Connectedness of Ordered Rings of Fractions of $C(X)$
with the $m$-Topology

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Abstract. An order is presented on the rings of fractions $S^{-1}C(X)$ of $C(X)$, where $S$ is a multiplicatively closed subset of $C(X)$, the ring of all continuous real-valued functions on a Tychonoff space $X$. Using this, a topology is defined on $S^{-1}C(X)$ and for a family of particular multiplicatively closed subsets of $C(X)$ namely $m.c.\ 5$-subsets, it is shown that $S^{-1}C(X)$ endowed with this topology is a Hausdorff topological ring. Finally, the connectedness of $S^{-1}C(X)$ via topological properties of $X$ is investigated.

1. Introduction

In this paper, the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space $X$, is denoted by $C(X)$ ($C^*(X)$). The space $X$ is called pseudocompact if $C(X) = C^*(X)$. For every $f \in C(X)$ the set $Z(f) = \{ f \in C(X) : f(x) = 0 \}$ is said to be zero-set of $f$ and it’s complement which is denoted by $\text{coz} f$, is called cozero-set of $f$. Moreover, an ideal $I \subseteq C(X)$ is said to be $z$-ideal if for every $f \in I$ and $g \in C(X)$, the inclusion $Z(f) \subseteq Z(g)$ implies that $g \in I$. $u \in C(X)$ is a unit (i.e., $u$ has multiplicative inverse) if and only if $Z(u) = 0$ and it is not hard to see that an element $f$ of $C(X)$ is zero-divisor if and only if $\text{int}_X Z(f) \neq \emptyset$. The set of all units and the set of all zero-divisors of $C(X)$ are denoted by $U(X)$ and $Zd(X)$ respectively.

Let $\beta X$ and $\nu X$ be the Stone-Čech compactification and the Hewitt realcompactification of the space $X$, respectively. For every $f \in C(X)$ the unique extension of $f$ to a continuous function in $C(\beta X)$ is denoted by $f^\beta$ and for each $p \in \beta X$, $M_p = \{ f \in C(X) : p \in \text{cl}_{\beta X} Z(f) \}$ ($M^p = \{ f \in C(X) : f^\beta(p) = 0 \}$) is a maximal ideal of $C(X)$ ($C^*(X)$) and also, every maximal ideal of $C(X)$ ($C^*(X)$) is precisely of the form $M^p$, for some $p \in \beta X$. Moreover, for every $p \in \beta X$, $O_p = \{ f \in C(X) : p \in \text{int}_{\beta X} \text{cl}_{\beta X} Z(f) \}$ is the intersection of all prime ideals of $C(X)$ which are contained in $M^p$. In fact, we have;

Lemma 1.1. ([7, Theorem 7.15]) Every prime ideal $P$ in $C(X)$ contains $O_p$ for some unique $p \in \beta X$, and $M^p$ is the unique maximal ideal containing $P$.

Whenever $p \in X$, the ideals $M_p$ and $O_p$ will be the sets $\{ f \in C(X) : p \in Z(f) \}$ and $\{ f \in C(X) : p \in \text{int}_X Z(f) \}$ respectively and in this case, they are denoted by $M_p$ and $O_p$. A maximal ideal $M$ of $C(X)$ is called real whenever the residue class field $\frac{C(X)}{M}$ is isomorphic with the real field $\mathbb{R}$. Thus, for every $p \in \nu X$, $M^p$ is a

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real maximal ideal, and conversely every real maximal ideal of \( C(X) \) is precisely of the form \( M^{p} \) for some \( p \in \nu X \). Moreover, \( M^{p} \cap C(X) = M^{p} \) if and only if \( p \in \nu X \), see 7.9 (c) in [7].

Let \( R \) be a commutative ring with unity and suppose that \( S \) is a multiplicatively closed subset or briefly an \( m.c. \) subset of \( R \). Here \( S^{-1}R \) is the ring of all equivalence classes of the formal fractions \( \frac{a}{s} \) for \( a \in R \) and \( s \in S \), where the equivalence relation is the obvious one. Whenever \( S \) is the set of all non-zero-divisors of \( R \), then \( S^{-1}R \) is called the classical ring of quotients of \( R \).

An \( m.c. \) subset \( T \) of \( R \) is called saturated whenever \( a, b \in R \) and \( ab \in T \) imply that \( a \) and \( b \) belong to \( T \). For an arbitrary \( m.c. \) subset \( S \) of \( R \), the intersection of all saturated \( m.c. \) subsets of \( R \) which contain \( S \), is called saturation of \( S \) and is denoted by \( S \). Using 5.7 in [11] we have

\[
S = R \setminus \bigcup_{P \in \text{Spec}(R)} P.
\]

**Lemma 1.2.** ([11, Exercise 5.12(iv)]) For an arbitrary \( m.c. \) subset \( S \) of a commutative ring \( R \) with unity, two rings \( S^{-1}R \) and \( S^{-1}R \) are isomorphic.

In sequel, for every \( m.c. \) subset \( S \) of \( C(X) \), the ring of fractions \( S^{-1}C(X) \) is often abbreviated as \( S^{-1}C \).

2. An Order Relation on \( S^{-1}C \)

The \( m \)-topology on \( C(X) \) is defined by taking the sets of the form

\[
B(f, u) = \{ g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X \}
\]
as a base for the neighborhood system at \( f \), for each \( f \in C(X) \), where \( u \) runs through the set of all positive units of \( C(X) \). This topology on \( C(X) \) which is denoted by \( C_{m}(X) \), was first introduced in [9] and studied more in [1–3, 5, 8, 12]. To define a topology on \( S^{-1}C \), similar to the \( m \)-topology on \( C(X) \), we need an ordering to make \( S^{-1}C \) a lattice-ordered ring. We define the order relation \( \leq \) on \( S^{-1}C \) as follows:

**Definition 2.1.** For \( \frac{f}{r} \in S^{-1}C \), we define

\[
0 \leq \frac{f}{r} \text{ if there exists } t \in S \text{ such that } 0 \leq (t^2rf)(x) \text{ for all } x \in X.
\]

Clearly \( 0 \leq \frac{f}{r} \) if and only if \( 0 \leq (rf)(x) \) for all \( x \in coz t \), for some \( t \in S \). This definition is similar to the familiar definition of order on \( C(X) \). But here we consider restriction of each \( \frac{f}{r} \) on a cozero-set of \( X \) instead of \( X \) itself. To see that the order \( \leq \) is well defined, let \( \frac{f}{r}, \frac{g}{s} \in S^{-1}C \), \( \frac{f}{r} \equiv \frac{g}{s} \) and \( 0 \leq \frac{f}{r} \). Then there exist \( p, q \in S \) such that \( qfs = qrg \) and \( 0 \leq p^2rf \). Now, the inequality \( 0 \leq (p^2rsq)(qfs) = (p^2rsq)(qrg) = (p^2rsq)(sg) \) and since \( pqr \in S \), we conclude that \( 0 \leq \frac{g}{s} \).

