Warped Product Bi-Slant Submanifolds of Cosymplectic Manifolds

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Abstract. In this paper, we study warped product bi-slant submanifolds of cosymplectic manifolds. It is shown that there is no proper warped product bi-slant submanifold other than pseudo-slant warped product. Finally, we give an example of warped product pseudo-slant submanifolds.

1. Introduction

In [6], Cabrerizo et al. introduced the notion of bi-slant submanifolds of almost contact metric manifolds as a generalization of contact CR-submanifolds, slant and semi-slant submanifolds. They have obtained non-trivial examples of such submanifolds. One of the class of such submanifolds is that of pseudo-slant submanifolds [8]. We note that the pseudo-slant submanifolds are also studied under the name of hemi-slant submanifolds [19].

Warped product submanifolds have been studied rapidly and actively, since Chen introduced the notion of CR-warped products of Kaehler manifolds [10, 11]. Different types of warped product submanifolds have been studied in several kinds of structures for last fifteen years (see [2, 15, 18, 20, 22]). The related studies on this topic can be found in Chen’s book and a survey article [12, 13].

Recently, warped product submanifolds of cosymplectic manifolds were studied in ([1], [15], [20–22]). In this paper, we study warped product bi-slant submanifolds of cosymplectic manifolds. We prove the non-existence of proper warped product bi-slant submanifolds of a cosymplectic manifold. Finally, we give an example of special class of warped product bi-slant submanifolds known as warped product pseudo-slant submanifolds studied in [23].

2. Preliminaries

Let $(\tilde{M}, g)$ be an odd dimensional Riemannian manifold with a tensor field $\varphi$ of type $(1, 1)$, a global vector field $\xi$ (structure vector field) and a dual 1-form $\eta$ of $\xi$ such that

\[ \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \] (1)

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for any \( X, Y \in \Gamma(T\tilde{M}) \), then \( \tilde{M} \) is called an \textit{almost contact metric manifold} \cite{4}, where \( \Gamma(T\tilde{M}) \) denotes the set all vector fields of \( \tilde{M} \) and \( \Gamma \) being the identity transformation on \( T\tilde{M} \). As a consequence, the dimension of \( \tilde{M} \) is odd (= 2\( m + 1 \)), \( \varphi(\xi) = 0 = \eta \circ \varphi \) and \( \eta(X) = g(X, \xi) \). The fundamental 2-form \( \Phi \) of \( \tilde{M} \) is defined \( \Phi(X, Y) = g(X, \varphi Y) \). An almost contact metric manifold \((\tilde{M}, \varphi, \xi, \eta, g)\) is said to be \textit{cosymplectic} if \([\varphi, \varphi] = 0 \) and \( d\eta = 0 \), \( d\Phi = 0 \), where \([\varphi, \varphi][X, Y] = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \) and \( d \) is an exterior differential operator.

Let \( \tilde{\nabla} \) denotes the Levi-Civita connection on \( \tilde{M} \) with respect to the Riemannian metric \( g \). Then in terms of the covariant derivative of \( \varphi \), the cosymplectic structure is characterized by the relation \((\tilde{\nabla}_X \varphi)Y = 0\), for any \( X, Y \in \Gamma(T\tilde{M}) \) \cite{16}. From the formula \((\tilde{\nabla}_X \varphi)Y = 0\), it follows that \( \tilde{\nabla}_X \xi = 0 \).

