Symmetric Difference Between Pseudo B-Fredholm Spectrum and Spectra Originated from Fredholm Theory

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Abstract. In this paper, we continue the study of the pseudo B-Fredholm operators of Boasso, and the pseudo B-Weyl spectrum of Zariouh and Zguitti; in particular we find that the pseudo B-Weyl spectrum is empty whenever the pseudo B-Fredholm spectrum is, and look at the symmetric differences between the pseudo B-Weyl and other spectra.

1. Introduction and Preliminaries

Throughout, $X$ denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on $X$. Berkani \cite{6} has defined $T \in \mathcal{B}(X)$ to be a “B-Fredholm operator” if there is an integer $n \geq 0$ for which the range $R(T^n) = T^n(X)$ is closed, while the restriction $T_n$ to $R(T^n)$ is Fredholm in the usual sense, and then $T$ to be “B-Weyl” if also the Fredholm operator $T_n$ has index zero. The “B-Fredholm” and “B-Weyl” spectrum of $T \in \mathcal{B}(X)$ are now defined in the obvious way, as the Fredholm and Weyl spectrum of $T_n$. Berkani \cite{4} also showed that $T$ is B-Fredholm if it has a direct sum decomposition $T = T_1 \oplus T_0$ with $T_1$ Fredholm and $T_0$ nilpotent; further \cite{5} this decomposition respects the index: $T$ is B-Weyl iff $T_1$ is Weyl. Boasso \cite{7} has used the decomposition to extend the Berkani concept to “pseudo B” Fredholm and Weyl operators, $T = T_1 \oplus T_0 \in \mathcal{B}(X)$ for which $T_1$ is Fredholm, respectively Weyl, while $T_0$ is only quasinilpotent, see also \cite{28}. The pseudo B-Fredholm and pseudo B-Weyl spectrum are defined by

\[ \sigma_{\text{pBF}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Fredholm} \} \]

\[ \sigma_{\text{pBW}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Weyl} \} \]

For $T \in \mathcal{B}(X)$ we denote by $T^*$, $R(T)$, $N(T)$, $\sigma(T)$, respectively the adjoint, the range, the null space and the spectrum of $T$. Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of $\lambda_0$, the only analytic function $f : U \rightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator $T$ is said to
have the SVEP if $T$ has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence $T$ and $T^*$ have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum.

An operator $T \in \mathcal{B}(X)$ is said to be semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^{\infty}(T) = \bigcap_{n \geq 0} R(T^n)$. The corresponding spectrum is the semi-regular spectrum $\sigma_{sr}(T)$ defined by $\sigma_{sr}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I$ is not semi-regular $\}$. see [1].

In the other hand, recall that an operator $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, ( GKD for short ), if there exists two closed $T$-invariant subspaces $X_1$, $X_2$ such that $X = X_1 \oplus X_2$, $T_1 = T|_{X_1}$ is semi-regular and $T_0 = T|_{X_2}$ is quasi-nilpotent, in this case $T$ is said a pseudo Fredholm operator. If we assume in the definition above that $T_0 = T|_{X_2}$ is nilpotent, then $T$ is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [18, 21] for more information about generalized Kato decomposition.

Recall that $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

1. $R(T^d) \cap N(T) = R(T^d) \cap N(T^d)$ for all $n \geq d$;
2. $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are closed in $X$.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree $d$. Note that semi-regular operators are quasi-Fredholm of degree 0 and by results of Labrousse [18], in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators. For every bounded operator $T \in \mathcal{B}(X)$, let us define the generalized Kato spectrum as follows :

$$\sigma_{GK}(T) := \{ \lambda \in \mathbb{C} : T - \lambda I$ does not admit a generalized Kato decomposition$\}.$$

It is know that $\sigma_{GK}(T)$ is always a compact subsets of the complex plane contained in the spectrum $\sigma(T)$ of $T$ [13, Corollary 2.3]. Note that $\sigma_{GK}(T)$ is not necessarily non-empty, see [13, 14] for more information about $\sigma_{GK}(T)$.

