On the Diophantine Equation $x^2 + 5^a \cdot p^b = y^n$

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Abstract. In this paper, all the solutions of the Diophantine equations $x^2 + 5^a \cdot p^b = y^n$ (for $p = 29, 41$) are given for nonnegative integers $a, b, x, y, n \geq 3$ with $x$ and $y$ coprime.

1. Introduction

Recently, there have been many papers dealing with by the generalized Lebesgue-Nagell equation

$$x^2 + C = y^n$$

where $C > 0$ is a fixed integer and $x, y, n$ are positive integer unknowns with $n \geq 3$. In 1850, V. A. Lebesque [14] proved that this equation has no solution for $C = 1$. Ljunggren [16] solved for $C = 2$ and Nagell [20], [21] solved it for $C = 3, 4$ and 5. J. H. E. Cohn [10] could solve (1) for 77 values of $C$ between 1 and 100. In [19], Mignotte and de Weger dealt with the cases $C = 74$ and 86, which had not been dealt with Cohn.

Finally the remaining cases up to 100 were dealt with by Bugeaud, Mignotte and Siksek in [7].

Here we consider the Diophantine equation (1) where $C = q_1^{a_1} \cdot q_2^{a_2} \cdots q_k^{a_k}$ or $C = 2^m \cdot q_1^{a_1} \cdot q_2^{a_2} \cdots q_k^{a_k}$ are fixed numbers satisfying the following three conditions:

(I) $q_i \equiv 1 \pmod{4}$ are primes for all $i = 1, 2, \ldots, k$.

Write $C = d \cdot z^2$ with $d$ is the square-free part of $C$. Let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let $\text{rad}(n)$ denote the radical of the positive integer $n$ (product of all prime divisors of $n$).

(II) $\text{rad}(h(-d)) | 6$ for any decomposition $C = d \cdot z^2$ as above.

(III) $\text{rad}(q_i + 1) | 2 \cdot 3 \cdot 5$ for all $i = 1, \ldots, k$.

In such cases we apply the method used in [4]. If we are able to determine all S-integral points (with $S$ is an explicit set of rational primes) on some associated elliptic curve, then we can completely solve such Diophantine equations. Conditions (I)-(III) above were suggested as a result of section 5 in [4].

In [11], all values of $C$ satisfying conditions (I)-(III) are determined (Lemma 2). Radicals of $C$ take exactly 41 values. Some of the equations $x^2 + C = y^n$ with $C$ listed in Lemma 2 were studied in the literature. These include the cases where $\text{rad}(C) \in \{5, 13, 17, 29, 41, 97, 2 \cdot 5, 2 \cdot 13, 5 \cdot 13, 2 \cdot 17, 5 \cdot 13, 2 \cdot 5 \cdot 13, 2 \cdot 5 \cdot 17, 2 \cdot 29, 2 \cdot 41\}$.

All solutions of the Diophantine equation (1) where found in [17] and [18] for $\text{rad}(C) = 10, 26$; in [11] for $\text{rad}(C) = 34, 58, 82$; in [4] for $\text{rad}(C) = 65$; in [22] for $\text{rad}(C) = 85$; and in [12], [13] for $\text{rad}(C) = 130, 170$.

\vspace{0.5cm}

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In [9], the authors gave the complete solutions \((n, a, b, x, y)\) of the Diophantine equation \(x^2 + 5^a \cdot 11^b = y^n\) when \(\gcd(x, y) = 1\), except for the case when \(x \cdot a \cdot b\) is odd.

In this paper, we obtain all solutions of the Diophantine equations
\[
x^2 + 5^a \cdot p^b = y^n \quad (p = 29, 41)
\]
in integers unknowns \(x, y, a, b, n\) under the conditions;
\[
x \geq 1, \ y > 1, \ n \geq 3, \ a \geq 0, \ b \geq 0 \quad \text{x and y are coprime.}
\]

We apply the method from [4]. For \(n = 3\) and \(n = 4\), the problem is reduced to finding all \(\{5, p\}\)-integral points on some elliptic curves. For \(n \geq 5\), we shall use the primitive divisors of Lucas sequences as in [6] to deduce that only cases \(n \in \{5, 7\}\) are possible. In these cases, we again reduce our problem to the computation of all \(\{5, p\}\)-integral points on some elliptic curves. The calculations were done using MAGMA, [5]. We now state the two main results of this paper:

**Theorem 1.1.** The only solutions of the equation
\[
x^2 + 5^a \cdot 29^b = y^n, \ x, y \geq 1, \ \gcd(x, y) = 1, \ n \geq 3, \ a, b \geq 0
\]
are
\[
(x, y, a, b) = (2, 9, 2, 1) \quad \text{when} \quad n = 3
\]
and
\[
(x, y, a, b) = (2, 3, 2, 1) \quad \text{when} \quad n = 6.
\]

**Theorem 1.2.** The only solutions of the equation
\[
x^2 + 5^a \cdot 41^b = y^n, \ x, y \geq 1, \ \gcd(x, y) = 1, \ n \geq 3, \ a, b \geq 0
\]
are
\[
(x, y, a, b) = (840, 29, 0, 2) \quad \text{when} \quad n = 4;
(x, y, a, b) = (38, 5, 0, 2) \quad \text{when} \quad n = 5
\]
and
\[
(x, y, a, b) = (278, 5, 0, 2) \quad \text{when} \quad n = 7.
\]

Note that when \(a = 0\), (3) becomes \(x^2 + 29^b = y^n\) and \(x^2 + 41^b = y^n\), respectively, all solutions of which are already known (see [11]), while when \(b = 0\), our equation becomes \(x^2 + 5^a = y^n\) and all solutions of which have been found in [2], [3] and [15]. Thus, from now on we shall assume that \(a \cdot b > 0\) in (2).

2. Preliminaries

We will determine all the primes \(p \equiv 1 \pmod{4}\) satisfying the condition (III). First we recall some results:

**Lemma 2.1.** ([11]) There are exactly eight primes \(p \equiv 1 \pmod{4}\) satisfying the condition (III): 5, 13, 17, 29, 41, 97, 449, 4801.

Now we are ready to determine all values of \(C\) satisfying (I)-(III).
Lemma 2.2. ([11]) (i) The prime power $p^n$ satisfies the conditions (I)-(III) iff $p \in \{5, 13, 17, 29, 41, 97\}$.
(ii) The number $C = 2^n \cdot p^n$ satisfies (I)-(III) iff $p \in \{5, 13, 17, 29, 41\}$.
(iii) The odd number $C = p^n \cdot 2^i$ with $p, q$ are different odd primes, satisfies (I)-(III) iff $p \cdot q \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 29, 5 \cdot 41, 13 \cdot 17, 13 \cdot 29, 13 \cdot 41, 17 \cdot 29, 17 \cdot 41, 17 \cdot 97, 29 \cdot 41\}$.
(iv) The number $C = 2^{n0} \cdot p^n \cdot q^i$ where $p, q$ are different odd primes satisfies (I)-(III) iff $p \cdot q \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 41, 13 \cdot 17, 13 \cdot 29, 13 \cdot 41, 17 \cdot 29, 17 \cdot 41, 17 \cdot 97, 29 \cdot 41\}$.
(v) The odd number $C = p^n \cdot 2^{i1} \cdot 3_{i1}^i$ with $p_1, p_2$ and $p_3$ are different odd primes satisfies (I)-(III) iff $p_1 \cdot p_2 \cdot p_3 \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 29, 5 \cdot 41, 13 \cdot 17, 13 \cdot 29, 13 \cdot 41, 17 \cdot 29, 17 \cdot 41, 17 \cdot 97, 29 \cdot 41\}$.
(vi) The number $C = 2^{n1} \cdot p_1^{i1} \cdot p_2^{i2} \cdot p_3^{i3}$ where $p_1, p_2$ and $p_3$ are different odd primes satisfies (I)-(III) iff $p_1 \cdot p_2 \cdot p_3 \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 29, 5 \cdot 41, 13 \cdot 17, 13 \cdot 29, 13 \cdot 41, 17 \cdot 29, 17 \cdot 41, 17 \cdot 97, 29 \cdot 41\}$.
(vii) The number $C$ with $\geq 4$ different odd prime factors satisfies (I)-(III) iff $C = 5^n \cdot 13^n \cdot 17^n \cdot 41^d$.