Let \( M \) be a Riemannian manifold isometrically immersed in \( \tilde{M} \) and denote by the same symbol \( g \) the Riemannian metric induced on \( M \). Let \( \Gamma(TM) \) be the Lie algebra of vector fields in \( M \) and \( \Gamma(T^2M) \), the set of all vector fields normal to \( M \). Let \( \nabla \) be the Levi-Civita connection on \( M \), then the Gauss and Weingarten formulas are respectively given by

\[
\tilde{\nabla}_XY = \nabla_XY + h(X, Y) \tag{2}
\]

and

\[
\tilde{\nabla}_XN = -A_NX + \nabla^\perp_XN \tag{3}
\]

for any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^2M) \), where \( \nabla^\perp \) is the normal connection in the normal bundle \( T^2M \) and \( A_N \) is the shape operator of \( M \) with respect to \( N \). Moreover, \( h : TM \times TM \rightarrow T^2M \) is the second fundamental form of \( M \) in \( \tilde{M} \). Furthermore, \( A_N \) and \( h \) are related by

\[
g(h(X, Y), N) = g(A_NX, Y) \tag{4}
\]

for any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^2M) \).

For any \( X \) tangent to \( M \), we write

\[
\varphi X = TX + FX \tag{5}
\]

where \( TX \) and \( FX \) are the tangential and normal components of \( \varphi X \), respectively. Then \( T \) is an endomorphism of tangent bundle \( TM \) and \( F \) is a normal bundle valued 1-form on \( TM \). Similarly, for any vector field \( N \) normal to \( M \), we put

\[
\varphi N = BN + CN \tag{6}
\]

where \( BN \) and \( CN \) are the tangential and normal components of \( \varphi N \), respectively. Moreover, from (1) and (5), we have

\[
g(TX, Y) = -g(X, TY) \tag{7}
\]

for any \( X, Y \in \Gamma(TM) \).

A submanifold \( M \) is said to be \( \varphi \)-\textit{invariant} if \( F \) is identically zero, i.e., \( \varphi X \in \Gamma(TM) \), for any \( X \in \Gamma(TM) \). On the other hand, \( M \) is said to be \( \varphi \)-\textit{anti-invariant} if \( T \) is identically zero i.e., \( \varphi X \in \Gamma(T^2M) \), for any \( X \in \Gamma(TM) \).

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in an almost contact metric manifold were considered. Throughout the paper we consider the structure vector field \( \xi \) is tangent to the submanifold otherwise it is a \( C \)-totally real submanifold.

(1) A submanifold \( M \) of an almost contact metric manifold \( \tilde{M} \) is called a \textit{contact CR-submanifold} \cite{11} of \( \tilde{M} \) if there exist a differentiable distribution \( \mathcal{D} : p \rightarrow D_p \subset T_pM \) such that \( \mathcal{D} \) is invariant with respect to \( \varphi \), i.e., \( \varphi(\mathcal{D}) = \mathcal{D} \) and the complementary distribution \( \mathcal{D}^\perp \) is anti-invariant with respect to \( \varphi \), i.e., \( \varphi(\mathcal{D}^\perp) \subset T^2M \) and \( TM \) has the orthogonal decomposition \( TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle \), where \( \langle \xi \rangle \) is a 1-dimensional distribution which is spanned by \( \xi \).
(2) A submanifold $M$ of an almost contact metric manifold $\widetilde{M}$ is said to be slant [7], if for each non-zero vector $X$ tangent to $M$ such that $X$ is not proportional to $\langle \xi, \cdot \rangle$, the angle $\theta(X)$ between $\varphi X$ and $T_p M$ is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_p M - \langle \xi_p \rangle$. A slant submanifold is said to be proper slant if $\theta \neq 0$ and $\neq \frac{\pi}{2}$.

(3) A submanifold $M$ of an almost contact metric manifold $\widetilde{M}$ is called semi-slant [6], if it is endowed with two orthogonal distributions $D$ and $D^0$, such that $TM = D \oplus D^0 \oplus \langle \xi \rangle$ where $D$ is invariant with respect to $\varphi$ and $D^0$ is proper slant, i.e., $\theta(X)$ is the angle between $\varphi X$ and $D^0_p$ is constant for any $X \in D^0_p$ and $p \in M$.