**Proposition 2.2.** Let \( S \) be an \( m.c. \) subset of \( C(X) \), then \( (S^{-1}C, \leq) \) is a lattice-ordered ring.

**Proof.** Clearly for every \( \frac{f}{r} \in S^{-1}C \) if \( 0 \leq \frac{f}{r} \) and \( 0 \leq \frac{g}{s} \), then \( \frac{f}{r} \leq \frac{g}{s} \). Now, suppose that \( \frac{f}{r}, \frac{g}{s} \in S^{-1}C, 0 \leq \frac{f}{r} \) and \( 0 \leq \frac{g}{s} \). There exist \( t_1, t_2 \in S \) such that \( 0 \leq rf \) on \( \text{coz } t_1 \) and \( 0 \leq sg \) on \( \text{coz } t_2 \). Therefore, \( 0 \leq r^2s^2 (rf + sg) \) and \( 0 \leq r^2s^2 (rfsg) \) on \( \text{coz } t_1t_2 \) and thus, \( 0 \leq \frac{r^2s^2rf + r^2sg}{r^2s^2} = \frac{f}{r} + \frac{g}{s} \) and \( 0 \leq \frac{rfsg}{r^2s^2} = \frac{f}{r} \cdot \frac{g}{s} \) on \( \text{coz } t_1t_2 \). To prove that \( S^{-1}C \) is lattice, it can be shown that

\[
\frac{f}{r} \wedge \frac{g}{s} = \frac{rf}{r^2} \wedge \frac{sg}{s^2} = \frac{s^2rf}{p^2s^2} \wedge \frac{r^2sg}{p^2s^2} = \frac{s^2rf}{p^2s^2} \wedge \frac{r^2sg}{s^2r^2}.
\]

\]
If $S$ is an $m.c.$ subset of a commutative ring $R$, then for every $n \in \mathbb{N}$, the set $S^n = \{s^n : s \in S\}$ is an $m.c.$ subset of $R$ and clearly two rings $(S^n)^{-1}R$ and $S^{-1}R$ are isomorphic. In fact, the map $i_n(f) = \frac{x^nf}{n!}$ is an isomorphism from $S^{-1}R$ onto $(S^n)^{-1}R$. Now we define an ordering $\leq'$ on $(S^2)^{-1}C$ as follows;

**Definition 2.3.** For every $f \in (S^2)^{-1}C$, we define

$$0 \leq' \frac{f}{r} \text{ if there exists } t \in S^2 \text{ such that } 0 \leq t(x)f(x) \text{ for all } x \in X.$$  

If $S$ is an $m.c.$ subset of $C(X)$ then $S^2 \subseteq \{f \in S : 0 \leq f\}$. Therefore $0 \leq \frac{f}{r}$ if and only if $0 \leq f$ on $\text{coz } t$ for some $t \in S$. Similar to Definition 2.1, it can be shown that $((S^2)^{-1}C, \leq')$ is a lattice-ordered ring. Moreover, we have the following result whose proof is left to the readers.

**Proposition 2.4.** Let $S$ be an $m.c.$ subset of $C(X)$. Two rings $(S^{-1}C, \leq)$ and $((S^2)^{-1}C, \leq')$ are lattice isomorphic. In fact, the map $i_2(\frac{f}{r}) = \frac{tr}{r}$ from $S^{-1}C$ onto $(S^2)^{-1}C$ is an isomorphism and also order-preserving, i.e., $\frac{f}{r} \leq \frac{g}{s}$ if and only if $\frac{tr}{r} \leq \frac{sr}{s}$. 

Now using the above proposition, without loss of generality, for every lattice-ordered ring $(S^{-1}C, \leq)$ we can assume that each member of $S$ is non-negative. In addition, we can consider $0 \leq \frac{f}{r}$ whenever $0 \leq f$ on $\text{coz } t$ for some $t \in S$.

**Definition 2.5.** A subset $S$ of $C(X)$ is called $3$-subset whenever $f, g \in C(X)$ and $f \in S$, then $Z(f) = Z(g)$ implies that $g \in S$.

**Example 2.6.** The set $C(X)\backslash Zd(X) = \{f \in C(X) : \text{int}_X Z(f) = \emptyset\}$ of all non-zero-divisor elements of $C(X)$, is a multiplicatively closed $3$-subset (or briefly an $m.c.3$-subset) of $C(X)$. Another example of $m.c.3$-subset is $U(X) = \{f \in C(X) : Z(f) = \emptyset\}$, the set of all units of $C(X)$. If $\{P_\lambda\}_{\lambda \in \Lambda}$ is a family of prime $\mathfrak{z}$-ideals of $C(X)$, then $S = C(X)\backslash \bigcup_{\lambda \in \Lambda} P_\lambda$ is also an $m.c.3$-subset of $C(X)$. Note that whenever $P$ is a prime ideal of $C(X)$ which is not $\mathfrak{z}$-ideal, then $S = C(X)\backslash P$ is a saturated $m.c.$ subset of $C(X)$ which is not a $3$-subset.

**Proposition 2.7.** If $S$ is an $m.c.3$-subset of $C(X)$, then the set $T := \{f \in C(X) : Z(f) \subseteq Z(s) \text{ for some } s \in S\}$ is the saturation of $S$.

