A Study on Fuzzy 2-absorbing Primary $\Gamma$-ideals in $\Gamma$-rings

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Abstract. In this paper, we initiate the study of a generalization of fuzzy primary $\Gamma$-ideals in $\Gamma$-rings by introducing fuzzy 2-absorbing primary $\Gamma$-ideals and fuzzy strongly 2-absorbing primary $\Gamma$-ideals. The notions of a fuzzy 2-absorbing primary $\Gamma$-ideal, fuzzy strongly 2-absorbing primary $\Gamma$-ideal and fuzzy weakly completely 2-absorbing primary $\Gamma$-ideal are defined and their structural characteristics and properties are investigated. The notion of a fuzzy $K$ – 2-absorbing primary $\Gamma$-ideal is introduced and several of its properties are investigated. Finally, the relationships between fuzzy 2-absorbing primary $\Gamma$-ideals of a $\Gamma$-ring are examined.

1. Introduction

The fuzzy set theory has been proposed in 1965 by Lofti A. Zadeh [33] from the University of Berkeley and since then this concept has been applied to various algebraic structures. Rosenfeld [31] was the first who applied this notion on algebraic structures. After the introduction of the concept of fuzzy sets by Zadeh, a lot of research took place regarding the generalization of the classical notions and results on algebraic structures applying fuzzy sets. (See [13, 14]) The concept of a $\Gamma$-ring has a special place among generalizations of rings. One of the most interesting examples of a ring would be the endomorphism ring of an abelian group, i.e., $\text{End}M$ or $\text{Hom}(M, M)$ where $M$ is an abelian group. Now if two abelian groups, say $A$ and $B$ instead of one are taken, then $\text{Hom}(A, B)$ is no longer a ring in the way as $\text{End}M$ becomes a ring because the composition is no longer defined. However, if one takes an element of $\text{Hom}(B, A)$ and put it in between two elements of $\text{Hom}(A, B)$, then the composition can be defined. This served as a motivating factor for introducing and studying the notion of a $\Gamma$-ring. The notion of a $\Gamma$-ring, a generalization of the concept of associative rings, has been introduced and studied by Nobusawa in [28]. Barnes [6] slightly weakened the conditions in the definition of a $\Gamma$-ring in the sense of Nobusawa. The structure of $\Gamma$-rings was investigated by several authors such as W.E.Barnes in [6], S.Kyuno in [19, 20] and J.Luh in [23] and were obtained various generalizations analogous to corresponding parts in ring theory. The concept of a fuzzy ideal of a ring was introduced by Liu in [22]. Y. B. Jun and C. Y. Lee [15] applied the concept of fuzzy sets to the theory of $\Gamma$-rings. They studied some properties of fuzzy ideals of $\Gamma$-rings. In fuzzy commutative algebra, primary ideals are the most significant structures. Dutta and Chanda [10], studied the structure of the set of fuzzy ideals of a $\Gamma$-ring. Jun [16] defined fuzzy prime ideal of a $\Gamma$-ring and obtained several
characterizations for a fuzzy ideal to be a fuzzy prime ideal. Fuzzy maximal, radical and primary ideal of a ring was studied by Malik and Mordeson in [24] and fuzzy prime ideal in \( \Gamma \)-rings was studied by Dutta and Chanda in [11]. Furthermore, Öztürk et al. [29, 30] characterized the Artinian and Noetherian \( \Gamma \)-rings in terms of fuzzy ideals.

The concept of a 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [2] and which was also studied in [1],[5]. At present, studies on the 2-absorbing ideal theory are progressing rapidly. It has been studied extensively by many authors (e.g.[3],[7],[17] ). Darani [9] investigated and examined the notion of \( L \)-fuzzy 2-absorbing ideals and he has obtained interesting results on these concepts. Darani and Hashempoor were focused on the concept of \( L \)-fuzzy 2-absorbing ideals in semiring [8]. Elkettani and Kasem [12] proposed the notion of 2-absorbing \( \delta \)-primary \( \Gamma \)-ideal of \( \Gamma \)-rings and obtained interesting results concerning these concepts.

In this paper, we introduce the fuzzy 2-absorbing \( \Gamma \)-ideals, fuzzy 2-absorbing primary \( \Gamma \)-ideals, fuzzy strongly 2-absorbing primary \( \Gamma \)-ideals and fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideals, some generalizations of 2-absorbing primary fuzzy ideals and describe some of their properties. The notion of a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal is introduced and several of its properties are investigated. Finally, the relationships between fuzzy 2-absorbing primary \( \Gamma \)-ideals of \( \Gamma \)-rings are examined. We also establish a diagram which transition between definitions of fuzzy 2-absorbing \( \Gamma \)-ideals of a \( \Gamma \)-ring as well as the relationships of these concepts with the concept of 2-absorbing \( \Gamma \)-ideal.

2. Preliminaries

In this section, for the sake of completeness, we first recall some useful definitions and results which are needed in the sequel. Throughout this paper, unless otherwise stated, \( R \) is a commutative \( \Gamma \)-ring with 1 \( \neq 0 \) and \( L = [0,1] \) stands for a complete lattice.

Definition 2.1. [33] A fuzzy subset \( \mu \) in a set \( X \) is a function \( \mu : X \to [0,1] \).

Definition 2.2. [25] Let \( \mu \) and \( \nu \) be fuzzy subsets of \( X \). We say that \( \mu \) is a subset of \( \nu \), and write \( \mu \subseteq \nu \), if and only if \( \mu(x) \leq \nu(x) \), for all \( x \in X \).

Definition 2.3. [25] Let \( \mu \) be any fuzzy subset of \( X \) and \( t \in L \). Then the set

\[
\mu_t = \{x \in X | \mu(x) \geq t \}
\]

is called the \( t \)-level subset of \( X \) with respect to \( \mu \).

Definition 2.4. [25] Let \( x \in X \) and \( r \in L - \{0\} \). A fuzzy point, written as \( x_r \), is defined to be a fuzzy subset of \( X \), given by

\[
x_r(y) = \begin{cases} 
  r, & \text{if } y = x; \\
  0, & \text{otherwise}.
\end{cases}
\]

If \( x_r \) is a fuzzy point of \( X \) and \( x_r \subseteq \mu \), where \( \mu \) is a fuzzy subset of \( X \), then we write \( x_r \in \mu \).