In [4], Berkani proved that every B-Fredholm operator in Hilbert space is a quasi-Fredholm operator. The proof is based on the decomposition of quasi-Fredholm operators of Labrousse [18] which was proved only for Hilbert-spaces operators. This gap was subsequently filled by Müller in [26, Theorem 7] and the result holds in more general setting of Banach space.

As a continuation of [7] and [28], in section two, we prove that every pseudo B-Fredholm operator is a pseudo-Fredholm operator. Also, we study the relationships between the class of pseudo B-Fredholm and other class of operator. we characterize when the pseudo B-Fredholm spectrum is empty or most countable. In section tree, we study the components of the complement of the pseudo B-Fredholm spectrum $\sigma_{pBF}(T)$, to obtain a classification of the components by using the constancy of the subspaces quasi-nilpotent part and analytic core. In the last section, we show that the symmetric difference $\sigma_{sr}(T) \Delta \sigma_{pBF}(T)$ is at most countable.

2. The Class of Pseudo B-Fredholm Operators

In the following theorem we prove that every pseudo B-Fredholm operator is pseudo Fredholm.

**Theorem 2.1.** Let $T \in \mathcal{B}(X)$. If $T$ is pseudo B-Fredholm, then $T$ is pseudo Fredholm.

**Proof.** Let $T \in \mathcal{B}(X)$. If $T$ is pseudo B-Fredholm operator, then there exists two closed subsets $M$ and $N$ of $X$ such that $X = M \oplus N$ and $T = T_1 \oplus T_2$ with $T_1 = T_{1M}$ is a Fredholm operator and $T_2 = T_{1N}$ is quasi-nilpotent. Since $T_1$ is Fredholm then $T$ admits a Kato decomposition, hence there exists $M'$, $M''$ two closed subsets of $M$ such that $M = M' \oplus M''$, $T_1 = T'_1 \oplus T''_1$ with $T'_1 = T'_{1M'}$ is a semi-regular operator and $T''_1 = T'_{1M''}$ is nilpotent. Then $X = M' \oplus M'' \oplus N$, and $T = S \oplus R$ where $S = T'_1$ is a semi-regular operator and $R = T''_1 \oplus T_2$ is a quasi-nilpotent operator, hence $T$ is a pseudo Fredholm operator. 

The following example (Müller [25]) shows that the pseudo B-Fredholm operators form a proper subclass of the pseudo Fredholm operators.
Example 2.2. Let $H$ be the Hilbert space with an orthonormal basis $(e_{ij})$, where $i$ and $j$ are integers such that $ij \leq 0$. Define operator $T \in \mathcal{B}(H)$ by:

$$T e_{ij} = \begin{cases} 
0 & \text{if } i = 0, j > 0 \\
e_{i+1,j} & \text{Otherwise}
\end{cases}$$

We have $N(T) = \bigcup_{j \geq 0} \{e_{0,j}\} \subset \mathbb{R}^\infty(T)$ and $R(T)$ is closed, then $T$ is a semi-regular operator but $T$ is not a Fredholm operator, since $\dim N(T) = \infty$.

Let $Q$ a quasinilpotent operator in $H$ which is not nilpotent and no commute with $T$, then $S = T \oplus Q$ is a pseudo Fredholm operator but is not pseudo $B$-Fredholm operator, hence the class of pseudo $B$-Fredholm operator is a proper subclass of pseudo Fredholm operator.

Remark 2.3. In [28, Remark 2.5] and [8, Proposition 1.2], if $T$ is a bilateral shift on $l^2$, then:

1. $T$ is pseudo $B$-Weyl if and only if $T$ is Weyl or $T$ is quasi-nilpotent operator.
2. $T$ is pseudo Fredholm if and only if $T$ is semi-regular or $T$ is quasi-nilpotent operator.

By the same argument we can prove:

1. $T$ is pseudo $B$-Fredholm if and only if $T$ is Fredholm or $T$ is quasi-nilpotent operator.
2. $T$ is generalized Drazin if and only if $T$ is invertible or $T$ is quasi-nilpotent operator.

