The Topology of $\theta_{\omega}$-Open Sets

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Abstract. We define the $\theta_{\omega}$-closure operator as a new topological operator. We show that $\theta_{\omega}$-closure of a subset of a topological space is strictly between its usual closure and its $\theta$-closure. Moreover, we give several sufficient conditions for the equivalence between $\theta_{\omega}$-closure and usual closure operators, and between $\theta_{\omega}$-closure and $\theta$-closure operators. Also, we use the $\theta_{\omega}$-closure operator to introduce $\theta_{\omega}$-open sets as a new class of sets and we prove that this class of sets lies strictly between the class of open sets and the class of $\theta$-open sets. We investigate $\theta_{\omega}$-open sets, in particular, we obtain a product theorem and several mapping theorems. Moreover, we introduce $\omega$-$T_2$ as a new separation axiom by utilizing $\omega$-open sets, we prove that the class of $\omega$-$T_2$ is strictly between the class of $T_2$ topological spaces and the class of $T_1$ topological spaces. We study relationship between $\omega$-$T_2$ and $\omega$-regularity. As main results of this paper, we give a characterization of $\omega$-$T_2$ via $\theta_{\omega}$-closure and we give characterizations of $\omega$-regularity via $\theta_{\omega}$-closure and via $\theta_{\omega}$-open sets.

1. Introduction

Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Denote the closure of $A$ by $\overline{A}$. A point $x \in X$ is in $\theta$-closure of $A$ [27] $(x \in Cl_\theta (A))$ if $\overline{U} \cap A \neq \emptyset$ for any $U \in \tau$ and with $x \in U$. A set $A$ is called $\theta$-closed [27] if $Cl_\theta (A) = A$. The complement of a $\theta$-closed set is called a $\theta$-open set. Denote the family of all $\theta$-open sets in $(X, \tau)$ by $\tau_\theta$. It is known that $\tau_\theta$ forms a topology on $X$ coarser than the topology $\tau$ and $\tau_\theta = \tau$ if and only if $(X, \tau)$ is regular. Authors in [6, 7, 17, 18, 21–25, 28] continued the study of $\theta$-closure operator, $\theta$-open sets, and their related topological concepts. Recently, authors in [8–10, 19] have studied several generalizations of $\theta$-open sets. A set $A$ is $\omega$-open set in $(X, \tau)$ [20] if for each $x \in A$, there is $U \in \tau$ such that $x \in U$ and $U \cap A$ is countable, or equivalently, $A$ is $\omega$-open set in $(X, \tau)$ [1] if for each $x \in A$, there is $U \in \tau$ and a countable set $C \subseteq X$ such that $x \in U \cap C \subseteq A$. Denote the family of all $\omega$-open sets in $(X, \tau)$ by $\tau_\omega$. It is known that $\tau_\omega$ forms a topology on $X$ finer than $\tau$. $\omega$-open sets played a vital role in general topology research see, [1, 4, 5, 11–16, 29]. Al Ghour in [1], used $\omega$-open sets to define $\omega$-regularity as a generalization of regularity as follows. A topological space $(X, \tau)$ is $\omega$-regular if for each closed set $F$ in $(X, \tau)$ and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U$ and $F \subseteq V$ with $U \cap V = \emptyset$. The closure of $A$ in the topological space $(X, \tau_\omega)$ is called the $\omega$-closure of $A$ in $(X, \tau)$ and is denoted by $\overline{A}_\omega$. In this work, we use the $\omega$-closure operator to define the $\theta_{\omega}$-closure operator in a similar way to that used in the definition of the $\theta$-closure operator as follows: A point $x \in X$ is in $\theta_{\omega}$-closure of $A$ $(x \in Cl_{\theta_{\omega}} (A)))$ if $\overline{U} \cap A \neq \emptyset$ for any $U \in \tau$ with
x ∈ U. A set A is called $\theta_\omega$-closed if $\text{Cl}_{\theta_\omega}(A) = A$. The complement of a $\theta_\omega$-closed set is called a $\theta_\omega$-open set. Denote the family of all $\theta_\omega$-open sets in $(X, \tau)$ by $\tau_{\theta_\omega}$. We will show that $\tau_{\theta_\omega}$ forms a topology on X which is strictly between $\tau_\theta$ and $\tau$. Moreover, $\tau_{\theta_\omega} = \tau$ if and only if $(X, \tau)$ is $\omega$-regular. In section 2, we define the $\theta_\omega$-closure operator as a new topological operator. We show that the $\theta_\omega$-closure of a subset of a topological space is strictly between its usual closure and its $\theta$-closure. Moreover, we give several sufficient conditions for the equivalence between $\theta_\omega$-closure and usual closure operators, and between $\theta_\omega$-closure and $\theta$-closure operators. Also, we use the $\theta_\omega$-closure operator to introduce $\theta_\omega$-open sets as a new class of sets and we prove that this class of sets lies strictly between the class of open sets and the class of $\theta$-open sets. We investigate $\theta_\omega$-open sets, in particular, we obtain a product theorem and several mapping theorems.