Definition 2.5. [6] Let \( R \) and \( \Gamma \) be two abelian groups. \( R \) is called a \( \Gamma \)-ring if there exists a mapping

\[
R \times \Gamma \times R \to R \\
(x, \alpha, y) \mapsto x \alpha y
\]

satisfying the following conditions:

1. \( (x + y) \alpha z = x \alpha z + y \alpha z \),
2. \( x \alpha (y + z) = x \alpha y + x \alpha z \),
3. \( x (\alpha + \beta) y = x \alpha y + x \beta y \)
4. \( x \alpha (y \beta z) = (x \alpha y) \beta z \) for all \( x, y, z \in R \) and all \( \alpha, \beta \in \Gamma \).
Definition 2.6. [10] A left (resp. right) $\Gamma$-ideal of a $\Gamma$-ring $R$ is a subset $A$ of $R$ which is an additive subgroup of $R$ and $\Gamma A \subseteq A$ (resp. $A \Gamma \subseteq A$) where,

$$R \Gamma A = \{xay \mid x \in R, \alpha \in \Gamma, y \in A\}.$$  

If $A$ is both a left and right ideal, then $A$ is called a $\Gamma$-ideal of $R$.

Definition 2.7. [11] A fuzzy set $\mu$ in $\Gamma$--ring $R$ is called a fuzzy $\Gamma$-ideal of $R$, if for all $x, y \in R$ and $\alpha \in \Gamma$, the following requirements are satisfied:

1. $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$
2. $\mu(x \alpha y) \geq \max \{\mu(x), \mu(y)\}$.

Definition 2.8. [10] Let $R$ and $S$ be two $\Gamma$-rings, and $f$ be a mapping of $R$ into $S$. Then $f$ is called $\Gamma$-homomorphism if

$$f(a + b) = f(a) + f(b) \text{ and } f(ab) = f(a)f(b)$$

for all $a, b \in R$ and $\alpha \in \Gamma$.

Proposition 2.9. [21] If $P$ is an ideal of a $\Gamma$-ring $R$, then the following conditions are equivalent:

1. $P$ is a prime ideal of $R$;
2. If $x, y \in R$ and $x \Gamma P \subseteq P$, then $x \in P$ or $y \in P$.

Definition 2.10. [16] A non-constant fuzzy $\Gamma$-ideal $\mu$ of a $\Gamma$-ring $R$ is called fuzzy prime $\Gamma$-ideal of $R$ if for any two fuzzy $\Gamma$-ideals $\alpha$ and $\theta$ of $R$,

$$\alpha \Gamma \theta \subseteq \mu \text{ implies that either } \alpha \subseteq \mu \text{ or } \theta \subseteq \mu.$$  

Lemma 2.11. [24] Let $R$ be a commutative $\Gamma$-ring with identity and let $x_r$ and $y_s$ be two fuzzy points of $R$. Then

1. $x_r \alpha y_s = (xay)_{r,s}$
2. $\langle x_r \rangle \alpha \langle y_s \rangle = \langle x_r \alpha y_s \rangle$, where $\langle x_r \rangle$ is fuzzy $\Gamma$-ideal of $R$ generated by $x_r$.

Theorem 2.12. [24] Let $R$ be a commutative $\Gamma$-ring and $\mu$ be a fuzzy $\Gamma$-ideal of $R$. Then the following statements are equivalent:

1. $x_r \Gamma y_s \subseteq \mu \Rightarrow x_r \subseteq \mu$ or $y_s \subseteq \mu$ where $x_r$ and $y_s$ are two fuzzy points of $R$.
2. $\mu$ is a fuzzy prime $\Gamma$-ideal of $R$.

Definition 2.13. [2] A proper ideal $I$ of a commutative ring $M$ is called a $2$-absorbing ideal of $M$ if whenever $x, y, z \in M$ and $xyz \in I$, then $xy \in I$ or $xz \in I$ or $yz \in I$.

Definition 2.14. [26] A fuzzy ideal $\mu$ of $R$ is said to be a fuzzy weakly completely prime ideal if $\mu$ is non-constant function and for all $x, y \in R$, $\mu(xy) = \max \{\mu(x), \mu(y)\}$.

Definition 2.15. [18] Let $\mu$ be a non-constant fuzzy ideal of $R$. $\mu$ is said to be a fuzzy $K$--prime ideal if $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$ for any $x, y \in R$.

Definition 2.16. [12] A proper $\Gamma$-ideal $I$ of a $\Gamma$-ring $R$ is called a $2$-absorbing $\Gamma$-ideal of $R$ if whenever $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $xay \beta z \in I$, then $xay \in I$ or $x \beta z \in I$ or $y \beta z \in I$.

Definition 2.17. [27] Let $\mu$ be a fuzzy ideal of $R$. Then $\sqrt[\mu]{\Gamma}$, called the radical of $\mu$, is defined by $\sqrt[\mu]{\Gamma}(x) = \sqrt[\mu]{\mu(x^a)}$ for some positive integer $n$.

Definition 2.18. [27] A fuzzy ideal $\mu$ of $R$ is called primary fuzzy ideal if for $x, y \in R$, $\mu(xy) > \mu(x)$ implies $\mu(xy) \leq \mu(y^n)$ for some positive integer $n$.
Theorem 2.19. [27] Let \( \mu \) be fuzzy ideal of a ring \( R \). Then \( \sqrt{\mu} \) is a fuzzy ideal of \( R \).

Theorem 2.20. [32] If \( \mu \) and \( \xi \) are two fuzzy ideals of \( R \), then \( \sqrt{\mu \cap \xi} = \sqrt{\mu} \cap \sqrt{\xi} \).

Theorem 2.21. [32] Let \( f : R \rightarrow S \) be a ring homomorphism and let \( \mu \) be a fuzzy ideal of \( R \) such that \( \mu \) is constant on \( \text{Ker} f \) and \( \xi \) be a fuzzy ideal of \( S \). Then,
\[
\sqrt{f(\mu)} = f(\sqrt{\mu}) \quad \text{and} \quad \sqrt{f^{-1}(\xi)} = f^{-1}(\sqrt{\xi}).
\]

Definition 2.22. [4] A proper ideal \( I \) of \( R \) is called 2-absorbing primary ideal of \( R \) if whenever \( a, b, c \in R \) with \( abc \in I \) then either \( ab \in I \) or \( ac \in \sqrt{I} \) or \( bc \in \sqrt{I} \).

Theorem 2.23. [4] If \( I \) is a 2-absorbing primary ideal of \( R \), then \( \sqrt{I} \) is a 2-absorbing ideal of \( R \).

3. Fuzzy 2-absorbing primary \( \Gamma \)-ideals of a \( \Gamma \)-ring

In this section, we introduce and study fuzzy 2-absorbing primary \( \Gamma \)-ideals of a \( \Gamma \)-ring. Firstly, we will give the structure of fuzzy primary \( \Gamma \)-ideals of a \( \Gamma \)-ring. Throughout this paper we assume that \( R \) is a commutative \( \Gamma \)-ring.

Definition 3.1. Let \( \mu \) be a non-constant fuzzy \( \Gamma \)-ideal of \( R \). Then \( \mu \) is said to be a fuzzy primary \( \Gamma \)-ideal of \( R \) if
\[
x_r y_s \alpha \implies \text{either } x_r \in \mu \text{ or } y_s \in \sqrt[\Gamma]{\mu}
\]
for any fuzzy points \( x_r, y_s \) of \( R \) and \( \alpha \in \Gamma \).