Corollary 2.4. Let $T \in \mathcal{B}(X)$. Then

$$\sigma_{\text{GK}}(T) \subset \sigma_{\text{pBF}}(T) \subset \sigma_{\text{pBW}}(T)$$

Lemma 2.5. [22] Let $T \in \mathcal{B}(X)$ and let $G$ a connected component of $\rho_{\infty}(T) = C \setminus \sigma_{\infty}(T)$. Then

$$G \setminus \sigma(T) \neq \emptyset \implies G \cap \sigma(T) = \emptyset$$

Lemma 2.6. [8] Let $T \in \mathcal{B}(X)$.

$$\sigma_{\infty}(T) \setminus \sigma_{\text{GK}}(T) \text{ is at most countable}$$

Since $\sigma_{\infty}(T) \setminus \sigma_{\text{pBF}}(T) \subset \sigma_{\infty}(T) \setminus \sigma_{\text{GK}}(T)$, we can easily obtain that:

Corollary 2.7. Let $T \in \mathcal{B}(X)$.

$$\sigma_{\infty}(T) \setminus \sigma_{\text{pBF}}(T) \text{ is at most countable}.$$

Proposition 2.8. Let $T \in \mathcal{B}(X)$. Then the following statements are equivalent:

1. $\sigma_{\text{pBF}}(T)$ is at most countable
2. $\sigma_{\text{pBW}}(T)$ is at most countable
3. $\sigma(T)$ is at most countable

Proof. 1) $\implies$ 3) Suppose that $\sigma_{\text{pBF}}(T)$ is at most countable then $\rho_{\text{pBF}}(T)$ is connexe, by corollary 2.7 $\rho_{\text{pBF}}(T) \setminus \rho_{\infty}(T)$ is at most countable. Hence $\rho_{\infty}(T) \cap \rho_{\text{pBF}}(T) = \rho_{\text{pBF}}(T) \setminus (\rho_{\text{pBF}}(T) \setminus \rho_{\infty}(T))$ is connexe. By lemma 2.5 $\sigma(T) = \sigma_{\text{pBF}}(T) \cup (\rho_{\text{pBF}}(T) \setminus \rho_{\infty}(T))$ is at most countable.

3) $\implies$ 1) Obvious.

2) $\implies$ 3) If $\sigma_{\text{pBW}}(T)$ is at most countable then $\rho_{\text{pBW}}(T)$ is connexe, since every pseudo $B$-Weyl operator is a pseudo $B$-Fredholm operator by corollary 2.7 $\rho_{\text{pBW}}(T) \setminus \rho_{\infty}(T)$ is at most countable. Hence $\rho_{\infty}(T) \cap \rho_{\text{pBW}}(T) = \rho_{\text{pBW}}(T) \setminus (\rho_{\text{pBW}}(T) \setminus \rho_{\infty}(T))$ is connexe. By lemma 2.5 $\sigma(T) = \sigma_{\infty}(T) \cup \rho_{\text{pBW}}(T)$. Therefore $\sigma(T) = \sigma_{\text{pBW}}(T) \cup \rho_{\text{pBW}}(T) \setminus \rho_{\infty}(T))$ is at most countable.

3) $\implies$ 2) Obvious.

Corollary 2.9. Let $T \in \mathcal{B}(X)$, if $\sigma_{\text{GK}}(T)$ is at most countable. Then:

$T$ is a spectral operator if and only if $T$ is similar to a paranormal operator.
Proof. See [23, Theorem 2.4 and Corollary 2.5].

Let $T \in \mathcal{B}(X)$. The operator range topology on $R(T)$ is the topology induced by the norm $\|\cdot\|_T$ defined by $\|y\|_T := \inf_{x \in X} \{\|x\| : y = Tx\}$. For a detailed discussion of operator ranges and their topology we refer the reader to [11].