**Proof.** We show that $T$ is the smallest saturated $m.c.$ subset containing $S$. First, note that $T$ is a saturated $m.c.$ subset of $C(X)$ containing $S$. In fact, if $f, g \in T$ then there exist $s_1, s_2$ in $S$ such that $Z(f) \subseteq Z(s_1)$ and $Z(g) \subseteq Z(s_2)$. Therefore, $Z(fg) = Z(f) \cup Z(g) \subseteq Z(s_1, s_2)$ which implies $fg \in T$. Moreover, if $fg \in T$ then $Z(fg) \subseteq Z(s)$, for some $s \in S$. Thus $Z(f) \subseteq Z(s)$ and also $Z(g) \subseteq Z(s)$ which imply that $f, g \in T$. Next, let $T'$ be a saturated $m.c.$ subset of $C(X)$ containing $S$ and suppose that $f \in T$. Hence $Z(f) \subseteq Z(s)$, for some $s \in S$ and thus $Z(fs) = Z(f) \cup Z(s) = Z(s)$. Since $S$ is a $3$-subset, $fs \in S \subseteq T'$ and so $f \in T'$, i.e., $T \subseteq T'$ which complete the proof. 

**Corollary 2.8.** Let $S$ be an $m.c.$ subset of $C(X)$. $S$ is a saturated $m.c.3$-subset if and only if for every $f \in C(X)$ and $s \in S$, the inclusion $Z(f) \subseteq Z(s)$ implies that $f \in S$.

**Corollary 2.9.** The saturation of every $m.c.3$-subset of $C(X)$ is a $3$-subset.

**Example 2.10.** Let $f(x) = |x| - 1$ be a function of $C(R)$. Then $S_1 = \{1, f, f^2, ...\}$ is an $m.c.$ subset of $X$ which is not $3$-subset nor saturated. In fact,

$$S_2 = \{g \in C(R) : Z(g) = \emptyset \text{ or } Z(g) = \{1, -1\}\}$$

is the smallest $m.c.3$-subset of $C(R)$ containing $S_1$ and for saturation of $S_2$ we have

$$S_2 = \{g \in C(R) : Z(g) \subseteq \{1, -1\}\}.$$  

Moreover, it is easy to see that $S_1 \subseteq S_2 \subseteq S_2$.  

Similarly to the order relation $\leq$, for every $\xi \in S^{-1}C$ we define $0 < \xi$ if $0 < f$ on $\text{coz } t$ for some $t \in S$.

**Proposition 2.11.** The set $U^* = \{ \xi \in S^{-1}C : 0 < \xi \}$ is closed with respect to the operations $\lor$ and $\land$. Moreover, if $S$ is an m.c. 3-subset, then every member of $U^*$ is a unit of $S^{-1}C$.

**Proof.** If $\xi, \eta \in U^*$, then there exist $t_1, t_2 \in S$ such that $0 < f$ on $\text{coz } t_1$ and $0 < g$ on $\text{coz } t_2$. Since $0 \leq r, s$ we have $0 < sf \land rg$ on $\text{coz } t_1t_2ys$ which implies that $0 < \frac{s^0t^0}{rs} = \xi \land \eta$. To prove the second part of the proposition, let $0 < \xi$. We have $0 < f$ on $\text{coz } t$ for some $t \in S$ and so $\text{coz } t \subseteq \text{coz } f$. Therefore, $\text{coz } t = \text{coz } tf$ and since $S$ is an m.c. 3-subset, then $tf \in S$. Now, $\xi = tf \in S^{-1}C$ implies that $\xi$ is a unit. \qed

3. The $m$-Topology on $S^{-1}C$

Before defining the $m$-topology on $S^{-1}C$, we note that $|\xi| = \frac{\xi}{r} \lor (-\frac{\xi}{r}) = \frac{f(r) - 0}{r} = \frac{1}{r}$. Now, for each $\xi \in S^{-1}C$ and each $\eta \in U^*$ if we consider the set $B(\xi, \eta) := \{ \frac{\xi}{r} : \frac{\xi}{r} - \frac{\eta}{r} < \frac{1}{r} \}$, then clearly we have:

$$B\left(\frac{\xi}{r}, \frac{u}{t} \right) = \left\{ \frac{\xi}{s} : \left| \frac{\xi}{s} - \frac{u}{t} \right| < \frac{1}{t} \right\}$$

for all $x \in \text{coz } q \subseteq \text{coz } rstu$ for some $q \in S$.

The collection $B = \{ B(\xi, \eta) : \xi \in S^{-1}C \text{ and } \eta \in U^* \}$ is a base for a topology on $S^{-1}C$. In fact, $\frac{\xi}{r} \in B(\xi, \eta)$ and $B(\frac{\xi}{r}, \frac{\eta}{r} \land \frac{\eta}{r}) \subseteq B(\frac{\xi}{r}, \frac{\eta}{r}) \cap B(\frac{\xi}{r}, 0)$ for every $\frac{\eta}{r}, \frac{\eta}{r} \in U^*$. Moreover, if $\frac{\xi}{r} \in B(\frac{\xi}{r}, \frac{\eta}{r})$, then $\frac{\eta}{r} := \frac{\eta}{r} - \frac{\xi}{r} < \frac{1}{r}$ and we have $B(\frac{\xi}{r}, \frac{\eta}{r}) \subseteq B(\frac{\xi}{r}, \frac{\eta}{r})$. As the $m$-topology on $C(X)$, this topology on $S^{-1}C$ is called the $m$-topology and $S^{-1}C$ endowed with this topology is denoted by $S^{-1}C$. This topology is in fact a generalization of the $m$-topology on $C(X)$. Note that whenever $S = U(X)$ then $S^{-1}C = C_m(X)$.

Recall that a topological ring is simply a ring furnished with a topology for which its algebraic operations are continuous, see [13]. We also notice that a Hausdorff topological ring is completely regular, see 8.1.17 in [6]. To prove that $S^{-1}C$ is a Hausdorff topological ring we need the following lemmas.

**Lemma 3.1.** Let $S$ be an m.c. 3-subset of $C(X)$. For every $0 \leq \frac{\xi}{r} \in S^{-1}C$ there exists $\frac{\eta}{r} \in S^{-1}C$ such that $0 \leq g, s \leq 1$ and $\frac{\xi}{r} = \frac{\eta}{r}$.

**Proof.** Consider $s = \frac{r}{1+r(t)}$ and $g = \frac{1}{1+r(t)}$. Clearly $Z(s) = Z(r)$ implies $s \in S$ and we have $\frac{\eta}{r} = \frac{1}{t} = \frac{\xi}{r}$. \qed

**Lemma 3.2.** If $S$ is an m.c. 3-subset of $C(X)$, then the set $\{ B(\frac{\xi}{r}, \frac{\eta}{s}) : f \in C(X), r, v \in S \text{ and } 0 \leq v \leq 1 \}$ is a base for the $m$-topology on $S^{-1}C$.