Let $\alpha, \beta$ be two algebraic integers. If $\alpha + \beta$ and $\alpha \cdot \beta$ are nonzero coprime integers and $\alpha/\beta$ is not a root of unity, then $(\alpha, \beta)$ is called a Lucas pair. Further, let $k = \alpha + \beta$ and $l = \alpha \cdot \beta$. Then we have
\[
\alpha = \frac{1}{2}(k + \sqrt{d}), \quad \beta = \frac{1}{2}(k - \sqrt{d}) \quad \text{with} \quad \lambda \in \{\pm 1\},
\]
where $d = k^2 - 4l$. We call $(k, l)$ the parameters of the Lucas pair $(\alpha, \beta)$. Two Lucas pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ are called equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \pm 1$. Given a Lucas pair $(\alpha, \beta)$, one defines the corresponding sequence of Lucas numbers by
\[
L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad n = 0, 1, 2, \ldots
\]
For two equivalent Lucas pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$, we have $L_n(\alpha_1, \beta_1) = \pm L_n(\alpha_2, \beta_2)$ for all $n \geq 0$.

A prime $r$ is called a primitive divisor of $L_n(\alpha, \beta)$, $(n > 1)$ if
\[
r \mid L_n(\alpha, \beta) \quad \text{and} \quad r \nmid d \cdot L_1(\alpha, \beta) \cdot \ldots \cdot L_{n-1}(\alpha, \beta).
\]

Lemma 2.3. ([8]) If $r$ is a primitive divisor of $L_n(\alpha, \beta)$, then
\[
r \equiv e \pmod{n}, \quad \text{where} \quad e = \left(\frac{-d}{n}\right).
\]

Now we give an important result of Bilu, Hanrot and Voutier [6] concerning the existence of primitive divisors of Lucas sequence:

Lemma 2.4. Let $L_n = L_n(\alpha, \beta)$ be a Lucas sequence. If $n \geq 5$ is a prime, then $L_n$ has a primitive divisor except for finitely many pairs $(\alpha, \beta)$ which are explicitly determined in Table 1 in [6].

Proof. Follows by Theorem 1.4 in [6] and Theorem 1 in [1].

3. The Case $n = 4$

We now consider the special case of $n = 4$. The situation is rather easy in this case:

Lemma 3.1. The equation (2) has no solution with $n = 4$ and $\alpha \cdot \beta > 0$.

Proof. Let $p \in \{29, 41\}$. Let us rewrite the equation $x^2 + 5^a \cdot p^b = y^4$ in the form $(x/z^2)^2 + A = (y/z)^4$ where $A$ is a 4th power-free positive integer, defined by $5^a \cdot p^b = A \cdot z^4$ for some integer $z$. Under these conditions, we can write, $A = 5^a \cdot p^b$ with $\alpha, \beta \in \{0, 1, 2, 3\}$ and we obtain the equation
\[
V^2 = U^4 - 5^a \cdot p^b
\]
with \( U = y/z, V = x/z^2 \). We now have to determine all \([5,p]\)-integral points on these 16 elliptic curves.

Recall that if \( S \) is a finite set of prime numbers, then an \([a,b]\)-integer is a rational number \( a/b \) with coprime integers \( a \) and \( b > 0 \), where the prime factors of \( b \) are in \( S \). We can always use MAGMA to determine the \([5,p]\)-integral points on the above elliptic curves (see [4], p. 176).

Now we give the results of our with MAGMA calculations:

(i) The only \([5,29]\)-integral point on \( V^2 = U^4 - 5^a \cdot 29^b \) is \((U,V,\alpha,\beta) = (1,0,0,0)\) with the conditions on \( x, y \) and the definition of \( U, V \) one can see that there is no solution for this equation.

(ii) The only \([5,41]\)-integral point on \( V^2 = U^4 - 5^a \cdot 41^b \) is \((U,V,\alpha,\beta) = (1,0,0,0),(29,840,0,2)\). Under the conditions on \( x, y \) the definition of \( U, V \) which are not convenient for us since they \( a = 0 \) or \( a = b = 0 \). This concludes the proof. \( \square \)

4. The Case \( n = 3 \)

Now we deal with the second separate case of \( n = 3 \):

Lemma 4.1. (i) The only solution of the equation (3) with \( n = 3 \) and \( ab > 0 \) is \((x,y,a,b) = (2,9,2,1)\). In particular, if \( n \geq 3 \) is a multiple of 3 and the Diophantine equation (2) has an integer valuation \((x,y,a,b)\), then \( n = 6 \). Furthermore when \( n = 6 \), the only solution \((x,y,a,b)\) is \((2,3,2,1)\).