In section 3, we introduce $\omega$-$T_2$ as a new separation axiom by utilizing $\omega$-open sets, we prove that the class of $\omega$-$T_2$ is strictly between the class of $T_2$ topological spaces and the class of $T_1$ topological spaces. We study relationships between $\omega$-$T_2$ and $\omega$-regularity. As the main results of this chapter, we give a characterization of $\omega$-$T_2$ via $\theta_\omega$-closure and we give characterizations of $\omega$-regularity via $\theta_\omega$-closure and via $\theta_\omega$-open sets.

In this paper, $\mathbb{R}, \mathbb{Q}, \mathbb{Q}^c$ and $\mathbb{N}$ denote, respectively the set of real numbers, the set of rational numbers, the set of irrational numbers and the set of natural numbers.

2. $\theta_\omega$-Closure Operator and the Topology of $\theta_\omega$-Open Sets

Let us start by the following definition:

**Definition 2.1.** ([27]) Let $(X, \tau)$ be a topological space and let $A \subseteq X$.

a. A point $x$ in $X$ is in the $\theta$-closure of $A$ ($x \in \text{Cl}_{\theta}(A)$) if $\overline{U} \cap A \neq \emptyset$ for any $U \in \tau$ and $x \in U$.

b. $A$ is $\theta$-closed if $\text{Cl}_{\theta}(A) = A$.

c. $A$ is $\theta$-open if the complement of $A$ is $\theta$-closed.

d. The family of all $\theta$-open sets in $(X, \tau)$ is denoted by $\tau_{\theta}$.

**Theorem 2.2.** ([27]) Let $(X, \tau)$ be a topological space. Then

a. $\tau_{\theta}$ forms a topology on $X$.

b. $\tau_\varnothing \subseteq \tau$ and $\tau_\varnothing \neq \tau$ in general.

**Definition 2.3.** ([20]) Let $(X, \tau)$ be a topological space and let $A \subseteq X$.

a. A point $x$ in $X$ is a condensation point of $A$ if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable.

b. A set $A$ is $\omega$-closed if it contains all its condensation points.

c. A set $A$ is $\omega$-open if the complement of $A$ is $\omega$-closed.

The family of all $\omega$-open sets in a topological space $(X, \tau)$ is denoted by $\tau_\omega$. For a subset $A$ of a topological space $(X, \tau)$, it is known that $A \in \tau_\omega$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U$ and $U \cap A$ is countable.

**Theorem 2.4.** ([2]) Let $(X, \tau)$ be a topological space. Then

a. $\tau_\omega$ is a topology on $X$.

b. $\tau \subseteq \tau_\omega$ and $\tau_\omega \neq \tau$ in general.

**Notation 2.5.** ([1]) Let $(X, \tau)$ be a topological space and let $A \subseteq X$. The closure of $A$ in $(X, \tau_\omega)$ will be denoted by $\overline{A}^\omega$.

**Theorem 2.6.** ([1]) Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then $\overline{A}^\omega \subseteq \overline{A}$ and $\overline{A}^\omega \neq \overline{A}$ in general.

The following is the main definition of this work:
Lemma 2.10. (1) Let \((X, \tau)\) be a topological space. Then
a. If \(x \in X\) is in the \(\theta_{\omega}\)-closure of \(A (x \in Cl_{\theta_{\omega}} (A))\) if \(\overline{U} \cap A \neq \emptyset\) for any \(U \in \tau\) with \(x \in U\).
b. A set \(A\) is called \(\theta_{\omega}\)-open if its complement is \(\theta_{\omega}\)-closed.
c. A set \(A\) is called \(\theta_{\omega}\)-closed if \(\overline{A} = A\).
d. The family of all \(\theta_{\omega}\)-open sets in \((X, \tau)\) will be denoted by \(\tau_{\theta_{\omega}}\).

Theorem 2.12. Let \((X, \tau)\) be a topological space and let \(A \subseteq X\).
a. Every locally indiscrete topological space is \(\omega\)-locally indiscrete.
b. Every locally countable topological space is \(\omega\)-locally indiscrete.

Proof. (a) Follows from the fact that every closed set in a topological space is \(\omega\)-closed.
(b) Let \((X, \tau)\) be locally countable. Then by Lemma 2.10 (b), \(\tau_{\omega}\) is the discrete topology. Thus, every open set in \((X, \tau)\) is \(\omega\)-closed and hence \((X, \tau)\) is \(\omega\)-locally indiscrete. \(\square\)

Example 2.13. Consider \((\mathbb{R}, \tau)\) where \(\tau = \{\emptyset, \mathbb{R}, \mathbb{N}\}\). Then \((\mathbb{R}, \tau)\) is \(\omega\)-locally indiscrete. On the other hand, since \(\mathbb{N}\) is open but not closed, then \((\mathbb{R}, \tau)\) is not locally indiscrete. Also, clearly that \((\mathbb{R}, \tau)\) is not locally countable.

Theorem 2.14. Let \((X, \tau)\) be an \(\omega\)-locally indiscrete topological space and let \(A \subseteq X\). Then
a. \(\overline{A} = Cl_{\theta_{\omega}} (A)\).
b. If \(A\) is closed in \((X, \tau)\), then \(A\) is \(\theta_{\omega}\)-closed in \((X, \tau)\).