**Proof.** By Lemma 3.1, for each $B(\frac{\xi}{r}, \frac{\eta}{s})$ there exist $v, s \in S$ such that $0 \leq v, s \leq 1$, and $\frac{\xi}{r} = \frac{\eta}{s}$. But $s(x)v(x) \leq v(x)$ for all $x \in \text{coz } sv$, then $\frac{\xi}{r} \leq \frac{\eta}{s}$ and so $\frac{\xi}{r} \in B(\frac{\xi}{r}, \frac{\eta}{s}) \subseteq B(\frac{\xi}{r}, \frac{\eta}{s}) = B(\frac{\xi}{r}, \frac{\eta}{s})$. \qed

**Proposition 3.3.** Let $S$ be an m.c. 3-subset of $C(X)$. Then $S^{-1}C$ is a Hausdorff topological ring.

**Proof.** To prove the continuity of addition and multiplication, let $\frac{\xi}{r}, \frac{\eta}{s} \in S^{-1}C$ and $\frac{\zeta}{t} \in U^*$. Then

$$+ \left( B\left(\frac{\xi}{r}, \frac{\eta}{s} \right) \times B\left(\frac{\eta}{s}, \frac{\zeta}{t} \right) \right) \subseteq B\left(\frac{\xi}{r} + \frac{\eta}{s}, \frac{\zeta}{t} \right)$$

and

$$\cdot \left( B\left(\frac{\xi}{r}, \frac{\eta}{s} \right) \times B\left(\frac{\eta}{s}, \frac{\zeta}{t} \right) \right) \subseteq B\left(\frac{\xi}{r} \cdot \frac{\eta}{s}, \frac{\zeta}{t} \right)$$. 
where \( \xi \in U^+ \) such that \( \left( \frac{1}{\xi} + \frac{1}{\xi} + |\frac{1}{\xi}r| + |\frac{1}{\xi}c| \right) < \frac{1}{\xi} \). In fact, if we consider \( w := \left( \frac{1}{\xi} + \frac{1}{\xi} + |\frac{1}{\xi}r| + |\frac{1}{\xi}c| \right) \), then \( \frac{1}{\xi} < w \in U^+ \) and \( w^{-1} < \frac{1}{\xi} \) and \( w^{-1} \in U^+ \). Now, it is enough to take \( \frac{1}{\xi} = w^{-1} \). To show that \( S_m^{-1}C \) is Hausdorff, let \( \frac{1}{\xi}, \frac{1}{\xi} \in S^{-1}C \) and \( \frac{1}{\xi} \neq \frac{1}{\xi} \). Thus, \( \frac{1}{\xi} \neq \frac{1}{\xi} \) on \( \text{coz} \) \( rs \) and so \( \text{coz} \) \( rs \) \( \subseteq \text{coz} \) \( (\frac{1}{\xi} - \frac{1}{\xi}) \). Therefore, \( \text{coz} \) \( rs \) = \( \text{coz} \) \( \frac{1}{\xi} \) \( rs \) \( \frac{1}{\xi} \) \( rs \) and since \( S \) is an \( m.c. \) \( 3 \)-subset, we have \( t := |\frac{1}{\xi} - \frac{1}{\xi}| \in S \). Now, it is not hard to see that \( B\left(\frac{1}{\xi}, \frac{1}{\xi}U^+\right) \) and \( B\left(\frac{1}{\xi}, \frac{1}{\xi}U^+\right) \) are disjoint. \( \square \)

**Corollary 3.4.** Let \( S \) be an \( m.c. \) \( 3 \)-subset of \( C(X) \). Then \( S^{-1}C \) with the \( m \)-topology is a completely regular Hausdorff space.

4. Connectedness of \( S_m^{-1}C \)

In this section, in imitate of [2], we first find the component of zero in \( S_m^{-1}C \), where \( S \) is an \( m.c. \) \( 3 \)-subset. Next using this, we give a necessary and sufficient condition for connectedness of \( S_m^{-1}C \).

**Definition 4.1.** A member \( \frac{1}{\xi} \in S^{-1}C \) is called bounded if there exists \( k \in \mathbb{N} \) such that \( |\frac{1}{\xi}| \leq \frac{1}{k} \), i.e., \( |f(x)| \leq k|r(x)| \) for all \( x \in \text{coz} \) \( t \) for some \( t \in S \).

Clearly the set \( (S^{-1}C)^* \) of all bounded elements of \( S^{-1}C \) is a subring of \( S^{-1}C \).

**Lemma 4.2.** \( (S^{-1}C)^* \) is a clopen subset of \( S_m^{-1}C \).

**Proof.** If \( \frac{1}{\xi} \in (S^{-1}C)^* \), then \( B\left(\frac{1}{\xi}, \frac{1}{k}\right) \subseteq (S^{-1}C)^* \). In fact, \( |\frac{1}{\xi} - \frac{1}{\xi}| < \frac{1}{k} \) implies that \( |\frac{1}{\xi}| < \frac{1}{k} + \frac{1}{k} \leq \frac{1}{k} + \frac{1}{k} \) for some \( k \in \mathbb{N} \) and hence \( \frac{1}{\xi} \) is bounded. On the other hand, if \( \frac{1}{\xi} \not\in (S^{-1}C)^* \), then \( B\left(\frac{1}{\xi}, \frac{1}{k}\right) \cap (S^{-1}C)^* = \emptyset \). \( \square \)

**Lemma 4.3.** \( J_\psi = \{ \frac{1}{\xi} \in S^{-1}C : \frac{1}{\xi} \cdot \frac{1}{\xi} \text{ is bounded for each } \frac{1}{\xi} \in U^+ \} \) is an ideal of \( S^{-1}C \).

