}}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using $(\Theta_2)$, we deduce that

$$\lim_{n \to \infty} d(T^n x, z) = 0,$$

which yields $z \in \left\{ \lim_{n \to \infty} T^n x \right\}$. Then we proved that

$$(x + X_0) \cap \text{Fix}(T) \subset \left\{ \lim_{n \to \infty} T^n x \right\}.$$

The proof is complete. $\square$

The following result follows immediately from Theorem 2.1 with $\theta(t) = e^{\sqrt{t}}$.

**Corollary 2.2.** Let $E$ be a group with respect to a certain operation $\neq$. Let $X$ be a subset of $E$ endowed with a certain metric $d$ such that $(X, d)$ is complete. Let $X_0 \subset X$ be a closed subset of $X$ such that $X_0$ is a subgroup of $E$. Let $T : X \to X$ be a given mapping satisfying

$$(x, y) \in X \times X, \quad x - y \in X_0 \implies d(Tx, Ty) \leq kd(x, y),$$

where $k \in (0, 1)$ is a constant. Suppose that the operation mapping $\pm : X \times X \to X$ defined by

$$\pm(x, y) = x \pm y, \quad (x, y) \in X \times X$$

is continuous with respect to the metric $d$. Moreover, suppose that

$$x - Tx \in X_0, \quad x \in X.$$

Then we have

(i) For every $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of $T$.

(ii) For every $x \in X$,

$$(x + X_0) \cap \text{Fix}(T) = \left\{ \lim_{n \to \infty} T^n x \right\}.$$
3. Applications: Iterates Properties of Some Polynomial Operators

In this section, as applications of Theorem 2.1, the iterates properties of some polynomial operators are investigated. Two types of polynomial operators are discussed: \( q \)-Bernstein-Stancu operators and \( q \)-Bernstein-Stancu operators of nonlinear type. For each kind of operators, a Kelisky-Rivlin type result is established. Let us mention some well known contributions in this topic. In [6], via some linear algebra tools, Kelisky and Rivlin studied the iterates properties of the class of Bernstein operators. Another proof of Kelisky-Rivlin theorem was presented by I.A. Rus [10] with the help of some trick with the Contraction principle. Another possibility to establish Kelisky-Rivlin theorem, which is based on a fixed point theorem for linear operators on a Banach space, was suggested by Jachymski [3]. For other related works, we refer to [1, 2, 8, 14, 15] and references therein.

The following basic notations in quantum calculus will be used. Let \( q > 0 \). For any \( n \in \mathbb{N} \), the \( q \)-integer \( n_q \) is defined by
\[
[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \quad (n \geq 1), \quad [0]_q = 0.
\]
The \( q \)-factorial \( n_q! \) is defined by
\[
[n]_q! = [1]_q[n]_q \cdots [n]_q \quad (n \geq 1), \quad [0]_q! = 1.
\]
For integers \( 0 \leq k \leq n \), the \( q \)-binomial is defined by
\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.
\]
It is clear that for \( q = 1 \), we have
\[
[n]_1 = n, \quad [n]_1! = n!, \quad \binom{n}{k}_1 = \binom{n}{k}.
\]
For more details on quantum calculus, we refer to [5].

3.1. A Kelisky-Rivlin type result for \( q \)-Bernstein-Stancu operators

Let \( C([0, 1]; \mathbb{R}) \) be the set of real valued and continuous functions \( f : [0, 1] \to \mathbb{R} \). For \( f \in C([0, 1]; \mathbb{R}) \), \( q > 0 \), \( \alpha \geq 0 \) and each \( n \in \mathbb{N}^* \), the \( q \)-Bernstein-Stancu operator of order \( n \) is defined by [7]
\[
B_n(q, \alpha)(f)(t) = \sum_{i=0}^{n} f\left( \frac{[i]_q}{[n]_q} \right) B_{n,i}^{q,\alpha}(t), \quad t \in [0, 1],
\]
where
\[
B_{n,i}^{q,\alpha}(t) = \binom{n}{i}_q \prod_{s=0}^{i-1} \left( t + \alpha[s]_q \right) \prod_{j=0}^{n-i-1} \left( 1 - q^j t + \alpha[j]_q \right).
\]
From here on an empty product is taken to be equal to 1.