$T$ is said to have uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for $n \geq d$. In addition, $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then $T$ is said to have topological uniform descent (TUD for brevity) for $n \geq d$. The topological uniform descent spectrum:

$$\sigma_{ud}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ does not have TUD}\}$$

Let $T \in \mathcal{B}(X)$, the ascent of $T$ is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such $p$ does not exists we let $a(T) = \infty$. Analogously the descent of $T$ is $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such $q$ does not exists we let $d(T) = \infty$ [20]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) = N(T^q)$ where $p = a(T) = d(T)$. The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by:

$$\sigma_{ds}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite descent}\}$$

$$\sigma_{as}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite ascent}\}$$

On the other hand, a bounded operator $T \in \mathcal{B}(X)$ is said to be a Drazin invertible if there exists a positive integer $k$ and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \quad T^{k+1}S = T^k \quad \text{and} \quad S^2T = S.$$ 

This is also equivalent to the fact that $T = T_1 \oplus T_2$ where $T_1$ is invertible and $T_2$ is nilpotent. Recall that an operator $T$ is Drazin invertible if it has a finite ascent and descent. The concept of Drazin invertible operators has been generalized by Koliha [17]. In fact $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc^c(T)$ the set of all points of accumulation of $\sigma(T)$, which is also equivalent to the fact that $T = T_1 \oplus T_2$ where $T_1$ is invertible and $T_2$ is quasinilpotent. The Drazin and generalized Drazin spectra of $T \in \mathcal{B}(X)$ are defined by:

$$\sigma_D(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ is not Drazin invertible}\}$$

$$\sigma_{GD}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ is not generalized Drazin}\}$$

We denote by, $\sigma_{des}(T)$, $\sigma_{GD}(T)$ and $\sigma_{IGD}(T)$ respectively the essential descent, right generalized Drazin and left generalized Drazin spectra of $T$. According to corollary 2.8, [8, Theorem 3.3] and [16, corollary 3.4], we have the following:

**Corollary 2.10.** Let $T \in \mathcal{B}(X)$. Then the following statements are equivalent

1. $\sigma(T)$ is at most countable;
2. $\sigma_{paf}(T)$ is at most countable;
3. $\sigma_{paq}(T)$ is at most countable;
4. $\sigma_{ud}(T)$ is at most countable;
5. $\sigma_{GD}(T)$ is at most countable;
6. $\sigma_{IGD}(T)$ is at most countable;
7. $\sigma_{DE}(T)$ is at most countable;
8. $\sigma_{DE}(T)$ is at most countable;
9. $\sigma_{DE}(T)$ is at most countable;
10. $\sigma_{DE}(T)$ is at most countable;
11. $\sigma_{DE}(T)$ is at most countable;
12. $\sigma_{DE}(T)$ is at most countable;
13. $\sigma_{DE}(T)$ is at most countable;
14. $\sigma_{DE}(T)$ is at most countable;
In [15], they showed that an operator with TUD for \( n \geq d \), \( K(T) = R^\infty(T) \) and \( \overline{H_0(V)} = \overline{N^\infty(V)} \), where \( K(T) \) and \( H_0(T) \) are the analytic core and the quasinilpotent part of \( T \). For a pseudo B-Fredholm operator, these properties do not necessarily hold. Indeed: let \( X \) be the Banach space of continuous functions on \([0,1]\), denoted by \( C([0,1]) \), provided with the infinity norm. We define by \( V \), the Volterra operator, \( X \) by:

\[
Vf(x) := \int_0^x f(x) \, dx.
\]

\( V \) is injective and quasi-nilpotent. In addition, \( N^\infty(V) = \{0\} \), \( K(V) = \{0\} \) and we have \( R^\infty(V) = \{f \in C^\infty[0,1] : f^{(n)}(0) = 0, \ n \in \mathbb{N}\} \), thus \( R^\infty(V) \) is not closed. Hence:

1. \( K(V) \neq R^\infty(V) \)
2. \( H_0(V) \neq N^\infty(V) \)
3. \( R(V) \) is not closed.