**Proof.** It is not hard to see that \( J_\psi \) is closed with respect to addition. Let \( \frac{1}{\xi} \in J_\psi, \frac{1}{\xi} \in S^{-1}C \) and \( \frac{1}{\xi} \in U^+ \). We claim that \( \frac{1}{\xi} \) is bounded and so \( \frac{1}{\xi} \in J_\psi \). Since \( \frac{1}{\xi} \in U^+ \), \( 0 < p \) on \( \text{coz} \) \( t \) for some \( t \in S \) and so \( 0 < (1 + |g|)p \) on \( \text{coz} \) \( t \). Therefore, \( \frac{1}{\xi} \cdot \frac{(1+|g|)p}{q} \in U^+ \). Now by our hypothesis, \( \frac{1}{\xi} \cdot \frac{(1+|g|)p}{q} \) is bounded which implies that \( \frac{1}{\xi} \cdot \frac{(1+|g|)p}{q} \) is bounded and \( \frac{1}{\xi} \cdot \frac{(1+|g|)p}{q} \) is bounded as well. \( \square \)

Using Lemmas 3.1 and 3.2 we have \( J_\psi = \{ \frac{1}{\xi} \in S^{-1}C : \frac{1}{\xi} \text{ is bounded, } \forall t \in U^+, 0 < t < 1 \} \)

**Lemma 4.4.** Let \( S \) be an \( m.c. \) \( 3 \)-subset of \( C(X) \) and consider \( \frac{1}{\xi} \in J_\psi \). The function \( \varphi_{\frac{1}{\xi}} : \mathbb{R} \rightarrow S^{-1}C \) defined by \( \varphi_{\frac{1}{\xi}}(a) = \frac{\varphi_{\frac{1}{\xi}}(a)}{\varphi_{\frac{1}{\xi}}(a)} \) is continuous.

**Proof.** Using Lemma 3.2, for every \( a \in \mathbb{R} \) and \( \frac{1}{\xi} \in U^+ \), we must show that \( \varphi_{\frac{1}{\xi}}^{-1}(B\left(\frac{1}{\xi}, \frac{1}{\xi}\right)) \) contains a neighborhood of \( a \) in \( \mathbb{R} \). Since \( \frac{1}{\xi} \in J_\psi \), there exists \( k \in \mathbb{N} \) such that \( |\frac{1}{\xi}a| \leq \frac{1}{k} \). Now, we show that the interval \((a - \frac{1}{k}, a + \frac{1}{k})\) is contained in \( \varphi_{\frac{1}{\xi}}^{-1}(B\left(\frac{1}{\xi}, \frac{1}{\xi}\right)) \). In fact, \( b \in (a - \frac{1}{k}, a + \frac{1}{k}) \) implies that \( \frac{1}{\xi} |\frac{1}{\xi}a| \leq \frac{1}{k} \cdot \frac{1}{\xi} = \frac{1}{k} \) and hence \( |\frac{1}{\xi}b - \frac{1}{\xi}a| < \frac{1}{k} \), i.e., \( b \in \varphi_{\frac{1}{\xi}}^{-1}(B\left(\frac{1}{\xi}, \frac{1}{\xi}\right)) \). \( \square \)

The following theorem is in fact a generalization of Corollary 3.3 in [2].

**Theorem 4.5.** Let \( S \) be an \( m.c. \) \( 3 \)-subset of \( C(X) \). The ideal \( J_\psi \) is the component of zero in \( S_m^{-1}C \).
Proof. First, since $\mathbb{R}$ is connected, using Lemma 4.4, $\varphi_\omega(\mathbb{R})$ is a connected set containing $0$ for every $\frac{r}{s} \in I_f$. Therefore, $I_f = \bigcup_{x \in \mathbb{R}} \varphi_\omega(\mathbb{R})$ is a connected set containing $0$. Next, if $I$ is the component of $0$ in $S_m^1C$, then $I_f \subseteq I$. Moreover, since $S_m^1C$ is topological ring, $I$ is an ideal of $S_m^1C$. To complete the proof, it is enough to show that $I \subseteq I_f$. On the contrary, let $\frac{r}{s} \in I \setminus I_f$. By Lemma 4.3, there exists $\frac{t}{v} \in U^*$ such that $\frac{r}{s} \not\in (S^1C)^*$. Consider the sets $I \cap (S^1C)^*$ and $I \setminus (S^1C)^*$. By Lemma 4.2, these two sets are open in $I$ and since $0 \in I \cap (S^1C)^*$ and $\frac{r}{s} \in I \setminus (S^1C)^*$, they are non-empty disjoint open subsets of the connected set $I$, a contradiction. \qed

Corollary 4.6. Let $S$ be an m.c. 3-subset of $C(X)$. $S_m^1C$ is connected if and only if $S_m^1C = I_f$, i.e., for every $f \in C(X)$ and each $r \in S$, there exist $k \in \mathbb{N}$ and $t \in S$ such that $|f(x)| \leq kr(x)$ for all $x \in \text{coz} f$.

Motivated by the previous corollary, we are going to investigate the connectedness of $S_m^1C$ via topological properties of $X$ for some particular m.c. 3-subsets of $X$. For example, let $p \in \beta X$ and put $S_p = C(X)\setminus M_p$ or more generally, suppose that $A \subseteq \beta X$ and $S_A := C(X)\setminus \bigcup_{\lambda \in \Lambda} M_\lambda$. Clearly $S_A$ is an m.c. 3-subset of $C(X)$ and $S_A = \{f \in C(X) : p \not\in \text{cl}_{\beta X}Z(f)\}$ for each $p \in A$. Let $A = \{f \in C(X) : A \cap \text{cl}_{\beta X}Z(f) = \emptyset\}$. Now, it is natural to ask the following questions.

When is the topological ring $(S_A)^{-1}C$ connected? what can we say about the connectedness of $(S_A)^{-1}C$ if we replace $\bigcup_{\lambda \in \Lambda} M_\lambda$ in $S_A$ by an arbitrary union of family of particular prime ideals of $C(X)$? We will address such questions in the next section.

5. Connectedness of $S_m^1C$ with the m-Topology

In this section, we study the connectedness of $S_m^1C$, where $S = C(X)\setminus \bigcup_{\lambda \in \Lambda} P_\lambda$, and $(P_\lambda)_{\lambda \in \Lambda}$ is a family of prime z-ideals of $C(X)$. Using this, we conclude that $C(X)$ with the m-topology is connected if and only if $X$ is pseudocompact. Also, it is shown that the classical ring of quotients of $C(X)$ endowed with the m-topology, is connected if and only if every dense cozero-set of $C(X)$ is pseudocompact.