If \( \alpha = 0 \), \( B_n(q, 0) \) reduces to the \( q \)-Bernstein polynomial of order \( n \) introduced by Phillips [9]
\[
B_n(q, 0)(f)(t) = \sum_{i=0}^{n} f\left( \frac{[i]_q}{[n]_q} \right) \binom{n}{i}_q t^i \prod_{j=0}^{n-i-1} \left( 1 - q^j t \right), \quad t \in [0, 1].
\]
If \( q = 1 \), \( B_n(1, \alpha) \) reduces to the Bernstein-Stancu polynomial of order \( n \) introduced by Stancu [11]
\[
B_n(1, \alpha)(f)(t) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) \left( \begin{array}{c} n \\ i \end{array} \right) \frac{t^i (1 + \alpha f)^n}{\prod_{j=0}^{n-i-1} (1 - t + \alpha f)} , \quad t \in [0, 1].
\]
If \((\alpha, q) = (0, 1)\), we obtain the standard Bernstein polynomial of order \( n \)
\[
B_n(1, 0)(f)(t) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) t^i (1 - t)^{n-i} , \quad t \in [0, 1].
\]

The following lemmas will be useful later (see [2, 15]).

**Lemma 3.1.** Let \( n \in \mathbb{N}^* \), \( q \in (0, 1) \) and \( \alpha \geq 0 \). Then
\[
\sum_{i=0}^{n} B_{n,i}^{q,\alpha}(t) = 1.
\]

**Lemma 3.2.** Let \( n \in \mathbb{N}^* \), \( q \in (0, 1) \) and \( \alpha \geq 0 \). Then
\[
\min \{ B_{n,0}^{q,\alpha}(t) + B_{n,n}^{q,\alpha}(t) : t \in [0, 1] \} > 0.
\]

We have the following Kelisky-Rivlin type result for \( q \)-Bernstein-Stancu operators.

**Theorem 3.3.** Let \( n \in \mathbb{N}^* \), \( \alpha \geq 0 \) and \( 0 < q < 1 \). Then, for every \( f \in C([0, 1]; \mathbb{R}) \),
\[
\lim_{n \to \infty} [B_n(q, \alpha)]^{\otimes n}(f)(t) = f(0) + [f(1) - f(0)]t , \quad t \in [0, 1].
\]

**Proof.** Let \( X = E = C([0, 1]; \mathbb{R}) \). We endow \( X \) with the metric \( d \) defined by
\[
d(f, g) = \max \{|f(t) - g(t)| : t \in [0, 1]\}, \quad (f, g) \in X \times X.
\]
Then \((X, d)\) is a complete metric space. Let \( X_0 \) be the subset of \( X \) defined by
\[
X_0 = \{ f \in X : f(0) = f(1) = 0 \}.
\]
Then \( X_0 \) is a closed linear subspace of \( X \). Let \((f, g) \in X \times X\) be such that \( f - g \in X_0 \), that is,
\[
(f, g) \in X \times X \quad \text{and} \quad f(0) = g(0), f(1) = g(1).
\]
Let \( t \in [0, 1] \) be fixed. Then we have
\[
|B_n(q, \alpha)(f)(t) - B_n(q, \alpha)(g)(t)|
\]
\[
= \sum_{i=0}^{n} f \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t) - \sum_{i=0}^{n} g \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t)
\]
\[
\leq \sum_{i=0}^{n} \left| f \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t) - g \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t) \right|
\]
\[
= \sum_{i=1}^{n-1} \left| f \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t) - g \left( \frac{i}{n} \right) \left[ \frac{[\alpha]}{n} \right] B_{n,i}^{q,\alpha}(t) \right|
\]
\[
\leq \sum_{i=1}^{n-1} B_{n,i}^{q,\alpha}(t) d(f, g).
\]
On the other hand, using Lemmas 3.1 and 3.2, we get
\[ \sum_{i=1}^{n-1} B_{n,q}^{\alpha,\gamma}(t) = 1 - \left( B_{n,q}^{\alpha}\left(1\right) + B_{n,q}^{\alpha}(t) \right) \]
\[ \leq 1 - \lambda, \]
where
\[ \lambda = \min \left\{ B_{n,q}^{\alpha}(t) + B_{n,q}^{\alpha}(t) : t \in [0,1] \right\} > 0. \tag{10} \]