(4) A submanifold $M$ of an almost contact metric manifold $\widetilde{M}$ is said to be pseudo-slant (or hemi–slant) [8], if it is endowed with two orthogonal distributions $D^\perp$ and $D^0$ such that $TM = D^\perp \oplus D^0 \oplus \langle \xi \rangle$, where $D^\perp$ is anti-invariant with respect to $\varphi$ and $D^0$ is proper slant.

We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

It is known that [7] if $M$ is a submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$, then $M$ is a slant submanifold with slant angle $\theta$ if and only if

$$T^2 = \cos^2 \theta (-I + \eta \otimes \xi)$$

The following relations are the consequences of (8) as

$$g(TX, TY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)),\quad g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y))$$

for any $X, Y \in \Gamma(TM)$. Another characterization of a slant submanifold of an almost contact metric manifold is obtained by using (5), (6) and (8) as

$$BFX = \sin^2 \theta (-X + \eta(X)\xi), \quad CFX = -FTX$$

for any $X \in \Gamma(TM)$.

In [6], Cabrerizo et al defined and studied bi-slant submanifolds of almost contact metric manifolds as follows:

**Definition 2.1.** Let $\widetilde{M}$ be an almost contact metric manifold and $M$ a real submanifold of $\widetilde{M}$, then we say that $M$ is a bi-slant submanifold if there exists a pair of orthogonal distributions $D_1$ and $D_2$ on $M$ such that

(i) The tangent space $TM$ admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$.

(ii) $TD_1 \perp D_2$ and $TD_2 \perp D_1$

(iii) For any $i = 1, 2$, $D_i$ is a slant distribution with slant angle $\theta_i$.

Let $d_1$ and $d_2$ denote the dimensions of $D_1$ and $D_2$, respectively. Then from the above definition, it is clear that

(i) If $d_1 = 0$ or $d_2 = 0$, then $M$ is a slant submanifold.

(ii) If $d_1 = 0$ and $\theta_2 = 0$, then $M$ is invariant.

(iii) If $d_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then $M$ is an invariant submanifold.

(iv) If neither $d_1 = 0$ nor $d_2 = 0$ and $\theta_1 = 0$, then $M$ is a semi-slant submanifold with slant angle $\theta_2$.

(v) If neither $d_1 = 0$ nor $d_2 = 0$ and $\theta_1 = \frac{\pi}{2}$, then $M$ is a pseudo-slant submanifold with slant angle $\theta_2$.

A bi-slant submanifold of an almost contact metric manifold $\widetilde{M}$ is called proper if the slant distributions $D_1, D_2$ are of slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

We refer to [6] and [14] for non-trivial examples of bi-slant submanifolds.
3. Warped product bi-slant submanifolds

In [3], Bishop and O’Neill introduced the notion of warped product manifolds as follows: Let $M_1$ and $M_2$ be two Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$, respectively, and a positive differentiable function $f$ on $M_1$. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_1_* X, \pi_1_* Y) + (f \circ \pi_1)^2 g_2(\pi_2_* X, \pi_2_* Y)$$

for any vector field $X, Y$ tangent to $M$, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is called trivial if the warping function $f$ is constant. Let $X$ be an unit vector field tangent to $M_1$ and $Z$ be another unit vector field on $M_2$, then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z$$  \hspace{1cm} (12)

where $\nabla$ is the Levi-Civita connection on $M$. If $M = M_1 \times_f M_2$ be a warped product manifold then $M_1$ is a totally geodesic submanifold of $M$ and $M_2$ is a totally umbilical submanifold of $M$ [3, 10].

**Definition 3.1.** A warped product $M_1 \times_f M_2$ of two slant submanifolds $M_1$ and $M_2$ with slant angles $\theta_1$ and $\theta_2$ of a cosymplectic manifold $\tilde{M}$ is called a warped product bi-slant submanifold.