Proof. (a) By Theorem 2.8 (a), \(\overline{A} \subseteq Cl_{\theta_{\omega}} (A)\). To see that \(Cl_{\theta_{\omega}} (A) \subseteq \overline{A}\), let \(x \in Cl_{\theta_{\omega}} (A)\) and \(U \in \tau\) such that \(x \in U\). Then \(\overline{U} \cap A \neq \emptyset\). Since \((X, \tau)\) is \(\omega\)-locally indiscrete, it follows that \(\overline{U} = U\) and hence \(U \cap A \neq \emptyset\). It follows that \(x \in \overline{A}\).
(b) Suppose that \(A\) is closed in \((X, \tau)\), then \(A = \overline{A}\). Thus by (a), \(A = Cl_{\theta_{\omega}} (A)\) and hence \(A\) is \(\theta_{\omega}\)-closed in \((X, \tau)\). \(\square\)
Corollary 2.15. Let \((X, \tau)\) be locally indiscrete and let \(A \subseteq X\). Then:
\[ a. \overline{A} = \overline{\text{Cl}_{\theta_\omega}(A)}. \]
\[ b. \text{If } A \text{ is closed in } (X, \tau), \text{ then } A \text{ is } \theta_\omega\text{-closed in } (X, \tau). \]

Proof. Theorems 2.12 (a) and 2.14. \(\square\)

Corollary 2.16. Let \((X, \tau)\) be locally countable and let \(A \subseteq X\). Then:
\[ a. \overline{A} = \overline{\text{Cl}_{\theta_\omega}(A)}. \]
\[ b. \text{If } A \text{ is closed in } (X, \tau), \text{ then } A \text{ is } \theta_\omega\text{-closed in } (X, \tau). \]

Proof. Theorems 2.12 (b) and 2.14. \(\square\)

Theorem 2.17. Let \((X, \tau)\) be an anti-locally countable topological space and let \(A \subseteq X\). Then
\[ a. \text{Cl}_\theta(A) = \text{Cl}_{\theta_\omega}(A). \]
\[ b. \text{If } A \text{ is } \theta_\omega\text{-closed in } (X, \tau), \text{ then } A \text{ is } \theta\text{-closed in } (X, \tau). \]

Proof. (a) By Theorem 2.8 (a), \(\text{Cl}_{\theta_\omega}(A) \subseteq \text{Cl}_\theta(A)\). To see that \(\text{Cl}_\theta(A) \subseteq \text{Cl}_{\theta_\omega}(A)\) let \(x \in \text{Cl}_\theta(A)\) and \(U \in \tau\) such that \(x \in U\). Then \(U \cap A \neq \emptyset\). Since \((X, \tau)\) is anti-locally countable, then by Lemma 2.10 (a), \(\overline{U} = \overline{U} = U\) and hence \(\overline{U} \cap A \neq \emptyset\). It follows that \(x \in \text{Cl}_{\theta_\omega}(A)\).

(b) Suppose that \(A \text{ is } \theta_\omega\text{-closed in } (X, \tau), \text{ then } A = \text{Cl}_{\theta_\omega}(A). \) Thus by (a), \(A = \text{Cl}_\theta(A)\) and hence \(A \text{ is } \theta\text{-closed in } (X, \tau). \)

Theorem 2.18. Let \((X, \tau)\) be a topological space. Then \(\tau_0 \subseteq \tau_{\theta_\omega} \subseteq \tau\).

Proof. To see that \(\tau_0 \subseteq \tau_{\theta_\omega}\), let \(A \in \tau_0\). Then \(X - A \text{ is } \theta\text{-closed and by Theorem 2.8 (b), } X - A \text{ is } \theta_\omega\text{-closed. Thus } A \in \tau_{\theta_\omega}. \) To see that \(\tau_{\theta_\omega} \subseteq \tau\), let \(A \in \tau_{\theta_\omega}. \) Then \(X - A \text{ is } \theta_\omega\text{-closed and by Theorem 2.8 (c), } X - A \text{ is } \text{closed. Thus } A \in \tau. \)

Lemma 2.19. ([27]) Let \((X, \tau)\) be a topological space. Then for each \(A \in \tau, \text{Cl}_\theta(A) = \overline{A}. \)

Theorem 2.20. Let \((X, \tau)\) be a topological space.
\[ a. \text{If } A \subseteq B \subseteq X, \text{ then } \text{Cl}_\theta(A) \subseteq \text{Cl}_\theta(B). \]
\[ b. \text{For each subsets } A, B \subseteq X, \text{Cl}_\theta(A \cup B) = \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B). \]
\[ c. \text{For each subset } A \subseteq X, \text{Cl}_\theta(A) \text{ is closed in } (X, \tau). \]
\[ d. \text{For each } A \in \tau_{\theta_\omega}, \text{Cl}_\theta(A) = \overline{A}. \]
\[ e. \text{For each } A \in \tau, \text{Cl}_\theta(A) = \text{Clos}_\omega(A) = \overline{A}. \]