Therefore, we have
\[ (f, 1) \in X \times X, \quad f - g \in X_0 \implies d\left( B_{n,q}(q,\alpha)(f), B_{n,q}(q,\alpha)(g) \right) \leq kd(f, g), \]
where \( k = 1 - \lambda \in (0,1) \). Next, using lemma 3.1, for every \( f \in X \) we have
\[ \gamma(t) := f(t) - B_n(q,\alpha)(f)(t) = \sum_{i=0}^{n-1} \left( f(t) - f\left( \frac{i}{n} \right) \right) B_{n,q}^{\alpha}(t), \quad t \in [0,1]. \]
We can check easily that
\[ \gamma(0) = \gamma(1) = 0, \]
which yields
\[ f - B_n(q,\alpha)(f) \in X_0, \quad f \in X. \]

Applying Theorem 2.1 (or Corollary 2.2), we deduce that
\[ (f + X_0) \cap \text{Fix}(B_n(q,\alpha)) = \left\{ \lim_{N \to \infty} \left[ B_n(q,\alpha)^N(f) \right] \right\}, \quad f \in X. \]

Let \( f \in X \). It is not difficult to observe that the function \( \omega : [0,1] \to \mathbb{R} \) defined by
\[ \omega(t) = f(0)(1 - t) + f(1)t, \quad t \in [0,1] \]
belongs to \( \text{Fix}(B_n(q,\alpha)) \). Moreover, for all \( t \in [0,1] \),
\[ \mu(t) := \omega(t) - f(t) = f(0)(1 - t) + f(1)t - f(t). \]

Observe that
\[ \mu(0) = f(0) - f(0) = 0 \]
and
\[ \mu(1) = f(1) - f(1) = 0. \]

Therefore, \( \omega \in f + X_0 \). As consequence, we get
\[ \lim_{N \to \infty} d\left( B_n(q,\alpha)^N(f), \omega \right) = 0, \]
which yields the desired result. \( \square \)

**Remark 3.4.** Another proof of Theorem 3.3 can be found in [15]. This proof is based on some linear algebra tools. In our opinion, the presented proof in this paper is more easy and more simplified.
3.2. A Kelisky-Rivlin type result for nonlinear $q$-Bernstein-Stancu operators

For $f \in C([0, 1]; \mathbb{R})$, $q > 0$, $\alpha \geq 0$ and each $n \in \mathbb{N}^*$, we define the nonlinear $q$-Bernstein-Stancu operator of order $n$ by

$$T_n(q, \alpha)(f)(t) = \sum_{i=0}^{n} \left| f \left( \frac{[i]_q}{[n]_q} \right) \right| B_{n,j}^{\alpha,q}(t), \quad t \in [0, 1].$$

Using Theorem 2.1, we shall establish the following Kelisky-Rivlin type result.

**Theorem 3.5.** Let $n \in \mathbb{N}^*$, $\alpha \geq 0$ and $0 < q < 1$. Then, for every $f \in C([0, 1]; \mathbb{R})$ such that $f(0) \geq 0$ and $f(1) \geq 0$,

$$\lim_{N \to \infty} [T_n(q, \alpha)]^N(f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0, 1].$$

**Proof.** Let $E = C([0, 1]; \mathbb{R})$ and $X$ be the subset if $E$ defined by

$$X = \{ f \in E : f(0) \geq 0, f(1) \geq 0 \}.$$

We endow $X$ with the metric $d$ defined by

$$d(f, g) = \max \{|f(t) - g(t)| : t \in [0, 1] \}, \quad (f, g) \in X \times X.$$ Then $(X, d)$ is a complete metric space. Let $X_0$ be the subset of $X$ defined by

$$X_0 = \{ f \in E : f(0) = f(1) = 0 \}.$$ Then $X_0$ is a closed subgroup of $E$. Let $(f, g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f, g) \in X \times X \quad \text{and} \quad f(0) = g(0), f(1) = g(1).$$