**Theorem 2.11.** There exists a pseudo B-Fredholm operator \( T \) such that:

1. \( K(T) \neq R^\infty(T) \)
2. \( H_0(T) \neq N^\infty(T) \)
3. \( R(T) \) is not closed.

**Proposition 2.12.** Let \( T \in \mathcal{B}(X) \). Then the following statements are equivalent

1. \( \sigma_{pBF}(T) \) is empty
2. \( \sigma_{pBW}(T) \) is empty
3. \( \sigma_{GK}(T) \) is empty
4. \( \sigma(T) \) is finite

**Proof.** 3) \( \iff \) 4) see [8, Theorem 3.3].

1) \( \implies \) 4) If \( \sigma_{pBF}(T) \) is empty then \( \sigma(T) = \rho_{pBF}(T) \setminus \rho_\omega(T) \). By corollary 2.7 \( \rho_{pBF}(T) \setminus \rho_\omega(T) \) is at most countable and this set is bounded, hence it is finite.

4) \( \iff \) 1) Suppose that \( \sigma(T) \) is finite then every \( \lambda_0 \in \sigma(T) \) is isolated, then \( X = H_0(T - \lambda_0) \oplus K(T - \lambda_0) \), [27, Theorem 4] \( (T - \lambda_0)_{H_0(T - \lambda_0)} \) is quasi-nilpotent and \( (T - \lambda_0)_{K(T - \lambda_0)} \) is surjective, hence \( (T - \lambda_0)_{K(T - \lambda_0)} \) is Fredholm. Indeed, \( \lambda_0 \) is an isolated point, then \( T \) has the SVEP at \( \lambda_0 \), hence \( (T - \lambda_0)_{K(T - \lambda_0)} \) has the SVEP at 0 and it is surjective by [1, corollary 2.24] \( (T - \lambda_0)_{K(T - \lambda_0)} \) is bijective. Thus every \( \lambda_0 \in \sigma(T) \), \( T - \lambda_0I \) is pseudo B-Fredholm, so \( \sigma_{pBF}(T) \) is empty.

2) \( \iff \) 4) similar to 1) \( \iff \) 4).

A bounded operator \( T \in \mathcal{B}(X) \) is said to be a Riesz operator if \( T - \lambda I \) is a Fredholm operator for every \( \lambda \in \mathbb{C} \setminus \{0\} \).

**Corollary 2.13.** Let \( T \in \mathcal{B}(X) \) a Riesz operator, then the following statements are equivalent

1. \( \sigma_{pBF}(T) \) is empty,
2. \( \sigma_{pBW}(T) \) is empty,
3. \( \sigma_{GK}(T) \) is empty,
4. \( \sigma(T) \) is finite,
5. \( K(T) \) is closed,
6. \( K(T^*) \) is closed,
7. \( K(T) \) is finite-dimensional,
8. \( K(T - \lambda) \) is closed for all \( \lambda \in \mathbb{C} \),
9. \( \text{codim} H_0(T) < \infty \),
10. \( \text{codim} H_0(T^*) < \infty \),
11. \( T = Q + F \), with \( Q, F \in \mathcal{B}(X) \), \( QF = FQ = 0 \), \( \sigma(Q) = \{0\} \) and \( F \) is a finite rank operator.
implies that \( T \) is a nilpotent operator. Thus \( T \) has finite descent then Proposition 2.14. Let \( T \in \mathcal{B}(X) \) with finite descent. Then \( T \) is a pseudo B-Fredholm if and only if \( T \) is a B-Fredholm operator.

**Proposition 2.14.** Let \( T \in \mathcal{B}(X) \) with finite descent. Then \( T \) is a pseudo B-Fredholm if and only if \( T \) is a B-Fredholm operator.