Proof. (a) Let \(x \in \text{Cl}_\theta(A)\) and \(U \in \tau\) with \(x \in U\). Since \(x \in \text{Cl}_\theta(A)\), \(\overline{U} \cap A \neq \emptyset\). Since \(A \subseteq B, \overline{U} \cap B \neq \emptyset. \) This implies that \(x \in \text{Cl}_\theta(B). \)

(b) By (a), we have \(\text{Cl}_\theta(A) \cup \text{Cl}_\theta(B) \subseteq \text{Cl}_\theta(A \cup B). \) Let \(x \notin \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B). \) Then there are \(U, V \in \tau\) such that \(x \in U \cap V, \overline{U} \cap A = \emptyset\) and \(\overline{V} \cap B = \emptyset. \) Thus, we have \(x \in U \cap V \in \tau\) and
\[ U \cap \overline{V} \cap (A \cup B) = \left(\overline{U} \cap \overline{V} \cap A\right) \cup \left(\overline{U} \cap \overline{V} \cap B\right) \]
\[ \subseteq \left(\overline{U} \cap A\right) \cup \left(\overline{V} \cap B\right) \]
\[ = \emptyset \cup \emptyset \]
\[ = \emptyset. \]

It follows that \(x \notin \text{Cl}_\theta(A \cup B). \)

(c) We show that \(X - \text{Cl}_\theta(A) \in \tau. \) Let \(x \in X - \text{Cl}_\theta(A). \) Then there is \(U \in \tau\) such that \(x \in U \cap \overline{U} \cap A = \emptyset. \) Hence, \(U \cap \text{Cl}_\theta(A) = \emptyset. \) It follows that \(X - \text{Cl}_\theta(A) \in \tau. \)

(d) By Theorem 2.8 (a), \(\overline{A} \subseteq \text{Cl}_\theta(A). \) Conversely, suppose to the contrary that there is \(x \in \text{Cl}_\theta(A) \cap (X - \overline{A}). \) Since \(X - \overline{A} \in \tau, \) we must have \(X - \overline{A} \cap A \neq \emptyset. \) Choose \(y \in X - \overline{A} \cap A. \) Since \(A \in \tau_{\theta_\omega}, \) then \((X - \overline{A}) \cap A \neq \emptyset, \) a contradiction.

(e) Follows from (d) and Lemma 2.19. \(\square\)
Theorem 2.21. Let $(X, \tau)$ be a topological space. Then
a. $\emptyset$ and $X$ are $\theta_\omega$-closed sets.
b. Finite union of $\theta_\omega$-closed sets is $\theta_\omega$-closed.
c. Arbitrary intersection of $\theta_\omega$-closed sets is $\theta_\omega$-closed.

Proof. (a) Follows from Theorems 2.2 (a) and 2.8 (b).
(b) It is sufficient to see that the union of two $\theta_\omega$-closed sets is $\theta_\omega$-closed. Let $A$ and $B$ be any two $\theta_\omega$-closed sets in $(X, \tau)$. Then $\text{Cl}_{\theta_\omega} (A) = A$ and $\text{Cl}_{\theta_\omega} (B) = B$. By Theorem 2.20 (b),
\[
\text{Cl}_{\theta_\omega} (A \cup B) = \text{Cl}_{\theta_\omega} (A) \cup \text{Cl}_{\theta_\omega} (B)
\]
\[
= A \cup B.
\]
It follows that $A \cup B$ is $\theta_\omega$-closed.
(c) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of $\theta_\omega$-closed sets in $(X, \tau)$. Then for all $\alpha \in \Delta, \text{Cl}_{\theta_\omega} (A_\alpha) = A_\alpha$. We show that $\text{Cl}_{\theta_\omega} (\bigcap \{A_\alpha : \alpha \in \Delta\}) \subseteq \bigcap \{A_\alpha : \alpha \in \Delta\}$. Let $x \in \text{Cl}_{\theta_\omega} (\bigcap \{A_\alpha : \alpha \in \Delta\})$ and let $U \in \tau$ such that $x \in U$. Then $U \cap (\bigcap \{A_\alpha : \alpha \in \Delta\}) \neq \emptyset$. Therefore, $\bigcap \{A_\alpha : \alpha \in \Delta\}$ is $\tau_\omega$-closed.

Theorem 2.22. Let $(X, \tau)$ be a topological space. Then $\tau_{\theta_\omega}$ is a topology on $X$.

Proof. (1) The fact that $\emptyset, X \in \tau_{\theta_\omega}$ follows from Theorem 2.21.
(2) Let $A, B \in \tau_{\theta_\omega}$. Then $X - A$ and $X - B$ are $\theta_\omega$-closed sets. By Theorem 2.21 (b),
\[
X - (A \cap B) = (X - A) \cup (X - B)
\]
is $\theta_\omega$-closed. Hence $A \cap B \in \tau_{\theta_\omega}$.
(3) Let $\{A_\alpha : \alpha \in \Delta\} \subseteq \tau_{\theta_\omega}$. Then $\{X - A_\alpha : \alpha \in \Delta\}$ is a family of $\theta_\omega$-closed sets in $(X, \tau)$. Thus by Theorem 2.21 (c),
\[
X - \bigcup \{A_\alpha : \alpha \in \Delta\} = \bigcap \{X - A_\alpha : \alpha \in \Delta\}
\]
is $\theta_\omega$-closed. Hence $\bigcup \{A_\alpha : \alpha \in \Delta\} \in \tau_{\theta_\omega}$. \Box

Theorem 2.23. Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $A \in \tau_{\theta_\omega}$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U} \subseteq A$.