A warped product bi-slant submanifold $M_1 \times_f M_2$ is called proper if both $M_1$ and $M_2$ are proper slant submanifolds with slant angle $\theta_1$, $\theta_2 \neq 0, \frac{\pi}{2}$ of $\tilde{M}$. A warped product $M_1 \times_f M_2$ is contact CR-warped product if $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ discussed in [22]. Also, a warped product bi-slant submanifold $M = M_1 \times_f M_2$ is pseudo-slant warped product if $\theta_2 = \frac{\pi}{2}$ [23].

In this section, we investigate the geometry of warped product bi-slant submanifolds of the form $M_1 \times_f M_2$ of a cosymplectic manifold $\tilde{M}$, where $M_1$ and $M_2$ are slant submanifolds of $\tilde{M}$. It is noted that on a warped product submanifold $M = M_1 \times_f M_2$ of a cosymplectic manifold $\tilde{M}$ if the structure vector field $\xi$ is tangent to $M_2$, then warped product is simply a Riemannian product (trivial) [15]. Now, throughout we consider the structure vector field $\xi$ is tangent to the base manifold $M_1$.

First, we give the following lemma for later use.

**Lemma 3.2.** Let $M = M_1 \times_f M_2$ be a warped product bi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi$ is tangent to $M_1$, where $M_1$ and $M_2$ are slant submanifolds of $\tilde{M}$. Then

$$g(h(X, Y), FV) = g(h(X, V), FY)$$ \hspace{1cm} (13)

for any $X, Y \in \Gamma(TM_1)$ and $V \in \Gamma(TM_2)$.

**Proof.** For any $X, Y \in \Gamma(TM_1)$ and $V \in \Gamma(TM_2)$, we have

$$g(h(X, Y), FV) = g(\tilde{\nabla}_X Y, FV) = g(\tilde{\nabla}_X Y, \varphi V) - g(\tilde{\nabla}_X Y, TV) = -g(\tilde{\nabla}_X \varphi Y, V) + g(\tilde{\nabla}_X TV, Y).$$

Then from (2), (5) and (12), we obtain

$$g(h(X, Y), FV) = -g(\tilde{\nabla}_X TY, V) - g(\tilde{\nabla}_X FY, V) + (X \ln f) g(TV, Y).$$

Last term in the right hand side of above relation vanishes identically by the orthogonality of the vector fields, thus we have

$$g(h(X, Y), FV) = g(TY, \tilde{\nabla}_X V) + g(AFYX, V).$$

Again from (12), (4) and the orthogonality of vector fields, we get the desired result. \qed
Proof. When the structure vector field $\xi$ is tangent to $M_1$, where $M_1$ and $M_2$ are proper slant submanifolds of $\tilde{M}$ with slant angles $\theta_1$ and $\theta_2$, respectively. Then
\begin{equation}
    g(h(X, Z), FV) = g(h(X, V), FZ)
\end{equation}
for any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$.

Proof. For any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = (X \ln f) g(Z, V).
\end{equation}

On the other hand, we also have
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = g(\varphi \tilde{\nabla}_X Z, \varphi V) = g(\tilde{\nabla}_X \varphi Z, \varphi V).
\end{equation}

Using (5), we derive
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = g(\tilde{\nabla}_X \varphi Z, \varphi V) + g(\tilde{\nabla}_X FZ, \varphi V).
\end{equation}

Then from (1), (2), (12), (9) and the cosymplectic characteristic, we find that
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = \cos^2 \theta_2 (X \ln f) g(Z, V) + g(h(X, T), FV) - g(\tilde{\nabla}_X \varphi FZ, V).
\end{equation}

By using (6), we arrive at
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = \cos^2 \theta_2 (X \ln f) g(Z, V) + g(h(X, T), FV) - g(\tilde{\nabla}_X BFZ, V) - g(\tilde{\nabla}_X CFZ, V).
\end{equation}

Then from (11), we obtain
\begin{equation}
    g(\tilde{\nabla}_X Z, V) = \cos^2 \theta_2 (X \ln f) g(Z, V) + g(h(X, T), FV) + \sin^2 \theta_2 g(\tilde{\nabla}_X Z, V) + g(\tilde{\nabla}_X FTZ, V).
\end{equation}