Proposition 3.2. Let \( \mu \) be a fuzzy \( \Gamma \)-ideal of \( R \). If \( \mu \) is a fuzzy primary \( \Gamma \)-ideal of \( R \), then for all \( x, y \in R \) and \( \alpha \in \Gamma \)
\[
\mu(xay) > \mu(x) \implies \mu(xay) \leq \sqrt[\Gamma]{\mu(y)}.
\]

Proof. Let \( \mu(xay) = r > \mu(x) \). Then \( (xay)_r \in \mu \) and \( x_r \notin \mu \). Since \( \mu \) is a fuzzy primary \( \Gamma \)-ideal of \( R \), then \( y_s \in \sqrt[\Gamma]{\mu} \). Thus \( \mu(xay) = r \leq \sqrt[\Gamma]{\mu(y)} \). \( \Box \)

Example 3.3. Every fuzzy prime \( \Gamma \)-ideal of \( R \) is a fuzzy primary \( \Gamma \)-ideal of \( R \).

Now, we give the definition of a fuzzy 2-absorbing primary \( \Gamma \)-ideal of a \( \Gamma \)-ring.

Definition 3.4. Let \( \mu \) be a non-constant fuzzy \( \Gamma \)-ideal of a \( \Gamma \)-ring \( R \). Then \( \mu \) is called fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \) if for any fuzzy points \( x_r, y_s, z_t \) of \( R \) and \( \alpha, \beta \in \Gamma \),
\[
x_r, ay_s, \beta z_t \in \mu \implies \text{either } x_r, ay_s \in \mu \text{ or } x_r, \beta z_t \in \sqrt[\Gamma]{\mu} \text{ or } y_s, \beta z_t \in \sqrt[\Gamma]{\mu}.
\]

Proposition 3.5. Every fuzzy primary \( \Gamma \)-ideal of \( R \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. The proof is straightforward. \( \Box \)

Theorem 3.6. Let \( \mu \) be a fuzzy \( \Gamma \)-ideal of \( R \). If \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then \( \mu_a \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \), for every \( a \in [0, \mu(0)] \) with \( \mu_a \neq R \).
Proof. Let \( \mu \) be fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \) and suppose that \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \) are such that \( x\alpha y\beta z \in \mu \) for every \( a \in [0, \mu(0)] \) with \( \mu_a \neq R \). Then
\[
\mu(x\alpha y\beta z) \leq a \quad \text{and} \quad (x\alpha y\beta z)_a (x\alpha y\beta z) = a \leq \mu(x\alpha y\beta z),
\]
so we have \( (x\alpha y\beta z)_a = x,\alpha y,\beta z \in \mu \). Since \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), we have
\[
(x\alpha y)_a = x,\alpha y \in \mu \quad \text{or} \quad (y\beta z)_a = y,\beta z \in \sqrt{\mu}.
\]
Thus \( x\alpha y \in \mu \) or \( x\beta z \in \sqrt{\mu} \) or \( y\beta z \in \sqrt{\mu} \), and \( \mu \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \). \( \square \)

The following example shows that the converse of the theorem is not generally true.

**Example 3.7.** Let \( R = \mathbb{Z} \) and \( \Gamma = 2\mathbb{Z} \), so \( R \) is a \( \Gamma \)-ring. Define the fuzzy \( \Gamma \)-ideal \( \mu \) of \( R \) by
\[
\mu(x) = \begin{cases} 1, & \text{if } x = 0; \\ \frac{1}{3}, & \text{if } x \in 15\mathbb{Z} - \{0\}; \\ 0, & \text{if } x \in \mathbb{Z} - 15\mathbb{Z}. \end{cases}
\]
Since \( \mu(0) = \mathbb{Z}, \mu_\frac{1}{3} = 15\mathbb{Z} \) and \( \mu_1 = 0 \), then we get \( \mu_a \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \). But, for \( \alpha, \beta \in 2\mathbb{Z} \), we get
\[
3,\alpha 5,\beta 1_\frac{1}{3} = (3\alpha 5\beta 1)_\frac{1}{3} \in \mu, \quad \text{and} \quad 3,\alpha 5_\frac{1}{3} = (3\alpha 5)_\frac{1}{3} = \frac{1}{3} > \mu(3\alpha 5) = \frac{1}{3},
\]
\[
3,\beta 1_\frac{1}{3} = (3\beta 1)_\frac{1}{3} = \frac{1}{3} > \sqrt{\mu(3\beta 1)} = 0;
\]
\[
5,\beta 1_\frac{1}{3} = (5\beta 1)_\frac{1}{3} = \frac{1}{3} > \sqrt{\mu(5\beta 1)} = 0.
\]
Thus \( \mu \) is not a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Corollary 3.8.** If \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then
\[
\mu_* = \{ x \in R \mid \mu(x) = \mu(0) \}
\]
is a 2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. Since \( \mu \) is a non-constant fuzzy \( \Gamma \)-ideal of \( R \), then \( \mu_* \neq R \). Now the result follows from the above theorem. \( \square \)

In the sequel of the paper, for the sake of simplicity, we denote \( x^m = x^{\gamma_1}x^{\gamma_2}x^{\gamma_m-1}x \) for some \( \gamma_1, \gamma_2, ..., \gamma_m-1 \in \Gamma \) and for some \( m \in \mathbb{Z}^+ \).

**Theorem 3.9.** Let \( I \) be a 2-absorbing primary \( \Gamma \)-ideal of \( R \). Then the fuzzy subset \( \mu \) of \( R \) defined by
\[
\mu(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{otherwise} \end{cases}
\]
is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. We have \( I \neq R \) and so \( \mu \) is non-constant because \( I \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \). Assume that \( x,\alpha y,\beta z \in \mu \), but \( x,\alpha y \notin \mu, x,\beta z \notin \sqrt{\mu} \) and \( y,\beta z \notin \sqrt{\mu} \) where \( x, y, z \) are fuzzy points of \( R \) and \( \alpha, \beta \in \Gamma \). Then
\[
\mu(x\alpha y)_s < r \wedge s \quad \mu((x\beta z)^t) < \sqrt{\mu}(x\beta z) < r \wedge t \quad \mu((y\beta z)^t) < \sqrt{\mu}(y\beta z) < s \wedge t
\]
Proposition 3.12. If \( x \land r \in \Gamma \) for some \( x \in R \), then \( \mu \) is a fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \).

Proof. Suppose that \( x \land r \in \Gamma \) for all \( n \geq 1 \). Hence
\[
\mu(xay) = 0 \text{ and } xay \notin I
\]
\[
\mu((x \land r)y) = 0 \text{ and } (x \land r)y \notin I \text{ so } x \land r \notin \sqrt{I}
\]
\[
\mu((y \land r)x) = 0 \text{ and } (y \land r)x \notin I \text{ so } y \land r \notin \sqrt{I}.
\]
Since \( I \) is a 2-absorbing \( \Gamma \)-ideal of \( R \), we have \( xay \land r \notin I \) and so \( \mu(xay) = 0 \) for \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \).