(ii) The equation (4) has no solution with \( n = 3 \) and \( ab > 0 \).

Proof. Let \( p \in \{29,41\} \). Rewrite the equation \( x^2 + 5^a \cdot p^b = y^3 \) in the form \((x/z^3)^2 + A = (y/z)^3\), where \( A \) is a 6th power-free positive integer, defined by \( 5^a \cdot p^b = Az^6 \), with some integer \( z \). Of course, \( A = 5^a \cdot p^b \) with \( a, b \in \{0,1,2,3,4,5\} \) and we obtain the equations:

\[
V^2 = U^3 - 5^a \cdot p^b,
\]

with \( U = y/z^2, V = x/z^3 \). We now have to determine the \([5,p]\)-integral points on these 36 elliptic curves, and to do that, we use again MAGMA.

(i) The only \([5,29]\)-integral points on \( V^2 = U^3 - 5^a \cdot 29^b \) are \((U,V,\alpha,\beta) \in \{(1,0,0,0),(29,0,0,3),(5,10,2,0),(9,2,2,1),(29,58,2,2),(125,1390,2,2),(145,1740,2,2),(865,25440,2,2),(145,0,3,3)\}\). As the numbers \( x \) and \( y \) are coprime positive integers, the above solutions lead to only one solution of the original equation, which is \((x,y,a,b) = (2,9,2,1)\).

When \( n = 6 \), replace \( n \) by \( 3 \) and \( y \) by \( y^2 \) to get a solution of equation (3) with \( n = 3 \) where the value of \( y \) being a perfect square. We have only the possibility \((2,9,2,1)\) for \((x,y,a,b)\). Therefore, the only solution of equation (3) with \( n = 6 \) is \((2,3,2,1)\).

(ii) The only \([5,41]\)-integral points \((u,v,\alpha,\beta)\) on the curve \( V^2 = U^3 - 5^a \cdot 41^b \) are \((1,0,0,0),(41,0,0,3),(41,246,1,2),(5,10,2,0),(41,164,2,2),(5,0,3,0),(205,0,3,3),(125,950,4,2)\) and \((1025,32800,4,2)\) with the conditions on \( x, y \) and the definition of \( U, V \) one can easily see that none of these leads to a solution of the equation in (1) in the case \( n = 3 \). This is the required result. \( \square \)

5. The Case \( n \geq 5 \) is prime

Lemma 5.1. Equations (4) and (5) have no solution with \( n \geq 5 \) prime and \( a.b > 0 \).

Proof. Suppose that (1) holds with \( n \geq 5 \), prime. We first rewrite the Diophantine equation \( x^2 + 5^a \cdot p^b = y^n \) as \( x^2 + d \cdot z^3 = y^n \), where \( d \in \{1,5,p,5p\} \), \( p = 29,41 \), \( z = 5^a \cdot p^b \) and the relation between \( a \) and \( b \) with \( a \) and \( b \), respectively, is clear.

If in (4) and (5), \( y > 1 \) is taken as an even number, we obviously have that \( x \) is odd. Since for any odd integer \( t \), we have \( t^2 \equiv 1 \pmod{8} \) we get that \( 1 + d \equiv 0 \pmod{8} \) by reducing (4) and (5) modulo 8. This leads to \( d \equiv 7 \pmod{8} \) for \( d \in \{1,5,29,145\} \) or \( d \in \{1,5,41,205\} \) which gives a contradiction. Hence in what follows we may assume \( y > 1 \) is odd in (4) and (5) (and hence \( x \geq 1 \) is even).

We work with the field \( K = Q(\sqrt{-d}) \). Since \( x \) is even, both factors on the left hand side of the equation \((x+z \sqrt{-d})(x-z \sqrt{-d}) = y^n \) are relatively prime. Hence, the ideal \( x+z \sqrt{-d} \) is a \( q \)-th power of some element
in \(\mathbb{Q}_K\), for a prime \(q\). The cardinality of the group of units of \(\mathbb{Q}_K\) is 2 or 6, both coprime to \(q\). Furthermore \(\{1, (1 + \sqrt{-d})/2\}\) is always an integral base for \(\mathbb{Q}_K\). Thus, we can finally write the relations

\[ x + z \sqrt{-d} = q^v, \quad q = u + v \sqrt{-d} \tag{6} \]

where \(u, v \in \mathbb{Z}\).