Then from (15) and (16), we compute
\begin{equation}
    g(h(X, T), FV) = g((X, V), FTZ)
\end{equation}

Interchanging $Z$ by $TZ$ and using (8), we obtain
\begin{equation}
    \cos^2 \theta_2 g(h(X, Z), FV) = \cos^2 \theta_2 g((X, V), FZ).
\end{equation}

Since $M$ is proper, then $\cos^2 \theta_2 \neq 0$, thus from the above relation we get (14), which proves the lemma completely. $\square$

Theorem 3.4. There does not exist any proper warped product bi-slant submanifold $M = M_1 \times M_2$ of a cosymplectic manifold $\tilde{M}$ such that $M_1$ and $M_2$ are proper slant submanifolds of $\tilde{M}$.

Proof. When the structure vector field $\xi$ is tangent to $M_2$, then warped product is trivial. Now, we consider $\xi \in \Gamma(TM_1)$ and for any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have
\begin{equation}
    g(h(X, Z), FV) = g(\tilde{\nabla}_Z X, FV) = g(\tilde{\nabla}_Z X, \varphi V) - g(\tilde{\nabla}_Z X, TV).
\end{equation}

Using (1), (2), (5), (12) and the cosymplectic characteristic equation, we derive
\begin{equation}
    g(h(X, Z), FV) = -g(\tilde{\nabla}_Z \varphi X, V) - (X \ln f) g(Z, TV)
\end{equation}
\begin{equation}
    = -g(\tilde{\nabla}_Z TX, V) - g(\tilde{\nabla}_Z FX, V) - (X \ln f) g(Z, TV).
\end{equation}
Again, from (2), (3), (4) and (12), we find that
\[
g(h(X, Z), FV) = -(TX \ln f) g(Z, V) + g(A_{XZ}, V) - (X \ln f) g(Z, TV)
\]
\[
= -(TX \ln f) g(Z, V) + g(h(Z, V), FX) - (X \ln f) g(Z, TV).
\]  
(17)

Interchanging \( Z \) by \( V \) in (17) and using (1), we obtain
\[
g(h(X, V), FZ) = -(TX \ln f) g(Z, V) + g(h(Z, V), FX) + (X \ln f) g(Z, TV).
\]  
(18)

Then from (17), (18) and Lemma 3.3, we arrive at
\[
(X \ln f) g(Z, TV) = 0.
\]  
(19)

Interchanging \( Z \) by \( TZ \) in (19) and using (9), we get
\[
\cos^2 \theta_2 (X \ln f) g(Z, V) = 0.
\]  
(20)

Since \( M \) is proper, then \( \cos^2 \theta_2 \neq 0 \), thus from (20) we conclude that \( f \) is constant. Hence, the theorem is proved completely. \( \square \)

From relation (20) of Theorem 3.4, if \( M \) is not proper and \( \theta_1 = \frac{\pi}{2} \), then \( M \) is a pseudo-slant warped product of the form \( M_1 \times_f M_2 \) and this a case which has been discussed in [23] for its characterisation and inequality.

Now, we give an example of such warped products.

**Example 3.5.** Let \( \mathbb{R}^7 \) be the Euclidean 7-space endowed with the standard metric and cartesian coordinates \((x_1, x_2, x_3, y_1, y_2, y_3, t)\) and with the canonical structure given by
\[
\varphi \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, \quad \varphi \left( \frac{\partial}{\partial y_i} \right) = -\frac{\partial}{\partial x_j}, \quad \varphi \left( \frac{\partial}{\partial t} \right) = 0, \quad 1 \leq i, j \leq 3.
\]