We use the following lemma frequently. But, before that, we review some results which are needed in sequel. First, notice that for every $f \in C(X)$ we have

$$\text{coz} f \subseteq \beta X \setminus \text{cl}_{\beta X}Z(f) \subseteq \text{cl}_{\beta X}\text{coz} f. \quad (1)$$

The proof of the first inclusion is clear. To prove the second, let $x \not\in \text{cl}_{\beta X}Z(f)$. There exists an open neighborhood $G$ of $x$ in $\beta X$ such that $G \subseteq \beta X \setminus Z(f)$. Now, for an arbitrary open subset $H$ of $\beta X$ containing $x$, we have

$$\emptyset \neq X \cap (G \cap H) \subseteq (X \cap H) \cap (\beta X \setminus Z(f)) = H \cap \text{coz} f$$

which implies that $x \in \text{cl}_{\beta X}\text{coz} f$. Next, by part (1), we conclude that

$$\text{cl}_{\beta X}\text{coz} f = \text{cl}_{\beta X}(\beta X \setminus \text{cl}_{\beta X}Z(f)) = \beta X \setminus \text{int}_{\beta X}\text{cl}_{\beta X}Z(f). \quad (2)$$

Finally, if $f \in C^*(X)$, then $\text{coz} f = X \cap \text{coz} f^\delta$ and in this case, we have

$$\text{cl}_{\beta X}\text{coz} f = \text{cl}_{\beta X}\text{coz} f^\delta = \beta X \setminus \text{int}_{\beta X}Z(f^\delta).$$

Using part (2), the next lemma is now evident.

Lemma 5.1. Let $f \in C(X)$ and $p \in \beta X$. $f \not\in O^\delta$ if and only if $p \in \text{cl}_{\beta X}\text{coz} f$.

Proposition 5.2. Let $S$ be an m.c. 3-subset of $C(X)$ and consider $S_p := C(X)\setminus M_p$, for some $p \in \beta X \setminus \nu X$. If $S_p \subseteq S$ then $S_m^1C$ is disconnected.
Proof. Let \( p \in \beta X \setminus vX \). By 8.7.(b) in [7], there exists \( r \in C'(X) \) such that \( Z(r) = \emptyset \), while \( r^\circ(p) = 0 \). Since \( S \) is a 3-subset and \( Z(r) = Z(1) \), then \( r \in S \) and so \( \frac{1}{r} \in S^{-1}C \). To complete the proof, we claim that \( \frac{1}{r} \) is unbounded on \( coz t \) for every \( t \in S \). Let \( \mathcal{S} \) be the saturation of \( S \). Recall that \( \mathcal{S} = C(X) \setminus \bigcup_{p \in S^{-1}C} P \) where each \( P \) is a prime ideal of \( C(X) \). Furthermore, for every prime ideal \( P \) which doesn’t intersect \( S \), we have \( P \subseteq C(X) \setminus S \subseteq C(X) \setminus S_p = M^p \). Now, for every \( t \in S \), \( tr \in S \subseteq \mathcal{S} \) and so, there exists a prime ideal \( P \) such that \( tr \notin P \) and consequently \( tr \notin O^p \). Thus by Lemma 5.1, \( p \in cl_{mX}coz tr \) and hence there exists a net \( \{x_i\} \) contained in \( coz tr = coz t \cap coz r \) which converges to \( p \). Since \( r^\circ \) is continuous, \( r^\circ(x_i) \rightarrow r^\circ(p) = 0 \) and this implies that the function \( r \) converges to zero on \( coz tr \subseteq coz r \) and so the fraction \( \frac{1}{r} \in S^{-1}C \) is not bounded on \( coz tr \subseteq coz r \). Therefore, the claim is true and so \( S^{-1}C \) is disconnected. \( \square \)

Proposition 5.3. Let \( P \) be a prime \( z \)-ideal of \( C(X) \) and suppose that \( S = C(X) \setminus P \). The topological ring \( S^{-1}C \) is connected if and only if \( P \) is a real maximal ideal.

Proof. We first prove the necessity. By contrary, assume that \( P \) is not real maximal ideal. Now, using 7.15 in [7], \( p \in \beta X \) and \( M^p \) be the unique maximal ideal of \( C(X) \) containing \( P \). We consider two cases:

Case 1. \( p \in \beta X \setminus vX \). In this case, Proposition 5.2 implies that \( S^{-1}C \) is disconnected, a contradiction.

Case 2. Let \( p \in vX \). In this case, using 7.9.(c) in [7], we have \( M^p \cap C(X \setminus P) = M^p \). On the other hand, \( P \subseteq M^p \) by our assumption. Then there exists a function \( r \in C'(X) \) such that \( r \in M^p \setminus P \) and so \( \frac{1}{r} \in S^{-1}C \). Moreover, \( r \in M^p \cap C(X) \) implies \( r^\circ(p) = 0 \). Now, for every \( t \in S \), \( tr \notin P \) which shows that \( tr \notin O^p \) and hence \( p \in cl_{mX}coz tr \), by Lemma 5.1. Finally, similar to the proof of the Proposition 5.2, we conclude that \( \frac{1}{r} \) is unbounded on \( coz tr \subseteq coz t \) and consequently using the Corollary 4.6, \( S^{-1}C \) is not connected, a contradiction.

Next, to prove the sufficiency, let \( p \in \beta X \), \( M^p \) be a real maximal ideal of \( C(X) \) and \( S = C(X) \setminus M^p \). Suppose that \( \frac{1}{r} \in S^{-1}C \). By Lemma 3.1, we can assume that \( f, r \in C'(X) \). Since \( r \notin M^p \) and \( M^p \) is real, we have \( r \notin M^p \), by 7.9.(c) in [7], and hence \( r^\circ(p) \neq 0 \). Moreover, for every \( f \in C(X) \), \( f^\circ(p) \) does not approach to infinity. Now, consider the open subset \( H = \{ x \in coz r^\circ : |\frac{f}{r^\circ}(x) - \frac{f}{r^\circ}(p)| < 1 \} \) of \( coz r^\circ \subseteq \beta X \). We observe that \( H \) is an open neighborhood of \( p \) in \( \beta X \) and since \( coz r^\circ : t \in C'(X) \) is a base for the space \( \beta X \), there exists \( t \in C'(X) \) such that \( p \in coz t^\circ \subseteq \beta X \). Thus, for every \( x \in coz t^\circ \), \( |\frac{f}{r^\circ}(x) - \frac{f}{r^\circ}(p)| < |\frac{f}{r^\circ}(p)| + 1 \) and hence for every \( x \in X \cap coz t^\circ \), we have \( |\frac{f}{r^\circ}(x)| < |\frac{f}{r^\circ}(p)| + 1 \) which implies that \( \frac{1}{r} \) is bounded on \( coz t \). Therefore, by Corollary 4.6, \( S^{-1}C \) with the \( m \)-topology, i.e., \( S^{-1}C \) is connected. \( \square \)

The following result is an immediate consequence of the previous proposition.