By our hypothesis, we have \( \mu(xayr) = x \land ay \land rz \in \mu \land r \) and \( s \land t \leq \mu(xayz) = 0 \). Hence \( r \land s = 0 \) or \( r \land t = 0 \) or \( s \land t = 0 \), which is a contradiction. Hence \( x, ay \in \mu \land ay \land rz \in \sqrt{I} \) and \( x, ay \land rz \in \sqrt{I} \) and \( z \in \mu \) is a fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \).

Theorem 3.10. Every fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. The proof is straightforward.

Example 3.11. Let \( R = \mathbb{Z} \) and \( \Gamma = 5\mathbb{Z} \), so \( R \) is a \( \Gamma \)-ring. Define the fuzzy \( \Gamma \)-ideal \( \mu \) of \( R \) by

\[
\mu(x) = \begin{cases} 
1, & \text{if } x \in 12\mathbb{Z}, \\
0, & \text{otherwise}. 
\end{cases}
\]

Then \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), but it is not fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \) because for \( \alpha, \beta \in 2\mathbb{Z} \) and \( r, s, t \in [0, 1] \),
\[
2, \alpha 2, \beta 3, 1 = \mu, \text{ but } 2, \alpha 2, \beta 3, 1 = \mu \text{ and } 2, \beta 3, 1 = \mu.
\]

Proposition 3.12. If \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then \( \sqrt{\mu} \) is a fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \).

Proof. Suppose that \( x, ay, bz \in \sqrt{\mu} \) and \( x, ay, bz \notin \sqrt{\mu} \) where \( x, y, z \) are fuzzy points of \( R \) and \( \alpha, \beta \in \Gamma \). Since \( x, ay, bz \in \sqrt{\mu} \), then
\[
r \land s \land t = (xaybz)(xaybz) = x, ay, bz, (xaybz) \leq \sqrt{\mu} (xaybz).
\]
From the definition of \( \sqrt{\mu} \), we have
\[
\sqrt{\mu} (xaybz) = \bigvee_{n \geq 1} \mu((xaybz)^n) \geq \bigvee_{n \geq 1} \mu((x^n y^n z^n) \geq r \land s \land t.
\]
for some \( \gamma_1, \gamma_2 \in \Gamma \). Then there exists \( k \in \mathbb{Z}^+ \) such that for some \( \gamma_1', \gamma_2' \in \Gamma \),
\[
r \land s \land t \leq \mu((x^n y^n z^n) \leq \mu((xaybz)^k)
\]
which implies that \( (x, ay, bz)^k \in \mu \). If \( x, ay, bz \notin \sqrt{\mu} \), then for all \( k \in \mathbb{Z}^+ \) and for some \( \gamma \in \Gamma \),
\[
(x, ay, bz)^k \leq x^n y^n z^n \notin \mu.
\]
Since \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then
\[
x \land bz, y, bz \in \sqrt{\mu} \text{ or } y, bz \in \sqrt{\mu}.
\]
Hence \( \sqrt{\mu} \) is fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \).

Definition 3.13. Let \( \mu \) be a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \) and \( \gamma = \sqrt{\mu} \) which is a fuzzy 2-absorbing \( \Gamma \)-ideal of \( R \). Then \( \mu \) is called a fuzzy \( \gamma \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).

Theorem 3.14. Let \( \mu_1, \mu_2, ..., \mu_n \) be fuzzy \( \gamma \)-2-absorbing primary \( \Gamma \)-ideals of \( R \) for some fuzzy 2-absorbing \( \Gamma \)-ideal \( \gamma \) of \( R \). Then \( \mu = \bigcap_{i=1}^n \mu_i \) is a fuzzy \( \gamma \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).
Proof. Assume that \( x_\alpha y_\beta z_i \in \mu \) and \( x_\alpha y_\beta \notin \mu \) where \( x_\alpha, y_\beta, z_i \) are fuzzy points of \( R \) and \( \alpha, \beta \in \Gamma \). Then \( x_\alpha y_\beta \notin \mu_j \) for some \( n \geq j \geq 1 \) and \( x_\alpha y_\beta z_i \in \mu_j \) for all \( n \geq j \geq 1 \). Since \( \mu_j \) is a fuzzy \( \gamma \)-2-absorbing primary ideal of \( R \), we have \( y_\beta z_i \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^n \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^n \mu_i} = \sqrt{\mu} \) or \( x_\alpha y_\beta z_i \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^n \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^n \mu_i} = \sqrt{\mu} \).

Thus \( \mu \) is a fuzzy \( \gamma \)-2-absorbing primary ideal of \( R \). \( \square \)

In the following example, we show that if \( \mu_1, \mu_2 \) are fuzzy 2-absorbing primary ideals of a \( \Gamma \)-ring \( R \), then \( \mu_1 \cap \mu_2 \) need not to be a fuzzy 2-absorbing primary ideal of \( R \).

**Example 3.15.** Let \( R = \mathbb{Z} \) and \( \Gamma = p_i \mathbb{Z} \), so \( R \) is a \( \Gamma \)-ring. Define the fuzzy \( \Gamma \)-ideals \( \mu_1 \) and \( \mu_2 \) of \( R \) by

\[
\mu_1(x) = \begin{cases} 
1, & \text{if } x \in 50\mathbb{Z}, \\
0, & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\mu_2(x) = \begin{cases} 
1, & \text{if } x \in 75\mathbb{Z}, \\
0, & \text{otherwise,}
\end{cases}
\]

such that \( p_i \neq 2, 3, 5 \) is a prime integer. Hence \( \mu_1 \) and \( \mu_2 \) are fuzzy 2-absorbing primary ideals of a \( \Gamma \)-ring \( R \). Since

\[
(\mu_1 \cap \mu_2)(x) = \begin{cases} 
1, & \text{if } x \in 150\mathbb{Z}, \\
0, & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\sqrt{\mu_1 \cap \mu_2}(x) = \begin{cases} 
1, & \text{if } x \in 30\mathbb{Z}, \\
0, & \text{otherwise,}
\end{cases}
\]

then for \( \alpha, \beta \in \Gamma \) and \( r, s, t \in [0, 1] \), \( 25, 3 \alpha, 3 \beta, 25, 3 \alpha \beta, 25, 3 \alpha \beta \notin \mu_1 \cap \mu_2 \), but \( 25, 3 \alpha, 3 \beta, 25, 3 \alpha \beta, 25, 3 \alpha \beta \notin \sqrt{\mu_1 \cap \mu_2} \). Therefore, \( \mu_1 \cap \mu_2 \) is not a fuzzy 2-absorbing primary ideal of \( R \).