**Proof.** Obviously if \( T \) is B-Fredholm then \( T \) is pseudo B-Fredholm.

If \( T \) is a pseudo B-Fredholm then \( T = T_1 \oplus T_2 \) with \( T_1 \) is Fredholm operator and \( T_2 \) is quasi-nilpotent. Since \( T \) has finite descent then \( T_1 \) and \( T_2 \) have finite descent. We have \( T_2 \) is quasi-nilpotent with finite descent implies that is a nilpotent operator. Thus \( T \) is a B-Fredholm operator. \( \square \)

3. Classification of the Components of Pseudo B-Fredholm Resolvent

We begin this section by the following lemmas which will be needed in the sequel.

**Lemma 3.1.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm, then there exists \( \varepsilon > 0 \) such that for all \( |\lambda| < \varepsilon \), we have:

1. \( K(T - \lambda) + H_0(T - \lambda) = K(T) + H_0(T) \).
2. \( K(T - \lambda) \cap H_0(T - \lambda) = K(T) \cap H_0(T) \).

**Proof.** By Theorem 2.1, \( T \) is a pseudo Fredholm operator, hence we conclude by [8, Theorem 4.2] the result. \( \square \)

The pseudo B-Fredholm resolvent set is defined as \( \rho_{pBF}(T) = \mathcal{C} \setminus \sigma_{pBF}(T) \).

**Corollary 3.2.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm operator, then the mappings

\[ \lambda \rightarrow K(T - \lambda) + H_0(T - \lambda), \lambda \rightarrow K(T - \lambda) \cap H_0(T - \lambda) \]

are constant on the components of \( \rho_{pBF}(T) \).

We denote by \( \sigma_{ap}(T) \) and \( \sigma_{ap}(T) \) respectively the approximate point spectrum and the surjectivity spectrum of \( T \).

**Lemma 3.3.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm operator. Then the following statements are equivalent:

1. \( T \) has the SVEP at 0,
2. \( \sigma_{ap}(T) \) does not cluster at 0.

**Proof.** Without loss of generality, we can assume that \( \lambda_0 = 0 \).

1) \( \Rightarrow \) 2) Suppose that \( T \) is a pseudo B-Fredholm operator, then there exists two closed \( T \)-invariant subspaces \( X_1, X_2 \subset X \) such that \( X = X_1 \oplus X_2, T_{X_1} \) is Fredholm, \( T_{X_2} \) is quasi-nilpotent and \( T = T_{X_1} \oplus T_{X_2} \). Since \( T_{X_1} \) is Fredholm, then \( T_{X_1} \) is of Kato type by [2, Theorem 2.2] there exists a constant \( \varepsilon > 0 \) such that for all \( \lambda \in D^*(0, \varepsilon), \lambda I - T \) is bounded below. Since \( T_{X_2} \) is quasi-nilpotent, \( \lambda I - T \) is bounded below for all \( \lambda \neq 0 \). Hence \( \lambda I - T \) is bounded below for all \( \lambda \in D^*(0, \varepsilon) \). Therefore \( \sigma_{ap}(T) \) does not cluster at \( \lambda_0 \). \( \square \)

By duality we have :

**Lemma 3.4.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm operator. Then the following statements are equivalent:

1. \( T \) has the SVEP at 0,
2. \( \sigma_{ap}(T) \) does not cluster at 0.

**Theorem 3.5.** Let \( T \in \mathcal{B}(X) \) and \( \Omega \) a component of \( \rho_{pBF}(T) \). Then the following alternative holds:

1. \( T \) has the SVEP for every point of \( \Omega \). In this case, \( \sigma_{ap}(T) \) does not have limit points in \( \Omega \), every point of \( \Omega \) is not an eigenvalue of \( T \) except a subset of \( \Omega \) which consists of at most countably many isolated points.
2. \( T \) has the SVEP at no point of \( \Omega \). In this case, every point of \( \Omega \) is an eigenvalue of \( T \).
Proof. 1) Assume that $T$ has the SVEP at $\lambda_0 \in \Omega$. By [1, Theorem 3.14] we have $K(T - \lambda_0) \cap H_0(T - \lambda_0) = K(T - \lambda_0) \cap \overline{H_0}(T - \lambda_0) = \{0\}$. According to corollary 3.2, we have $K(T - \lambda_0) \cap \overline{H_0}(T - \lambda_0) = K(T - \lambda) \cap \overline{H_0}(T - \lambda) = \{0\}$ for all $\lambda \in \Omega$. Hence $K(T - \lambda) \cap \overline{H_0}(T - \lambda) = \{0\}$ and therefore $T$ has the SVEP at every $\lambda \in \Omega$ [1, Theorem 3.14]. By Lemma 3.3, $\sigma_{\rho_T}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of $\Omega$ is not an eigenvalue of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.