Corollary 5.4. Let \( p \in \beta X \). \( S^{-1}C \) with the \( m \)-topology is connected if and only if \( p \in vX \).

By 8A.4 in [7], \( vX = \beta X \) if and only if \( X \) is pseudocompact. Using this and corollary 5.4 the following result is now evident.

Corollary 5.5. \( S_{P}^{-1}C \) with the \( m \)-topology is connected for every \( p \in \beta X \) if and only if \( X \) is pseudocompact.

Recall that whenever \( \mathcal{S} \) is the saturation of an \( m.c. \) subset \( S \) of \( C(X) \), then two rings \( S^{-1}C \) and \( (\mathcal{S})^{-1}C \) are isomorphic. By Corollary 2.9, the saturation of every \( m.c. \) \( 3 \)-subset \( S \) of \( C(X) \) is a \( 3 \)-subset. If we consider \( S = C(X) \setminus \bigcup_{\lambda \in X} P_{\lambda} \) where \( \{P_{\lambda}\}_{\lambda \in X} \) is a family of prime ideals of \( C(X) \), then for every \( \lambda \in X \), we have \( P_{\lambda} \cap S = \emptyset \) and conversely, for each prime ideal \( P \) disjoint from \( S \) there exists \( \lambda \in X \) such that \( P = P_{\lambda} \).

Definition 5.6. An ideal \( I \) of \( C(X) \) is called real whenever every maximal ideal containing \( I \), is real.

As 7O in [7], for an ideal \( I \) in \( C(X) \) if we define \( \theta(I) = \{ p \in \beta X : I \subseteq M^p \} \), then \( \theta(I) = \bigcap_{f \in I} cl_{mX}Z(f) \). Thus, an ideal of \( C(X) \) is real ideal if and only if \( \theta(I) \subseteq vX \) or equivalently \( \bigcap_{f \in I} cl_{mX}Z(f) \subseteq vX \).

Proposition 5.7. Let \( \{P_{\lambda}\}_{\lambda \in X} \) be a family of prime \( z \)-ideals of \( C(X) \) and take \( S := C(X) \setminus \bigcup_{\lambda \in X} P_{\lambda} \). Then \( S \) is an \( m.c. \) \( 3 \)-subset of \( C(X) \) and if \( S_{P}^{-1}C \) is connected, then for every \( \lambda \in X \) the ideal \( P_{\lambda} \) is real. Moreover, \( \bigcup_{\lambda \in X} P_{\lambda} = \bigcup_{p \in A} M^p \) where \( A = \bigcup_{\lambda \in X} \theta(P_{\lambda}) \).
Proof. By contrary, suppose that \( S^{-1}_m C \) is connected but at least one of the prime ideals is not real. Thus, there exists \( \lambda \in \Lambda \) and \( p \in \beta X \setminus vX \) such that \( P_{\lambda'} \subseteq M^p \). Since \( p \notin vX \), there is a function \( r \in C(X) \) such that \( Z(r) = \emptyset \) and \( r^p(0) = 0 \). Now, similar to the proof of Proposition 5.2, we conclude that \( \frac{1}{r} \in S^{-1}_m C \) and for every \( t \in S \) we have \( tr \notin O^r \), since \( tr \notin P_{\lambda'} \). So by Lemma 5.1, \( p \in \cl_{\beta X} tr \). Therefore, \( \frac{1}{r} \) is not bounded on \( \coz t \) and thus \( S^{-1}_m C \) is not connected, by Corollary 4.6, a contradiction.

To prove the last part of the proposition, by contrary, let \( r \in \bigcup_{p \in A} M^p \setminus \bigcup_{p \in A} P_1 \). As above, for every \( t \in S \) it can be shown that \( \frac{1}{r} \) is unbounded on \( \coz t \) and so \( S^{-1}_m C \) is disconnected, a contradiction. \( \square \)

**Corollary 5.8.** Let \( p \in \beta X \) and \( \{ P^p_{\lambda} \}_{\lambda \in \Lambda} \) be a family of prime \( z \)-ideals of \( C(X) \) contained in the maximal ideal \( M^p \) and suppose that \( S = C(X) \setminus \bigcup_{\lambda \in \Lambda} P^p_{\lambda} \). Then \( S^{-1}_m C \) is connected if and only if \( p \in vX \) and \( M^p = \bigcup_{\lambda \in \Lambda} P^p_{\lambda} \).

**Corollary 5.9.** Let \( A \subseteq \beta X \) and suppose that \( S_A = C(X) \setminus \bigcup_{p \in A} M^p \). If \( S^{-1}_m C \) with the \( m \)-topology is connected, then \( A \subseteq vX \).

The following theorem which is in fact a generalization of Corollary 5.4, shows that whenever \( A \) is a compact subset of \( \beta X \), the converse of the previous corollary is also true. But, we were unable to answer the converse of the corollary.

**Theorem 5.10.** Let \( A \) be a compact subset of \( \beta X \) and consider \( S_A = C(X) \setminus \bigcup_{p \in A} M^p \). Then \( S^{-1}_m C \) with the \( m \)-topology is connected if and only if \( A \subseteq vX \).