Conjugating (7) and subtracting the two relations, we get

\[ 2 \sqrt{-d} \cdot 5^a \cdot p^b = q^v - \overline{q}^v. \tag{7} \]

5.1. The Diophantine equation \(x^2 + 5^a \cdot 29^b = y^n\)

Since \(n \geq 5, 29\) is primitive for \(L_n\) by Lemma 3 (\(n\) is prime). Thus, \(29 \equiv \pm 1 \pmod{n}\) and we conclude that the only possibilities are \(n = 7\) and \(d = 1\) or \(n = 5\) and \(d = 2\).

5.1.1. The Case \(n = 7\)

By means of (8) with \(n = 7\) and \(d = 1\), we obtain the relation

\[ v(7u^6 - 35u^4v^2 + 21u^2v^4 - v^6) = 5^a \cdot 29^b \tag{8} \]

Since \(u\) and \(v\) are coprime, we have the following possibilities:

(a) \(v = \pm 5^a \cdot 29^b\),  \quad (b) \(v = \pm 29^b\),  \quad (c) \(v = \pm 5^a\),  \quad (d) \(v = \pm 1\).

We need only look at the last two possibilities.

Case 1: \(v = \pm 5^a\).

In this case, equation (9) becomes

\[ 7u^6 - 35u^4v^2 + 21u^2v^4 - v^6 = \pm 29^b. \]

Dividing both sides by \(v^b\), we obtain

\[ 7U^3 - 35U^2 + 21U - 1 = D_1 \cdot V^2 \tag{9} \]

where \(U = u^2/v^2\),  \(V = 29^b/v^3\),  \(\beta_1 = [\beta/2]\),  \(D_1 = \pm 1, \pm 29\). In this case, as \(D_1 = \pm 1\), we have to find the \(\{5\}\)-integral points on the elliptic curves:

\[ 7U^3 - 35\gamma U^2 + 21U - \gamma = D_1 \cdot V^2, \quad \gamma = \pm 1. \tag{10} \]

We multiply both sides of (10) by \(7^2\) to obtain

\[ X^3 - 35\gamma X^2 + 147X - 49\gamma = Y^2, \]

where \((X, Y) = (7\gamma U, 7V)\) are \(\{5\}\)-integral points on the above elliptic curves.

Using MAGMA, we find \((X, Y) \in \{(1, 8), (58, -293)\}\) (hence \((U, V) \in \{(1/7, 8/7), (58/7, -293/7)\}\) for \(\gamma = 1\). These do not lead to any solutions of the equation (4), either.

Consider the case \(D_1 = \pm 29\). The unique \(\{5\}\)-integral point \((2349, -87464)\) on the elliptic curve

\[ X^3 - 35 \cdot 29X^2 + 21 \cdot 7 \cdot 29^2 X - 7^2 \cdot 29^3 = Y^2 \]

does not lead us to a solution of (4). With MAGMA, we find the following \(\{5\}\)-integral points \((-812, 5887), (-377, 6728), (-5, -776), (91, 4648), (1015, 47096), (-340103561/390625, 420852069512/244140625)\) on the elliptic curve

\[ X^3 + 35 \cdot 29X^2 + 21 \cdot 7 \cdot 29^2 X + 7^2 \cdot 29^3 = Y^2. \]

Only the point \((-812, 5887)\) leads to the solution \((x, y, a, b) = (278, 5, 0, 2)\) of our original equation (4), which is not convenient for us since it has \(a = 0\).
Case 2. \( v = \pm 1 \).

We have to find the integral points on

\[
7U^3 - 35U^2 + 21U - 1 = D_1 \cdot V^2
\]

where \( D_1 = \pm 1, \pm 5, \pm 29, \pm 145 \).

The cases \( D_1 = \pm 5 \) lead to no solutions of our original equation (4).

Consider the case \( D_1 = \pm 29 \) where treated above.

These do not lead to any solutions of equation (4).