2) Suppose that $T$ has the SVEP at no point of $\Omega$. From [1, Theorem 2.22], we have $N(T - \lambda) \neq \{0\}$, for all $\lambda \in \Omega$, hence every point of $\Omega$ is an eigenvalue of $T$. □

Theorem 3.6. Let $T \in \mathcal{B}(X)$ and $\Omega$ a component of $\rho_{pBF}(T)$. Then the following alternative holds:

1. $T^*$ has the SVEP for every point of $\Omega$. In this case, $\sigma_{\text{in}}(T)$ does not have limit points in $\Omega$, every point of $\Omega$ is not a deficiency value of $T$ except a subset of $\Omega$ which consists of at most countably many points.

2. $T^*$ has the SVEP at no point of $\Omega$. In this case, every point of $\Omega$ is a deficiency value of $T$.

Proof. 1) Assume that $T^*$ has the SVEP at $\lambda_0 \in \Omega$, by [1, Theorem 3.15] we have $K(T - \lambda_0) + H_0(T - \lambda_0) = X$. According to corollary 3.2, we have $K(T - \lambda_0) + H_0(T - \lambda_0) = K(T - \lambda) + H_0(T - \lambda) = X$ for all $\lambda \in \Omega$. Hence $K(T - \lambda) + H_0(T - \lambda) = X$ and therefore $T$ has the SVEP at every $\lambda \in \Omega$ [1, Theorem 3.15]. By lemma 3.4, $\sigma_{\text{in}}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of $\Omega$ is not a deficiency value of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.

2) Suppose that $T^*$ has the SVEP at no point of $\Omega$. Assume that there exists a $\lambda_0 \in \Omega$ such that $T - \lambda_0$ is surjective, then $T^* - \lambda_0$ is injective this implies that $T^*$ has the SVEP at $\lambda_0$. Contradiction and hence every point of $\Omega$ is a deficiency value of $T$. □

Remark 3.7. We have $\sigma_{pBF}(\cdot) \subset \sigma_{\text{gD}}(\cdot)$, this inclusion is proper. Indeed: Consider the operator $T$ defined in $l^2(\mathbb{N})$ by $T(x_1, x_2, ...), T^*(x_1, x_2, ...) = (x_2, x_3, ...)$. Let $S = T \oplus T^*$. Then $\sigma_{\text{gD}}(S) = \{ \lambda \in \mathbb{C};|\lambda| \leq 1 \}$ and we have $0 \notin \sigma_{pBF}(S)$. This shows that the inclusion $\sigma_{pBF}(S) \subset \sigma_{\text{gD}}(S)$ is proper.

Next we obtain a condition on an operator such that its pseudo B-Fredholm spectrum coincide with the generalized Drazin spectrum.

Theorem 3.8. Suppose that $T \in \mathcal{B}(X)$ and $\rho_{pBF}(T)$ has only one component. Then

$$\sigma_{pBF}(T) = \sigma_{\text{gD}}(T)$$

Proof. $\rho_{pBF}(T)$ has only one component, then $\rho_{pBF}(T)$ is the unique component. Since $T$ has the SVEP on $\rho(T) \subset \rho_{pBF}(T)$. By Theorem 3.5, $T$ has the SVEP on $\rho_{pBF}(T)$. Similar $T^*$ also has the SVEP on $\rho_{pBF}(T)$ by Theorem 3.6. (This since $\rho(T^*) = \rho(T) \subset \rho_{pBF}(T)$). From Lemma 3.3 and Lemma 3.4, $\sigma(T)$ does not cluster at any $\lambda \in \rho_{pBF}(T)$. Therefore $\rho_{pBF}(T) \subset \text{iso}(T) \cup \rho(T) = \rho_{\text{gD}}(T)$, hence $\rho_{pBF}(T) = \rho_{\text{gD}}(T)$. □