**Theorem 3.16.** Let \( \mu \) be a fuzzy \( \Gamma \)-ideal of \( R \). If \( \sqrt{\mu} \) is a fuzzy prime \( \Gamma \)-ideal of \( R \), then \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Corollary 3.17.** If \( \mu \) is a fuzzy prime \( \Gamma \)-ideal of \( R \), then \( \sqrt{\mu} \) is fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), for any \( n \in Z^+ \).

**Theorem 3.18.** Let \( \{\mu_i \mid i \in I\} \) be a directed collection of fuzzy 2-absorbing primary \( \Gamma \)-ideals of \( R \). Then the fuzzy ideal \( \mu = \bigcup_{i \in I} \mu_i \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Proof.** Assume that \( x_\alpha y_\beta z_i \in \mu \) and \( x_\alpha y_\beta \notin \mu \) for some \( x_\alpha, y_\beta, z_i \) fuzzy points of \( R \) and \( \alpha, \beta \in \Gamma \). Then there exists \( j \in I \) such that \( x_\alpha y_\beta z_i \in \mu_j \) and \( x_\alpha y_\beta \notin \mu_j \) for all \( j \in I \). Since \( \mu_j \) is a fuzzy 2-absorbing primary ideal of \( R \), then

\[
y_\beta z_i \in \sqrt{\mu_j} \quad \text{or} \quad x_\alpha y_\beta z_i \in \sqrt{\mu_j}.
\]

Thus

\[
y_\beta z_i \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu \quad \text{or} \quad x_\alpha y_\beta z_i \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu.
\]

Hence \( \mu = \bigcup_{i \in I} \mu_i \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \). \( \square \)
**Theorem 3.19.** Let \( f : R \to S \) be a surjective \( \Gamma \)-ring homomorphism. If \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \) which is constant on \( \text{Ker} f \), then \( f(\mu) \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( S \).

**Proof.** Suppose that \( x, \alpha y, \beta z_1 = f(\mu) \), where \( x, y, z, \alpha, \beta \in \Gamma \). Since \( f \) is a surjective \( \Gamma \)-ring homomorphism, then there exist \( a, b, c \in R \) such that \( f(a) = x, f(b) = y, f(c) = z \). Thus

\[
x, \alpha y, \beta z_1 (xy)z_2 = r \wedge s \wedge t
\]

\[
\leq f(\mu) (xayb)
\]

\[
= f(\mu) (f(a)af(b))
\]

\[
= f(\mu) (f(aab\beta c))
\]

\[
= \mu (aab\beta c)
\]

because \( \mu \) is constant on \( \text{Ker} f \). Then we get \( a, ab, \beta c = f(\mu) \). Since \( \mu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then

\[
a, ab, c \in \mu \text{ or } a, ab, c \in \sqrt{\mu}
\]

Thus,

\[
r \wedge s \leq \mu (aab) = f(\mu) (f(aab))
\]

\[
= f(\mu) (f(a)af(b))
\]

\[
= f(\mu) (f(aab\beta c))
\]

and so, \( x, \alpha y = f(\mu) \) or

\[
r \wedge t \leq \sqrt{\mu} (a\beta c) = f(\sqrt{\mu})(f(a\beta c))
\]

\[
= f(\sqrt{\mu})(f(a)\beta f(c))
\]

\[
= f(\sqrt{\mu})(xy)z_2
\]

so \( x, \beta z_1 \in f(\sqrt{\mu}) \) or

\[
s \wedge t \leq \sqrt{\mu} (b\beta c) = f(\sqrt{\mu})(f(b\beta c))
\]

\[
= f(\sqrt{\mu})(f(b)\beta f(c))
\]

\[
= f(\sqrt{\mu})(yz_2)
\]

so \( y, \beta z_1 \in f(\sqrt{\mu}) \). Hence \( f(\mu) \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( S \). \( \square \)

**Theorem 3.20.** Let \( f : R \to S \) be a \( \Gamma \)-ring homomorphism. If \( \nu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( S \), then \( f^{-1}(\nu) \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Proof.** Suppose that \( x, \alpha y, \beta z_1 \in f^{-1}(\nu) \), where \( x, y, z, \alpha, \beta \in \Gamma \). Then

\[
r \wedge s \wedge t \leq f^{-1}(\nu)((xay)z_2)
\]

\[
= \nu(f(xayb)z_2)
\]

\[
= \nu(f(x)af(y)\beta f(z))
\]

Let \( f(x) = a, f(y) = b, f(z) = c \in S \). Hence we have that \( r \wedge s \wedge t \leq \nu(aab\beta c) \) and \( a, ab, \beta \in \nu \). Since \( \nu \) is a fuzzy 2-absorbing primary \( \Gamma \)-ideal of \( R \), then \( a, ab, \beta \in \sqrt{\nu} \) or \( a, ab, \beta \in \sqrt{\nu} \). If \( a, ab, \beta \in \nu \), then

\[
r \wedge s \leq \nu(aab) = \nu(f(x)af(y))
\]

\[
= \nu(f(xay))
\]

\[
= f^{-1}(\nu(xay))
\]
Thus we get $x_\alpha y_\beta z_\gamma \in f^{-1}(\nu)$. If $a_\alpha b_\beta c_\gamma \in \sqrt{\nu}$, then

$\mu(x_\alpha y_\beta z_\gamma) = \sqrt{\nu}(x_\alpha y_\beta z_\gamma) \subseteq \sqrt{\nu}$.

Therefore, we see that $f^{-1}(\nu)$ is a fuzzy 2-absorbing primary $\Gamma$-ideal of $R$. $\square$

**Definition 3.21.** Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$. $\mu$ is called a fuzzy strongly 2-absorbing primary $\Gamma$-ideal of $R$ if it is non-constant and whenever $\lambda, \eta, \nu$ are fuzzy $\Gamma$-ideals of $R$ with $\lambda \eta \nu \subseteq \mu$, then $\lambda \Gamma \eta \subseteq \mu$ or $\lambda \Gamma \nu \subseteq \sqrt{\mu}$ or $\eta \Gamma \nu \subseteq \sqrt{\mu}$.

**Theorem 3.22.** Every fuzzy primary $\Gamma$-ideal of $R$ is a fuzzy strongly 2-absorbing primary $\Gamma$-ideal of $R$.

**Proof.** The proof is straightforward. $\square$

**Theorem 3.23.** Every fuzzy strongly 2-absorbing primary $\Gamma$-ideal of $R$ is a fuzzy 2-absorbing primary $\Gamma$-ideal of $R$.