Proof. Suppose that $A \in \tau_{\theta_\omega}$ and let $x \in A$. Then $X - A$ is $\theta_\omega$-closed and $x \notin X - A$. Thus, $x \notin \text{Cl}_{\theta_\omega} (X - A)$ and hence there is $U \in \tau$ such that $x \in U$ and $U \cap (X - A) = \emptyset$. Therefore, we have $x \in U \subseteq \overline{U} \subseteq A$.

Conversely, suppose for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U} \subseteq A$ and suppose on that contrary that $A \notin \tau_{\theta_\omega}$. Then $X - A$ is not $\theta_\omega$-closed and $\text{Cl}_{\theta_\omega} (X - A) \neq X - A$. Choose $x \in \text{Cl}_{\theta_\omega} (X - A) - (X - A)$. Since $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U} \subseteq A$. Thus we have $x \in U \in \tau$ and hence $\overline{U} \cap (X - A) = \emptyset$. Therefore $x \notin \text{Cl}_{\theta_\omega} (X - A)$, a contradiction. \Box

Corollary 2.24. Every open $\omega$-closed set in a topological space is $\theta_\omega$-open.

Proof. Let $(X, \tau)$ be a topological space and let $A$ be open and $\omega$-closed set in $(X, \tau)$. Let $x \in A$. Since $A$ is $\omega$-closed, then $\overline{A} = A$. Take $U = A$. Then $U \in \tau$ and $x \in U = \overline{U} = A \subseteq A$. Thus by Theorem 2.23, it follows that $A$ is $\theta_\omega$-open. \Box

Corollary 2.25. Every countable open set in a topological space is $\theta_\omega$-open.

Proof. Follows directly from Corollary 2.24 since countable sets in a topological space are $\omega$-closed. \Box

The following example shows that $\theta_\omega$-open sets are strictly between $\theta$-open sets and open sets:
Example 2.26. Consider \((\mathbb{R}, \tau)\) where \(\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}', \mathbb{N} \cup \mathbb{Q}'\}\). Then
(a) \(\tau_{\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}\).
(b) \(\tau_{\theta} = \{\emptyset, \mathbb{R}\}\).

Proof. (a) Note that \(\tau_{\omega} = \tau_{\text{cov}} \cup \{A : A \subseteq \mathbb{N}\}\) where \(\tau_{\text{cov}}\) is the co-countable topology on \(\mathbb{R}\). Then a subset \(B \subseteq \mathbb{R}\) is closed in \((\mathbb{R}, \tau_{\omega})\) if and only if either \(B\) is countable or \(B = \mathbb{R} - A\) where \(A \subseteq \mathbb{N}\). So \(\overline{\mathbb{Q}'} = \mathbb{R} - \mathbb{N}\). If \(\mathbb{Q}' \subseteq \tau_{\theta, \omega}\), then there is \(U \in \tau\) such that \(\sqrt{2} \in U \subseteq \overline{U} \subseteq \mathbb{Q}'\). Since \(\sqrt{2} \in U \subseteq \tau\), then \(\mathbb{Q}' \subseteq U\) and so \(\mathbb{R} - \mathbb{N} = \overline{\mathbb{Q}'} \subseteq \overline{U} \subseteq \mathbb{Q}'\) which is impossible, it follows that \(\mathbb{Q}' \in \tau - \tau_{\theta, \omega}\). If \(\mathbb{N} \cup \mathbb{Q}' \in \tau_{\theta, \omega}\), then there is \(U \in \tau\) such that \(\sqrt{2} \in U \subseteq \overline{U} \subseteq \mathbb{N} \cup \mathbb{Q}'\). Since \(\sqrt{2} \in U \subseteq \tau\), then \(\mathbb{Q}' \subseteq U\) and so \(\mathbb{R} - \mathbb{N} = \overline{\mathbb{Q}'} \subseteq \overline{U} \subseteq \mathbb{N} \cup \mathbb{Q}'\) which is impossible and it follows that \(\mathbb{N} \cup \mathbb{Q}' \in \tau - \tau_{\theta, \omega}\). By Corollary 2.25, \(\tau_{\theta} \in \tau_{\theta, \omega}\). This ends the proof that \(\tau_{\theta, \omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}\).