4. Symmetric Difference for Pseudo B-Fredholm Spectrum

Let in the following we give symmetric difference between $\sigma_{pBF}(T)$ and other parts of the spectrum. Denoted by $\rho_{FK}(T) = \{ \lambda \in \mathbb{C}; K(T - \lambda) \text{ is not closed} \}$, $\sigma_{FK}(T) = \mathbb{C} \setminus \rho_{FK}(T)$ and $\rho_{cr}(T) = \{ \lambda \in \mathbb{C}; R(T - \lambda) \text{ is closed} \}$, $\sigma_{cr}(T) = \mathbb{C} \setminus \rho_{cr}(T)$ the Goldberg spectrum. Most of the classes of operators, for example, in Fredholm theory, require that the operators have closed ranges. Thus, it is natural to consider the closed-range spectrum or Goldberg spectrum of an operator.

Proposition 4.1. If $\lambda \in \sigma(T)$ is non-isolated point then $\lambda \in \sigma_{pBF}(T)$, where $* \in \{FK, cr\}$.

Proof. Let $\lambda \in \sigma(T)$ an isolated point. Suppose that $T - \lambda$ is a pseudo B-Fredholm, by Lemma 2.6 there exists a constant $\epsilon > 0$ such that for all $\mu \in D^*(\lambda, \epsilon)$, $\mu - T$ is semi-regular. Then $R(T - \mu)$ and $K(T - \mu)$ are closed for all $\mu \in D^*(\lambda, \epsilon)$, then $\lambda$ is an isolated point of $\sigma(T)$, contradiction. □
Corollary 4.2. \( \sigma_\ast(T) \setminus \sigma_{pBF}(T) \) is at most countable, where \( \ast \in \{ K, cr \} \).

Proposition 4.3. Let \( T \in B(X) \) such that \( \sigma_\ast(T) = \sigma(T) \) and every \( \lambda \) is non-isolated in \( \sigma(T) \). Then
\[
\sigma(T) = \sigma_{cr}(T) = \sigma_{pBF}(T) = \sigma_{pBV}(T) = \sigma_{se}(T) = \sigma_{ap}(T)
\]

Proof. Since every \( \lambda \in \sigma(T) = \sigma_{cr}(T) \) is non-isolated then by Proposition 4.1, we have \( \sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{pBF}(T) \subseteq \sigma_{pBV}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T) \), and since \( \sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T) \), we deduce the statement of the theorem. \( \blacksquare \)

Proposition 4.4. Let \( T \in B(X) \). The symmetric difference \( \sigma_\ast(T) \setminus \sigma_{pBF}(T) \) is at most countable.

Proof. By corollary 2.7, \( \sigma_{se}(T) \setminus \sigma_{pBF}(T) \) is at most countable. We have \( \sigma_\ast(T) \setminus \sigma_{se}(T) \) consists of at most countably many isolated points (see [1, Theorem 1.65] and \( \sigma_{pBF}(T) \setminus \sigma_{se}(T) \subseteq \sigma_\ast(T) \setminus \sigma_{se}(T) \), hence \( \sigma_{pBF}(T) \setminus \sigma_{se}(T) \) is at most countable. Since
\[
\sigma_{se}(T) \setminus \sigma_{pBF}(T) = (\sigma_{se}(T) \setminus \sigma_{pBF}(T)) \bigcup (\sigma_{pBF}(T) \setminus \sigma_{se}(T))
\]
Therefore \( \sigma_{se}(T) \setminus \sigma_{pBF}(T) \) is at most countable. \( \blacksquare \)

Acknowledgement The authors thank the referees for his suggestions, remarks and comments thorough reading of the manuscript.

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