**Proof.** Assume that $\mu$ is a fuzzy strongly 2-absorbing primary $\Gamma$-ideal of $R$. Suppose that $x_\alpha y_\beta z_\gamma \in \mu$ for some fuzzy points $x_\alpha y_\beta z_\gamma \in R$. We get $\langle x_\alpha y_\beta z_\gamma \rangle \subseteq \mu$. Since $\mu$ is a fuzzy strongly 2-absorbing primary $\Gamma$-ideal of $R$, then we get $\langle x_\alpha y_\beta z_\gamma \rangle \subseteq \mu$. Therefore, we see that $f^{-1}(\nu)$ is a fuzzy 2-absorbing primary $\Gamma$-ideal of $R$. $\square$

4. Fuzzy Weakly Completely 2-absorbing Primary $\Gamma$-ideals

In this section, we study fuzzy weakly completely 2-absorbing primary $\Gamma$-ideals of a $\Gamma$-ring. Firstly, we give the definitions of fuzzy weakly completely 2-absorbing $\Gamma$-ideal and fuzzy weakly completely primary $\Gamma$-ideal of a $\Gamma$-ring.

**Definition 4.1.** Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$. $\mu$ is called a fuzzy weakly completely 2-absorbing $\Gamma$-ideal of $R$ if for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,

$\mu(x_\alpha y_\beta z_\gamma) \subseteq \mu(x_\alpha y_\beta z_\gamma)$ or $\mu(x_\alpha y_\beta z_\gamma) \subseteq \mu(x_\beta z_\gamma)$ or $\mu(x_\alpha y_\beta z_\gamma) \subseteq \mu(y_\beta z_\gamma)$.

**Definition 4.2.** Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$. $\mu$ is said to be a fuzzy weakly completely primary $\Gamma$-ideal of $R$ if $\mu$ is non-constant fuzzy $\Gamma$-ideal of $R$ and for all $x, y \in R$ and $\alpha \in \Gamma$,

$\mu(x_\alpha y_\beta z_\gamma) \subseteq \mu(x_\alpha y_\beta z_\gamma) \subseteq \mu(x_\alpha y_\beta z_\gamma)$.

**Proposition 4.3.** Let $\mu$ be a non-constant fuzzy $\Gamma$-ideal of $R$. $\mu$ is a fuzzy weakly completely primary $\Gamma$-ideal of $R$ if and only if for every $x, y \in R$ and $\alpha \in \Gamma$,

$\mu(x_\alpha y_\beta z_\gamma) = \max \{\mu(x_\alpha y_\beta z_\gamma), \sqrt{\nu}(y_\beta z_\gamma)\}$.

Now, we give the definition of a fuzzy weakly completely 2-absorbing primary $\Gamma$-ideal of a $\Gamma$-ring.
Definition 4.4. Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$. $\mu$ is called a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$ if for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,
\[
\mu(x\beta z) \leq \mu(xay) \quad \text{or} \quad \mu(x\gamma yz) \leq \mu(xay\beta z) \quad \text{or} \quad \mu(xay\gamma z) \leq \mu(xay\beta z).
\]

Proposition 4.5. Let $\mu$ be a non-constant fuzzy $\Gamma$-ideal of $R$. $\mu$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$ if and only if for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,
\[
\mu(xay\beta z) = \max \{\mu(xay), \mu(xay\beta z), \mu(xay\gamma z)\}.
\]

Theorem 4.6. Every fuzzy weakly completely $2$-absorbing $\Gamma$-ideal of $R$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$.

Proof. The proof is straightforward. □

Theorem 4.7. Every fuzzy primary $\Gamma$-ideal of $R$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$.

Proof. Let $\mu$ be a fuzzy primary $\Gamma$-ideal of $R$. Suppose that $\mu(xay\beta z) > \mu(xay)$ for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. From the definition of a fuzzy primary $\Gamma$-ideal of $R$, we get $\mu(xay\beta z) \leq \sqrt{\mu}(z)$. Since $\sqrt{\mu}$ is a fuzzy $\Gamma$-ideal, then
\[
\sqrt{\mu}(x\beta z) \geq \sqrt{\mu}(z) \geq \mu(xay\beta z) \quad \text{or} \quad \sqrt{\mu}(y\beta z) \geq \sqrt{\mu}(z) \geq \mu(xay\beta z).
\]
Hence, $\mu$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$. □

Theorem 4.8. Every fuzzy weakly completely primary $\Gamma$-ideal of $R$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$.

Proof. Let $\mu$ be a fuzzy weakly completely primary $\Gamma$-ideal of $R$. Then for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,
\[
\mu(xay\beta z) \leq \mu(x) \quad \text{or} \quad \mu(xay\beta z) \leq \sqrt{\mu}(y) \quad \text{or} \quad \mu(xay\beta z) \leq \sqrt{\mu}(z).
\]
Suppose that $\mu(xay\beta z) \leq \mu(x)$. Since $\mu$ is a fuzzy $\Gamma$-ideal of $R$, then $\mu(xay\beta z) \leq \mu(xay)$, and we get $\mu(xay\beta z) \leq \mu(xay)$.

If $\mu(xay\beta z) \leq \sqrt{\mu}(y)$, then since $\sqrt{\mu}$ is a fuzzy $\Gamma$-ideal, we have $\mu(xay\beta z) \leq \sqrt{\mu}(y) \leq \sqrt{\mu}(y\beta z)$, and we get $\mu(xay\beta z) \leq \sqrt{\mu}(y\beta z)$, or if $\mu(xay\beta z) \leq \sqrt{\mu}(z)$, then $\mu(xay\beta z) \leq \sqrt{\mu}(z) \leq \sqrt{\mu}(x\beta z)$, and we get $\mu(xay\beta z) \leq \sqrt{\mu}(x\beta z)$.

Hence, $\mu$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$. □

Lemma 4.9. Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$ and $a \in [0, \mu(0)]$. Then $\mu$ is a fuzzy weakly completely $2$-absorbing $\Gamma$-ideal of $R$ if and only if $\mu_a$ is a $2$-absorbing $\Gamma$-ideal of $R$.

Theorem 4.10. Let $\mu$ be a fuzzy $\Gamma$-ideal of $R$. The following statements are equivalent:

1. $\mu$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$.
2. For every $a \in [0, \mu(0)]$, the $a$-level subset $\mu_a$ of $\mu$ is a $2$-absorbing primary $\Gamma$-ideal of $R$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\mu$ is a fuzzy weakly completely $2$-absorbing primary $\Gamma$-ideal of $R$. Let $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $xay\beta z \in \mu_a$ for some $a \in [0, \mu(0)]$. Then
\[
\max \{\mu(xay), \sqrt{\mu}(x\beta z), \sqrt{\mu}(y\beta z)\} = \mu(xay\beta z) \geq a.
\]
Hence $\mu(xay) \geq a$ or $\sqrt{\mu}(x\beta z) \geq a$ or $\sqrt{\mu}(y\beta z) \geq a$, which implies that $xay \in \mu_a$ or $x\beta z \in \sqrt{\mu_a} = \sqrt{\mu_a}$ or $y\beta z \in \sqrt{\mu_a} = \sqrt{\mu_a}$.