5.1.2. Case \( n = 5 \)

Using (8) with \( n = 5, d = 2 \), we obtain the relation

\[
v(5u^4 - 20u^2v^2 + 4v^4) = 5^a \cdot 29^b.
\]

As in the case \( n = 7 \), we only need to check the values \( v = \pm 5^a, v = \pm 1 \).

In the first case, the Diophantine equation (12) is \( 5u^4 - 20u^2v^2 + 4v^4 = \pm 29^b \). Dividing both sides by \( v^4 \), we obtain

\[
5U^4 - 20U^2 + 4 = D_1V^2,
\]

where \( U = u/v, V = 29^b/v^2, \beta_1 = [\beta/2] \) and \( D_1 = \pm 1, \pm 29 \). Using MAGMA, we find three \([5]\)-integral points \((0, 2), (2, 2), (-2, 2)\) on the curve (13) with \( D_1 = \pm 1 \), and no other points in the remaining cases. These points do not lead to any solutions of our original equation (1).

In the second case, the Diophantine equation (12) is \( 5U^4 - 20U^2 + 4 = \pm 5^a \cdot 29^b \). We need to find the integral points on the curves \( 5U^4 - 20U^2 + 4 = D_1V^2 \), for \( D_1 = \pm 1, \pm 5, \pm 29, \pm 145 \). MAGMA finds three solutions \((0, 2), (2, 2), (-2, 2)\). None of points leads to any solutions of equation (2).

5.2. The Diophantine equation \( x^2 + 5^a \cdot 41^b = y^4 \)

Since \( n \geq 5 \), by using Lemma 3, 41 is primitive for \( L_n \). Thus, \( 41 \equiv \pm 1 \pmod{n} \) and we now see that the only possibilities are \( n = 5 \) and \( d = 1 \) or \( n = 5 \) and \( d = 2 \).

Using (8) with \( n = 5, d = 2 \), we obtain

\[
v(5u^4 - 20u^2v^2 + 4v^4) = 5^a \cdot 41^b.
\]

Therefore we only need to check \( v = \pm 5^a, v = \pm 1 \).

In the first case the Diophantine equation is \( v(5u^4 - 20u^2v^2 + 4v^4) = \pm 41^b \). Dividing both sides by \( v^4 \), we obtain

\[
5U^4 - 20U^2 + 4 = D_1V^2,
\]
where $U = u/v$, $V = 41^\beta /v^2$, $\beta_1 = [\beta/2]$ and $D_1 = \pm 1, \pm 41$. Using MAGMA, we find three $[5]$-integral points $(0, 2), (2, 2), (2, -2)$ on (14) with $D_1 = 1$, and none in the remaining cases. These points do not lead to any solutions of equation (4).

In the second case the Diophantine equation is $v(5u^4 - 20u^2v^2 + 4v^4) = 5^a \cdot 41^\beta$. We need to find integral points on the curves $v(5U^4 - 20U^2 + 4) = D_1V^2$, for $D_1 = 1, \pm 5, \pm 41, \pm 205$. MAGMA finds three solutions $(0, 2), (2, 2), (2, -2)$. These points do not lead either to any solutions of our original equation (4).

Using (8) with $n = 5$, $d = 1$, we obtain the relation

$$v(5u^4 - 20u^2v^2 + 4v^4) = 5^a \cdot 41^\beta.$$  

In case $v = \pm 5^a$, we obtain $5u^4 - 10u^2v^2 + v^4 = 41^\beta$. MAGMA then finds the $[5]$-integral points on

$$5U^4 - 10U^2 + 1 = D_1V^2$$

which are $(0, 1)$ if $D_1 = 1, (1, -2), (-1, -2)$ if $D_1 = -1$, and finally $(2, 1), (-2, 1)$ if $D_1 = 41$. The point $(2, 1)$ gives a new solution $(x, y, a, b) = (38, 5, 0, 2)$ of the equation (4) which is not convenient for us since it has $a = 0$.

In case $v = \pm 1$, we obtain $5u^4 - 10u^2v^2 + v^4 = 41^\beta$ MAGMA finds the integral points on

$$5U^4 - 10U^2 + 1 = D_1V^2$$

These points are $(2, 41), (-2, 41)$ for $D_1 = 41$. The point $(2, 1)$ gives the solution $(x, y) = (38, 5)$ of (4) again. This solution is not convenient for us since it has $a = 0$. This completes the proof of lemma. □

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References