(b) By Theorem 2.18, \(\tau_{\theta} \subseteq \tau_{\theta, \omega}\). So to see that \(\tau_{\theta} = \{\emptyset, \mathbb{R}\}\), it is sufficient to show that \(\mathbb{N} \notin \tau_{\theta}\). If \(\mathbb{N} \in \tau_{\theta}\), then there is \(U \in \tau\) such that \(1 \in U \subseteq \overline{U} \subseteq \mathbb{N}\). Since \(1 \in U \subseteq \tau\), we have \(U = \mathbb{N}\) and so \(\overline{U} = \mathbb{N}\), but \(\mathbb{N} = \mathbb{Q}\). Therefore, \(\mathbb{N} \notin \tau_{\theta}\). □

If \((X, \tau)\) and \((Y, \sigma)\) are two topological spaces, then \(\tau \times \sigma\) will denote the product topology on \(X \times Y\), also \(\pi_x, \pi_y\) will denote the projection functions on \(X\) and \(Y\), respectively.

Lemma 2.27. ([3]) Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces.
(a) \((\tau \times \sigma)_{\omega} \subseteq \tau \times \sigma_{\omega}\).
(b) If \(A \subseteq X\) and \(B \subseteq Y\), then \(\overline{A \times B} \subseteq \overline{A} \times \overline{B} \).

Theorem 2.28. Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces. If \(G \in (\tau \times \sigma)_{\omega}\), then \(\pi_x(G) \in \tau_{\theta, \omega}\) and \(\pi_y(G) \in \sigma_{\theta, \omega}\).

Proof. Let \(x \in \pi_x(G)\). Choose \(y \in Y\) such that \((x, y) \in G\). Since \(G \in (\tau \times \sigma)_{\omega}\), there is \(H \in \tau \times \sigma\) such that \((x, y) \in H \subseteq \overline{H} \subseteq G\). Choose \(U \in \tau\) and \(V \in \sigma\) such that \((x, y) \in U \times V \subseteq H\). Thus, by Lemma 2.27 (b),

\[(x, y) \in U \times V \subseteq \overline{U} \times \overline{V} \subseteq \overline{U} \times \overline{V} \subseteq \overline{H} \subseteq G\]

and hence

\[x \in U \subseteq \overline{U} \subseteq \pi_x(G)\].

It follows that \(\pi_x(G) \in \tau_{\theta, \omega}\). Similarly, we can show that \(\pi_y(G) \in \sigma_{\theta, \omega}\). □

If \(f : (X, \tau) \longrightarrow (Y, \sigma)\) is a closed function, then \(f : (X, \tau) \longrightarrow (Y, \sigma_{\omega})\) is closed, but the converse is not true in general as the following example shows:

Example 2.29. Define \(f : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_u)\) by

\[f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}\].

For every closed subset \(C\) of \((\mathbb{R}, \tau_u)\), \(f(C) \subseteq \mathbb{Q}\), which shows that \(f\) is \(\omega\)-closed. Since \(\mathbb{R}\) is closed in \((\mathbb{R}, \tau_u)\) but \(f(\mathbb{R}) = \mathbb{Q}\) is not closed in \((\mathbb{R}, \tau_u)\), \(f\) is not closed.

Theorem 2.30. Let \(f : (X, \tau) \longrightarrow (Y, \sigma)\) be a function. If \(f : (X, \tau) \longrightarrow (Y, \sigma)\) is open and \(f : (X, \tau) \longrightarrow (Y, \sigma_{\omega})\) is closed, then \(f : (X, \tau_0) \longrightarrow (Y, \sigma_{\theta, \omega})\) is open.

Proof. Let \(A \in \tau_0\) and let \(y \in f(A)\). Choose \(x \in A\) such that \(y = f(x)\). Choose \(V \in \tau\) such that \(x \in V \subseteq \overline{V} \subseteq A\). Thus, \(f(x) = y \in f(V) \subseteq \overline{f(V)} \subseteq f(A)\). Since \(f\) is open, then \(f(V) \in \sigma\). Since \(f\) is \(\omega\)-closed, then \(f(\overline{V})\) is \(\omega\)-closed and so \(f(\overline{V})_\omega \subseteq f(V)_\omega \subseteq f(A)_\omega\). It follows that \(f(A)_\omega \in \sigma_{\theta, \omega}\). □
Theorem 2.31. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is open and \( f : (X, \tau_\omega) \rightarrow (Y, \sigma_\omega) \) is closed, then \( f : (X, \tau_\omega) \rightarrow (Y, \sigma_\omega) \) is open.

Proof. Let \( A \in \tau_\omega \) and let \( y \in f(A) \). Choose \( x \in A \) such that \( y = f(x) \). Choose \( V \in \tau \) such that \( x \in V \subseteq \overline{\omega} \subseteq A \). Thus, \( f(x) = y \in f(V) \subseteq f(\overline{\omega}) \subseteq f(A) \). Since \( f \) is open, then \( f(V) \in \sigma \). Since \( f \) is \( \omega \)-closed, then \( f(\overline{V}) \) is \( \omega \)-closed and so \( f(\overline{V}) \subseteq f(\overline{\omega}) \subseteq f(A) \). It follows that \( f(A) \in \sigma_\omega \). \( \square \)

Theorem 2.32. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( f : (X, \tau_\omega) \rightarrow (Y, \sigma_\omega) \) are both continuous, then \( f : (X, \tau_\omega) \rightarrow (Y, \sigma_\omega) \) is continuous.