Thus, $\mu_a$ is a $2$-absorbing primary $\Gamma$-ideal of $R$. 
(2) ⇒ (1). Assume that \( \mu_a \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \) for every \( a \in [0,1] \). Let \( \mu (xay\beta z) = a \) for any \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \). Then \( xay\beta z \in \mu_a \) and \( \mu_a \) is a 2-absorbing primary \( \Gamma \)-ideal. Thus it gives
\[
xay \in \mu_a \text{ or } \sqrt{\mu_a} \cap y\beta z \in \sqrt{\mu_a}.
\]
Hence \( \mu (xay) \geq a \) or \( \sqrt{\mu} (x\beta z) \geq a \) or \( \sqrt{\mu} (y\beta z) \geq a \), which implies that
\[
\mu (xay) \geq a = \mu (xay\beta z) \text{ or } \sqrt{\mu} (x\beta z) \geq a = \mu (xay\beta z) \text{ or } \sqrt{\mu} (y\beta z) \geq a = \mu (xay\beta z).
\]
Therefore, \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \). \qed

**Theorem 4.11.** If \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \), then \( \sqrt{\mu} \) is a fuzzy weakly completely 2-absorbing \( \Gamma \)-ideal of \( R \).

**Proof.** If \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \), then by the previous theorem, we get that \( \mu_a \) is a 2-absorbing primary \( \Gamma \)-ideal of \( R \) for any \( a \in [0, \mu (0)] \). Since \( \mu_a \) is 2-absorbing primary \( \Gamma \)-ideal of \( R \), then \( \sqrt{\mu_a} = \mu_a \) is a 2-absorbing \( \Gamma \)-ideal of \( R \). From the previous lemma, since \( \sqrt{\mu_a} \) is a 2-absorbing \( \Gamma \)-ideal of \( R \), we get that \( \sqrt{\mu} \) is a fuzzy weakly completely 2-absorbing \( \Gamma \)-ideal of \( R \). Hence we see that \( \sqrt{\mu} \) is a fuzzy weakly completely 2-absorbing \( \Gamma \)-ideal of \( R \). \qed

**Theorem 4.12.** Let \( f : R \to S \) be a surjective \( \Gamma \)-ring homomorphism. If \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \) which is constant on \( \text{Ker} f \), then \( f (\mu) \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( S \).

**Proof.** Suppose that \( f (\mu) (xay\beta z) > f (\mu) (xay) \) for any \( x, y, z \in S \) and \( \alpha, \beta \in \Gamma \). Since \( f \) is a surjective \( \Gamma \)-ring homomorphism, then
\[
f (a) = x, \ f (b) = y, \ f (c) = z \quad \text{for some } a, b, c \in R.
\]
Hence
\[
f (\mu) (xay\beta z) = f (\mu) (f (a) \alpha f (b) \beta f (c)) = f (\mu) (f (aab\beta c)) \neq f (\mu) (xay) = f (\mu) (f (a) \alpha f (b)) = f (\mu) (f (aab)).
\]
Since \( \mu \) is constant on \( \text{Ker} f \),
\[
f (\mu) (f (aab\beta c)) = \mu (aab\beta c) \quad \text{and}
\]
\[
f (\mu) (f (aab)) = \mu (aab).
\]
It means that
\[
f (\mu) (f (aab\beta c)) = \mu (aab\beta c) > \mu (aab) = f (\mu) (f (aab)).
\]
Since \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \), we have that
\[
\mu (aab\beta c) = f (\mu) (f (a) \alpha f (b) \beta f (c)) = f (\mu) (xay\beta z) \leq \sqrt{\mu} (a\beta c) = f \left( \sqrt{\mu} \right) (f (a\beta c)) = f \left( \sqrt{\mu} \right) (f (a) \beta f (c)) = f \left( \sqrt{\mu} \right) (x\beta z)
\]
so, we get \( f (\mu) (xay\beta z) \leq f \left( \sqrt{\mu} \right) (x\beta z) = \sqrt{f (\mu)} (x\beta z) \) or
\[
\mu (aab\beta c) = f (\mu) (f (a) \alpha f (b) \beta f (c)) = f (\mu) (xay\beta z) = \sqrt{\mu} (b\beta c) = f \left( \sqrt{\mu} \right) (f (b\beta c)) = f \left( \sqrt{\mu} \right) (f (b) \beta f (c)) = f \left( \sqrt{\mu} \right) (y\beta z)
\]
and we have \( f (\mu) (xay\beta z) \leq f \left( \sqrt{\mu} \right) (y\beta z) = \sqrt{f (\mu)} (y\beta z) \). Thus, \( f (\mu) \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( S \). \qed

**Theorem 4.13.** Let \( f : R \to S \) be a \( \Gamma \)-ring homomorphism. If \( v \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( S \), then \( f^{-1}(v) \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \).
Proof. Suppose that $f^{-1}(v)(xayβz) > f^{-1}(v)(xay)$ for any $x, y, z \in R$ and $α, β \in Γ$. Then

$$f^{-1}(v)(xayβz) = v(f(xayβz)) = v(f(x)αf(y)βf(z))$$
$$> f^{-1}(v)(xay) = v(f(xay)) = v(f(x)αf(y)).$$

Since $v$ is a fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $S$, we have that

$$f^{-1}(v)(xayβz) = v(f(x)αf(y)βf(z))$$
$$≤ \sqrt{v}(f(x)βf(z)) = \sqrt{v}(f(yβz))$$
$$= f^{-1}(√v)(yβz)$$
$$= \sqrt{f^{-1}(v)(yβz)}.$$

or

$$f^{-1}(v)(xayβz) = v(f(x)αf(y)βf(z))$$
$$≤ \sqrt{v}(f(y)βf(z)) = \sqrt{v}(f(yβz))$$
$$= f^{-1}(√v)(yβz)$$
$$= \sqrt{f^{-1}(v)(yβz)}.$$

Thus $f^{-1}(v)$ is a fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $R$. □

**Corollary 4.14.** Let $f$ be a $Γ$-ring homomorphism from $R$ onto $S$. $f$ induces a one-to-one inclusion preserving correspondence between fuzzy weakly completely 2-absorbing primary $Γ$-ideals of $S$ in such a way that if $μ$ is a fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $R$ constant on $\text{Ker} f$, then $f(μ)$ is the corresponding fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $S$, and if $v$ is a fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $S$, then $f^{-1}(v)$ is the corresponding fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $R$.

5. **Fuzzy $K$-2-absorbing primary $Γ$-ideals**

Let $μ$ be a fuzzy $Γ$-ideal of $R$. $μ$ is said to be a fuzzy $K$-$Γ$-ideal of $R$ if for $x, y \in R$ and $α, β \in Γ$

$$μ(xay) = μ(0)$$

implies that $μ(x) = μ(0)$ or $μ(y) = μ(0)$

and $μ$ is called a fuzzy $K$-primary $Γ$-ideal of $R$ if

$$μ(xay) = μ(0)$$

implies that $μ(x) = μ(0)$ or $√μ(y) = μ(0)$.