Let $t \in [0, 1]$ be fixed. Then we have

$$|T_n(q, \alpha)(f)(t) - T_n(q, \alpha)(g)(t)|$$

$$= \sum_{i=0}^{n} \left| f \left( \frac{[i]_q}{[n]_q} \right) \right| B_{n,j}^{\alpha,q}(t) - \sum_{i=0}^{n} \left| g \left( \frac{[i]_q}{[n]_q} \right) \right| B_{n,j}^{\alpha,q}(t)$$

$$= \sum_{i=0}^{n} \left| f \left( \frac{[i]_q}{[n]_q} \right) \right| - \left| g \left( \frac{[i]_q}{[n]_q} \right) \right| B_{n,j}^{\alpha,q}(t)$$

$$\leq \sum_{i=0}^{n} f \left( \frac{[i]_q}{[n]_q} \right) - g \left( \frac{[i]_q}{[n]_q} \right) B_{n,j}^{\alpha,q}(t)$$

$$= \sum_{i=1}^{n} f \left( \frac{[i]_q}{[n]_q} \right) - g \left( \frac{[i]_q}{[n]_q} \right) B_{n,j}^{\alpha,q}(t)$$

$$\leq \sum_{i=1}^{n-1} B_{n,j}^{\alpha,q}(t) d(f, g)$$

$$= (1 - \lambda) d(f, g),$$

where $\lambda$ is given by (10). Therefore, we have

$$(f, g) \in X \times X, \quad f - g \in X_0 \implies d(T_n(q, \alpha)(f), T_n(q, \alpha)(g)) \leq kd(f, g),$$

where $k = 1 - \lambda \in (0, 1)$. Next, for every $f \in X$ we have

$$\gamma'(t) := f(t) - T_n(q, \alpha)(f)(t) = \sum_{i=0}^{n} \left( f(t) - \left| f \left( \frac{[i]_q}{[n]_q} \right) \right| \right) B_{n,j}^{\alpha,q}(t), \quad t \in [0, 1].$$
Observe that
\[\gamma'(0) = f(0) - |f(0)| = f(0) - f(0) = 0\]
and
\[\gamma'(1) = f(1) - |f(1)| = f(1) - f(1) = 0.\]
Then
\[f - T_n(q, \alpha)(f) \in X_0, \quad f \in X.\]
Applying Theorem 2.1 (or Corollary 2.2), we deduce that
\[(f + X_0) \cap \text{Fix}(T_n(q, \alpha)) = \left\{ \lim_{N \to \infty} [T_n(q, \alpha)]^N(f) \right\}, \quad f \in X.\]
Let \(f \in X\). It is not difficult to observe that the function \(\omega : [0, 1] \to \mathbb{R}\) defined by
\[\omega(t) = f(0)(1 - t) + f(1)t, \quad t \in [0, 1]\]
belongs to \((f + X_0) \cap \text{Fix}(T_n(q, \alpha))\). As consequence, we get
\[\lim_{N \to \infty} d([T_n(q, \alpha)]^N(f), \omega) = 0,\]
which yields the desired result. \(\square\)

**Remark 3.6.** Note that Theorem 4.1 in [3] cannot be applied in our case since it requires linear operators defined on a certain Banach space \(X\). Observe that in our case, \(X\) is not a linear space.

**Remark 3.7.** The case \((\alpha, q) = (0, 1)\) was considered in [12]. The authors claimed that if \(n \in \mathbb{N}\), for every \(f \in X = C([0, 1]; \mathbb{R})\), the Picard sequence \([T_n(0, 1)]^N(f)\) converges uniformly to a fixed point of \(T_n(0, 1)\) (see Corollary 4 in [12]). For the proof of this claim, the authors used that \(f - T_n(0, 1)(f) \in X_0\) for every \(f \in X\), where \(X_0\) is the set of functions \(u \in X\) such that \(u(0) = u(1) = 0\). Unfortunately, the above property is not true. To observe this fact, we have just to consider a function \(f \in X\) such that \(f(0) < 0\) or \(f(1) < 0\). Our Theorem 3.5 for the case \((\alpha, q) = (0, 1)\) is a corrected version of Corollary 4 in [12].

**References**