Proof. Let \( B \in \sigma_\omega \) and let \( x \in f^{-1}(B) \). Then \( f(x) \in B \) and so there is \( V \in \sigma \) such that \( f(x) \in V \subseteq \overline{\omega} \subseteq B \). Thus, \( x \in f^{-1}(V) \subseteq f^{-1}(\overline{\omega}) \subseteq f^{-1}(B) \). Since \( f : (X, \tau) \rightarrow (Y, \sigma) \) is continuous, then \( f^{-1}(V) \in \tau \). Since \( f : (X, \tau_\omega) \rightarrow (Y, \sigma_\omega) \) is continuous, then \( f^{-1}\left(\overline{\omega}\right) \) is \( \omega \)-closed and so \( f^{-1}(\overline{\omega}) \subseteq f^{-1}(\overline{\omega}) \subseteq f^{-1}(B) \). It follows that \( f^{-1}(B) \in \tau_\omega \). \( \square \)

3. Separation Axioms

Definition 3.1. A topological space \((X, \tau)\) is said to be \(\omega\)-\(T_2\) if for any pair \((x, y)\) of distinct points in \(X\) there exist \(U \in \tau, V \in \tau_\omega\) such that \(x \in U, \ y \in V\) and \(U \cap V = \emptyset\).

Theorem 3.2. A topological space \((X, \tau)\) is \(\omega\)-\(T_2\) if and only if for each \(x \in X, Cl_\omega(\{x\}) = \{x\}\).

Proof. Suppose that \((X, \tau)\) is \(\omega\)-\(T_2\) and suppose on the contrary that for some \(x \in X, Cl_\omega(\{x\}) \neq \{x\}\). Choose \(y \in Cl_\omega(\{x\}) - \{x\}\). Then there exist \(U \in \tau_\omega\) and \(V \in \tau\) such that \(x \in U, \ y \in V\) and \(U \cap V = \emptyset\). Since \(y \in V \in \tau\) and \(y \in Cl_\omega(\{x\})\), then \(\overline{\omega} \cap \{x\} \neq \emptyset\). Thus we have \(x \in U \in \tau_\omega\) and \(x \in \overline{\omega}\) and hence \(U \cap V \neq \emptyset\), a contradiction.

Conversely, suppose for each \(x \in X, Cl_\omega(\{x\}) = \{x\}\). Let \(x, y \in X\) with \(x \neq y\). By assumption, \(Cl_\omega(\{y\}) = \{y\}\) and so we have \(x \notin Cl_\omega(\{y\})\). Thus there is \(U \in \tau\) such that \(x \in U\) and \(\overline{\omega} \cap \{y\} = \emptyset\). Take \(V = X - \overline{U}\). Then we have \(y \in V \in \tau_\omega\) and \(U \cap V = \emptyset\). This ends the proof that \((X, \tau)\) is \(\omega\)-\(T_2\). \( \square \)

Theorem 3.3. If \((X, \tau)\) is an \(\omega\)-\(T_2\) topological space, then \((X, \tau_\omega)\) is \(T_2\).

Proof. Obvious. \( \square \)

The converse of Theorem 3.3 is not true in general as the following example clarifies:

Example 3.4. Consider \((X, \tau)\) where \(X\) is any countable set which contains at least two distinct points and \(\tau\) is the indiscrete topology. It is obvious that \(\tau_\omega\) is the discrete topology and so \((X, \tau_\omega)\) is \(T_2\). Choose \(x, y \in X\) such that \(x \neq y\). If \(U \in \tau\) and \(V \in \tau_\omega\) such that \(x \in U, \ y \in V\). Then \(U = X\) and \(U \cap V \neq \emptyset\). It follows that \((X, \tau)\) is not \(\omega\)-\(T_2\).

Theorem 3.5. Every \(\omega\)-\(T_2\) topological space is \(T_1\).

Proof. Let \((X, \tau)\) be \(\omega\)-\(T_2\). We show for each \(x \in X, \overline{\{x\}} \subseteq \{x\}\). Let \(x \in X\). Since \((X, \tau)\) is \(\omega\)-\(T_2\), then by Theorem 3.2, \(Cl_\omega(\{x\}) = \{x\}\). By Theorem 2.8 (a), we have \(\overline{x} \subseteq Cl_\omega(\{x\}) = \{x\}\). \( \square \)

The following example shows that the converse of Theorem 3.5 is not true in general:

Example 3.6. Consider \((\mathbb{R}, \tau)\) where \(\tau\) is the cofinite topology. It is clear that \((\mathbb{R}, \tau)\) is \(T_1\). It is not difficult to check that \(\tau_\omega\) is the cocountable topology. Thus \((\mathbb{R}, \tau_\omega)\) is not \(T_2\) and by Theorem 3.3, \((\mathbb{R}, \tau)\) not \(\omega\)-\(T_2\).