If we assume a vector field \( X = \lambda_1 \frac{\partial}{\partial x_1} + \mu_1 \frac{\partial}{\partial x_2} + v^2 \frac{\partial}{\partial y_3} \) of \( \mathbb{R}^7 \), then \( \varphi X = \lambda_1 \frac{\partial}{\partial x_1} - \mu_1 \frac{\partial}{\partial x_2} \) and \( \varphi^2 (X) = -\lambda_1 \frac{\partial}{\partial x_2} + \mu_1 \frac{\partial}{\partial x_1} = -X + v^2 \frac{\partial}{\partial t} \). Also, we can see that \( g(X, X) = \lambda_1^2 + \mu_1^2 + v^2 \) and \( g(\varphi X, \varphi X) = \lambda_2^2 + \mu_2^2 \), where \( g \) is the Euclidean metric tensor of \( \mathbb{R}^7 \). Then, we have \( g(\varphi X, \varphi X) = g(X, X) - \eta(\varphi X, \xi) \), where \( \xi = \frac{\partial}{\partial t} \) and hence \( (\varphi, \xi, \eta, g) \) is an almost contact structure on \( \mathbb{R}^7 \). Consider a submanifold \( M \) of \( \mathbb{R}^7 \) defined by
\[
\phi(u, v, w, t) = (u \cos v, u \sin v, w, w \cos v, w \sin v, 2u, t), \quad v \neq 0, \quad \frac{\pi}{2}
\]
for non-zero \( u \) and \( w \). The tangent bundle \( TM \) of \( M \) is spanned by
\[
Z_1 = \cos v \frac{\partial}{\partial x_1} + \sin v \frac{\partial}{\partial x_2} + \frac{2}{\partial y_3},
\]
\[
Z_2 = -u \sin v \frac{\partial}{\partial x_1} + u \cos v \frac{\partial}{\partial x_2} - w \sin v \frac{\partial}{\partial y_1} + w \cos v \frac{\partial}{\partial y_2},
\]
\[
Z_3 = \frac{\partial}{\partial x_3} + \cos v \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2}, \quad Z_4 = \frac{\partial}{\partial t}.
\]
Then, we have
\[ qZ_1 = \cos v \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2} - 2 \frac{\partial}{\partial x_3}, \]
\[ qZ_2 = -u \sin v \frac{\partial}{\partial y_1} + u \cos v \frac{\partial}{\partial y_2} + w \sin v \frac{\partial}{\partial x_1} - w \cos v \frac{\partial}{\partial x_2}, \]
\[ qZ_3 = \frac{\partial}{\partial y_3} - \cos v \frac{\partial}{\partial x_1} - \sin v \frac{\partial}{\partial x_2}, \quad qZ_4 = 0. \]

Clearly, the vector fields \( qZ_2 \) is orthogonal to \( TM \). Then the anti-invariant and proper slant distributions of \( M \) respectively are \( D_1 = \text{Span}\{Z_2\} \) and \( D_2 = \text{Span}\{Z_1, Z_3\} \) with slant angle \( \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \) such that \( \xi = Z_4 \) is tangent to \( M \). Hence, \( M \) is a proper pseudo-slant submanifold of \( \mathbb{R}^7 \). Furthermore, it is easy to see that both the distributions \( D_1 \) and \( D_2 \) are integrable. We denote the integral manifolds of \( D_1 \) and \( D_2 \) by \( M_\perp \) and \( M_\theta \), respectively. Then the metric tensor \( g \) of the product manifold \( M = M_\perp \times M_\theta \) is
\[
\begin{align*}
g &= 5 du^2 + 2 dv^2 + dt^2 + \left(u^2 + w^2 \right) dv^2 \\
&= g_{M_\perp} + \left(\sqrt{u^2 + w^2} \right)^2 g_{M_\theta}.
\end{align*}
\]

Thus \( M \) is a warped product pseudo-slant submanifold of \( \mathbb{R}^7 \) of the form \( M_\theta \times f M_\perp \) with warping function \( f = \sqrt{u^2 + w^2} \) such that \( \xi \) is tangent to \( M_\theta \).

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