**Proof.** Necessity is clear by Corollary 5.9. To prove the sufficiency, let \( A \) be a compact subset of \( vX \). Using Corollary 4.6, it is enough to show that for every \( \frac{1}{r} \in S^{-1}_m C \), there exists \( t \in S_A \) such that \( \frac{1}{r} \) is bounded on \( \coz t \). Since \( r \in S_A \), then for every \( p \in A \subseteq vX \), \( r \notin M^p \cap C(X) = M^p \) and so \( r^p(0) \neq 0 \). Moreover, \( p \in vX \) implies that \( f^p(p) \neq \infty \) and thus for each \( p \in A \), \( f^p(p) \) is a real number. As in the proof of Proposition 5.3, the subset \( H = \{ x \in \coz r^p : |f^p(r)(x)| - f^p(p) < 1 \} \) is an open neighborhood of \( p \) in \( \coz r^p \) and hence in \( \beta X \) as well. Thus, there exists \( t \in C(X) \) such that \( p \in \coz f^p \subseteq H \subseteq \coz r^p \) and so we conclude that \( \frac{1}{r} \) is bounded on \( \coz t \). In fact, for every \( x \in \coz t \), we have \( \frac{1}{r^p}(x) < f^p(p) + 1 \). Now, since \( A \) is compact and \( A \subseteq \bigcup_{p \in A} \coz f^p \), there are functions \( t_{p_1}, ..., t_{p_n} \) in \( C(X) \) such that \( A \subseteq \bigcup_{i=1}^n \coz t_{p_i} \). We claim that \( t = t_{p_1}^2 + ... + t_{p_n}^2 \) is the function which we look for.

First, note that for every \( p \in A \) we have \( t \notin M^p \). Otherwise, if for some \( q \in A \) we have \( t \in M^q \), then \( Z(t) \subseteq Z(t_{p_i}) \) \((1 \leq i \leq n)\) implies that \( t_{p_i} \in M^q \) for every \( 1 \leq i \leq n \), (since \( M^q \) is a \( z \)-ideal) which contradicts \( q \in A \subseteq \bigcup_{p \in A} \coz t_{p_i} \).

Next, because \( \frac{1}{r} \) is bounded on every \( \coz t_{p_i} \) \((1 \leq i \leq n)\), it is bounded on \( \coz t = \coz (t_{p_1}^2 + ... + t_{p_n}^2) = \bigcup_{i=1}^n \coz t_{p_i} \), too, which completes the proof. \( \square \)

Whenever a subset \( A \) of \( X \) is completely separated from every zero-set disjoint from it, in particular, if \( A \) is a zero-set or a \( C \)-embedded subset of \( X \), then for every \( f \in C(X) \), \( \cl_{\beta X} A \cap \cl_{\beta X} Z(f) = \emptyset \) if and only if \( A \cap \cl_{\beta X} Z(f) = \emptyset \), see Theorems 1.18 and 6.5 in [7]. Therefore, \( S_A = S_{\cl_{\beta X} A} \) and since \( \cl_{\beta X} A \) is a compact subset of \( \beta X \), the following result is now evident by Theorem 5.10.

**Corollary 5.11.** Let a subset \( A \subseteq X \) be completely separated from every zero-set disjoint from it. Then \( S^{-1}_m C \) with the \( m \)-topology is connected if and only if \( \cl_{\beta X} A \subseteq vX \).

If we consider \( S = C(X) \setminus \bigcup_{p \in \beta X} M^p \), then \( S \) is the set of all units of \( C(X) \) and so \( S^{-1}_m C = \cl_{\beta X} C_m(X) \). Therefore, by Theorem 5.10, \( C_m(X) \) is connected if and only if \( \beta X \subseteq vX \). Now, using 8A.4 in [7], the following results is evident.

**Corollary 5.12.** ([2, Proposition 3.12]) \( C(X) \) with the \( m \)-topology is connected if and only if \( X \) is pseudocompact.
Using Proposition 5.7 and Theorem 5.10, we conclude the paper by another proof for Corollary 3.11 in [3]. First, we recall that a point \( p \in X \) is called an almost \( P \)-point, if every \( G_\delta \)-set (zero-set) containing \( p \) has nonempty interior and a space \( X \) is called an almost \( P \)-space if each point of \( X \) is an almost \( P \)-point. Thus, \( X \) is an almost \( P \)-space if and only if every non-zero-divisor of \( (X) \) is unit, i.e., \( \mathbb{U}(X) = C(X) \setminus \text{Zd}(X) = C(X) \setminus \bigcup P \), where \( P \) is a prime ideal of \( (X) \) contained in \( \text{Zd}(X) \). It is proved that \( p \in X \) is an almost \( P \)-point if and only if whenever \( f \in (X) \) and \( p \in Z(f) \) imply that \( p \in \text{cl}_X \text{int}_X Z(f) \). In fact, if \( p \) is an almost \( P \)-point, then \( M_p \subseteq \text{Zd}(X) \) and thus, for every \( f \in (X) \) if \( p \in Z(f) \), then the ideal \((O_p, f) \) generated by \( O_p \cup \{ f \} \) is contained in \( M_p \). Now, using Corollary 3.3 in [4] we conclude that \( p \in \text{cl}_X \text{int}_X Z(f) \). See [10] for more information about almost \( P \)-spaces.

**Corollary 5.13.** ([3, Corollary 3.11]) The classical ring of quotients of \( (X) \) with the \( m \)-topology is connected if and only if \( X \) is a pseudocompact almost \( P \)-space.

**Proof.** Let \( S^{-1}C \) be the classical ring of quotients of \( (X) \) and for every \( p \in \beta X \), suppose that \( \{ P^p_A \}_{A \in \Lambda} \) is the family of all prime ideals of \( (X) \) contained in \( M^p \cap \text{Zd}(X) \). It is not hard to see that \( \text{Zd}(X) = \bigcup_{A \in \Lambda} P^p_A \) and so, \( S = (X) \setminus \bigcup_{A \in \Lambda} P^p_A \). Now, Using Proposition 5.7, if \( S^{-1}C \) is connected, then every \( P^p_A \) is real ideal which implies that \( \beta X \subseteq \nu X \), i.e., \( X \) is pseudocompact. On the other hand, since for every \( \lambda \in \Lambda_p \) we have \( \theta(P^p_A) = \{ p \} \), using the same proposition, we conclude that \( \text{Zd}(X) = \bigcup_{p \in \text{cl}_X M^p} M^p \). Thus, each non-unit of \( (X) \) is zero-divisor and this means that \( X \) is almost \( P \)-space.

Conversely, let \( X \) be pseudocompact almost \( P \)-space. Since \( X \) is an almost \( P \)-space, \( \text{Zd}(X) = \bigcup_{p \in \text{cl}_X M^p} M^p \) and by pseudocompactness of \( X \) we conclude that \( \beta X = \nu X \). Now, by Theorem 5.10, \( S^{-1}C \) is connected for \( S = (X) \setminus \text{Zd}(X) \).

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