Theorem 3.7. Every locally countable \(T_1\) topological space is \(\omega\)-\(T_2\).
Proof. Let \((X, \tau)\) be locally countable and \(T_1\). Let \(x, y \in X\) with \(x \neq y\). Since \((X, \tau)\) is locally countable, then \(\tau_{\omega}\) is the discrete topology and so \([y] \in \tau_{\omega}\). On the other hand since \((X, \tau)\) is \(T_1\), then \([y]\) is closed in \((X, \tau)\) and \(X - [y] \in \tau\). Take \(U = X - [y]\) and \(V = [y]\). Then \(U \in \tau, V \in \tau_{\omega}, x \in U, y \in V\) and \(U \cap V = \emptyset\). This shows that \((X, \tau)\) is \(\omega\)-\(T_2\). 

**Theorem 3.8.** Every \(T_2\) topological space is \(\omega\)-\(T_2\).

Proof. Obvious.

The following example shows that the converse of Theorem 3.8 is not true in general:

**Example 3.9.** Consider \((N, \tau)\) where \(\tau\) is the cofinite topology. It is clear that \((N, \tau)\) is \(T_1\) and locally countable and thus by Theorem 3.7, it is \(\omega\)-\(T_2\). On the other hand, it is well known that \((N, \tau)\) is not \(T_2\).

**Definition 3.10.** ([1]) A topological space \((X, \tau)\) is called \(\omega\)-regular if for each closed set \(F\) in \((X, \tau)\) and \(x \in X - F\), there exist \(U \in \tau\) and \(V \in \tau_{\omega}\) such that \(x \in U, F \subseteq V\) and \(U \cap V = \emptyset\).

**Theorem 3.11.** ([1]) A topological space \((X, \tau)\) is \(\omega\)-regular if and only if for each \(U \in \tau\) and each \(x \in U\) there is \(V \in \tau\) such that \(x \in V \subseteq V_{\omega} \subseteq U\).

**Theorem 3.12.** ([27]) A topological space \((X, \tau)\) is regular if and only if \(\tau = \tau_\theta\).

**Theorem 3.13.** ([18]) A topological space \((X, \tau)\) is regular if and only if for each subset \(A \subseteq X\), \(Cl_\theta(A) = \overline{A}\).

The following result modify Theorems 3.12 and 3.13 for \(\omega\)-regular topological spaces:

**Theorem 3.14.** For any topological space \((X, \tau)\), the following are equivalent:

a. \((X, \tau)\) is \(\omega\)-regular.

b. \(\tau = \tau_{\omega}\).

c. For each subset \(A \subseteq X\), \(Cl_{\omega}(A) = \overline{A}\).

Proof. It follows from Theorems 2.18, 3.11 and 2.23.

**Corollary 3.15.** Every \(\omega\)-locally indiscrete topological space is \(\omega\)-regular.


**Corollary 3.16.** Every locally indiscrete topological space is \(\omega\)-regular.

Proof. Theorem 2.12 (a) and Corollary 3.15.

**Corollary 3.17.** Every locally countable topological space is \(\omega\)-regular.

Proof. Theorem 2.12 (b) and Corollary 3.15.

**Theorem 3.18.** ([1]) Every regular topological space is \(\omega\)-regular.

The converse of Theorem 3.18 is not true in general: Consider the topological space in Example 3.9. By Corollary 3.17, \((N, \tau)\) is \(\omega\)-regular. On the other hand, it is well known that this topological space is not regular.

**Theorem 3.19.** Every anti-locally countable \(\omega\)-regular topological space is regular.

Proof. Let \((X, \tau)\) be anti-locally countable and \(\omega\)-regular. We will apply Theorem 3.13. Let \(A \subseteq X\). Since \((X, \tau)\) is anti-locally countable, then by Theorem 2.17 (a) \(Cl_\theta(A) = Cl_{\omega}(A)\). Also, by Theorem 3.14 we have \(Cl_{\omega}(A) = \overline{A}\). It follows that \(Cl_\theta(A) = \overline{A}\).
The topological space in Example 3.9 is $\omega$-regular but not $\omega$-$T_2$. Thus, $\omega$-regularity does not imply $\omega$-$T_2$ in general, however we have the following result:

**Theorem 3.20.** Every $\omega$-regular $T_1$ topological space is $\omega$-$T_2$.

**Proof.** Let $(X, \tau)$ be $\omega$-regular and $T_1$. We apply Theorem 3.2. Let $x \in X$. Since $(X, \tau)$ is $\omega$-regular, then by Theorem 3.14, $Cl_{\omega}$ ($\{x\}$) = $\{x\}$. Since $(X, \tau)$ is $T_1$, then $[x] = \{x\}$. Therefore, $Cl_{\omega}$ ($\{x\}$) = $\{x\}$. \qed

To give an example on an $\omega$-$T_2$ topological space that is not $\omega$-regular, by Theorems 3.8 and 3.19 it is sufficient to give an example of an anti-locally countable $T_2$ topological space that is not regular. Consider $(\mathbb{R}, \tau_\omega)$ where $\tau$ is the usual topology on $\mathbb{R}$. Clearly that $(\mathbb{R}, \tau_\omega)$ is anti-locally countable. On the other hand it is well known that $(\mathbb{R}, \tau_\omega)$ is $T_2$ but not regular.

**References**