Also, $μ$ is called a fuzzy $K$-2-absorbing $Γ$-ideal of $R$ if

$$μ(xayβz) = μ(0)$$

implies that $μ(xay) = μ(0)$ or $μ(xβz) = μ(0)$ or $μ(yβz) = μ(0)$

for $x, y, z \in R$ and $α, β \in Γ$. Now, we give the definition of fuzzy $K$-2-absorbing primary $Γ$-ideal of $R$.

**Definition 5.1.** Let $μ$ be a fuzzy $Γ$-ideal of $R$. $μ$ is called a fuzzy $K$-2-absorbing primary $Γ$-ideal of $R$ if for $x, y, z \in R$ and $α, β \in Γ$

$$μ(xayβz) = μ(0)$$

implies that $μ(xay) = μ(0)$ or $√μ(xβz) = μ(0)$ or $√μ(yβz) = μ(0)$.

**Theorem 5.2.** Every fuzzy weakly completely 2-absorbing primary $Γ$-ideal of $R$ is a fuzzy $K$-2-absorbing primary $Γ$-ideal of $R$. 
Proof. Suppose that \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \). If \( \mu (xayb)z = \mu (0) \) for any \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \), then since \( \mu \) is a fuzzy weakly completely 2-absorbing primary \( \Gamma \)-ideal of \( R \), we have
\[
\begin{align*}
\mu (0) &\geq \mu (xay) \geq \mu (xayb)z = \mu (0) \quad \text{or} \\
\mu (0) &\geq \sqrt{\mu} (xzb) \geq \mu (xayb)z = \mu (0) \quad \text{or} \\
\mu (0) &\geq \sqrt{\mu} (ybz) \geq \mu (xayb)z = \mu (0).
\end{align*}
\]
Then,
\[
\mu (xay) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (xzb) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (ybz) = \mu (0).
\]
Therefore \( \mu \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \). \( \square \)

**Theorem 5.3.** Every fuzzy \( K \)-primary \( \Gamma \)-ideal of \( R \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. Let \( \mu \) be a fuzzy \( K \)-primary \( \Gamma \)-ideal of \( R \). Then for every \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \),
\[
\mu (xayb) = \mu (0) \quad \text{implies that} \quad \mu (x) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (y) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (z) = \mu (0).
\]
Suppose that \( \mu (x) = \mu (0) \). Then from
\[
\mu (0) = \mu (xayb) \geq \mu (xay) \geq \mu (x) = \mu (0),
\]
we get \( \mu (xay) = \mu (0) \). If \( \sqrt{\mu} (y) = \mu (0) \), then since \( \mu \) is a fuzzy \( K \)-primary \( \Gamma \)-ideal of \( R \), we have
\[
\mu (0) = \mu (xayb) \geq \sqrt{\mu} (ybz) \geq \sqrt{\mu} (y) = \mu (0).
\]
Thus, \( \sqrt{\mu} (ybz) = \mu (0) \) or if \( \sqrt{\mu} (z) = \mu (0) \). Then
\[
\mu (0) = \mu (xayb) \geq \sqrt{\mu} (xzb) \geq \sqrt{\mu} (z) = \mu (0)
\]
and we get \( \sqrt{\mu} (xzb) = \mu (0) \). We conclude that \( \mu \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \). \( \square \)

**Theorem 5.4.** Every fuzzy \( K \)-2-absorbing \( \Gamma \)-ideal of \( R \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).

Proof. The proof is obvious. \( \square \)

**Theorem 5.5.** Let \( f : R \to S \) be a surjective \( \Gamma \)-ring homomorphism. If \( \mu \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \) which is constant on \( \text{Ker} f \), then \( f (\mu) \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \).

Proof. Suppose that \( f (\mu) (aabc) = f (\mu) (0) \) for any \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \). Since \( f \) is a surjective \( \Gamma \)-ring homomorphism, then \( f (x) = a, f (y) = b, f (z) = c \) for some \( x, y, z \in R \). Hence
\[
\begin{align*}
f (\mu) (aabc) &= f (\mu) (f (x) f (y) f (z)) \\
&= f (\mu) (f (xayb))
\end{align*}
\]
and
\[
f (\mu) (0) = \vee \{ \mu (x) : f (x) = 0 \}.
\]
Thus we have \( x \in \text{Ker} f \) and so \( \mu \) is constant on \( \text{Ker} f \), \( \mu (x) = \mu (0) \)
\[
f (\mu) (0) = \vee \{ \mu (x) : \mu (x) = \mu (0) \}.
\]
Therefore we get
\[
f (\mu) (f (xayb)) = \mu (xayb) = \mu (0).
\]
Since \( \mu \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \),
\[
\mu (xayb) = \mu (0) \quad \text{implies that} \quad \mu (xay) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (xzb) = \mu (0) \quad \text{or} \quad \sqrt{\mu} (ybz) = \mu (0).
\]
Then the rest of the proof can easily be made similar to the proof of the previous theorems and we can see that \( f (\mu) \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \). \( \square \)
**Theorem 5.6.** Let \( f : R \to S \) be a \( \Gamma \)-ring homomorphism. If \( v \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \), then \( f^{-1}(v) \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Proof.** Assume that \( f^{-1}(v)(xaybaz) = f^{-1}(v)(0) \) for any \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \). Then from

\[
\begin{align*}
f^{-1}(v)(xaybaz) &= v(f(xaybaz)) = v(f(x)af(y) bf(z)) \\
&= f^{-1}(v)(0) = v(f(0)) = v(0),
\end{align*}
\]

we have \( v(f(x)af(y) bf(z)) = v(0) \). Since \( v \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \), then we get

\[
\begin{align*}
v(f(x)af(y)) &= v(0) \text{ implies that} \\
v(f(x)af(y)) &= v(0) \text{ or } \sqrt{v}(f(x)bf(z)) = v(0) \text{ or } \sqrt{v}(f(y)bf(z)) = v(0).
\end{align*}
\]

From this, we get

\[
\begin{align*}
v(f(x)af(y)) &= v(f(xay)) = f^{-1}(v)(xay) \\
v(0) &= v(f(0)) = f^{-1}(v)(0)
\end{align*}
\]

or

\[
\begin{align*}
\sqrt{v}(f(x)bf(z)) &= \sqrt{v}(f(xaybaz)) = f^{-1}(v)(\sqrt{v}(xaybaz)) \\
v(0) &= v(f(0)) = f^{-1}(v)(0)
\end{align*}
\]

or

\[
\begin{align*}
\sqrt{v}(f(y)bf(z)) &= \sqrt{v}(f(ybaz)) = f^{-1}(v)(\sqrt{v}(ybaz)) \\
v(0) &= v(f(0)) = f^{-1}(v)(0)
\end{align*}
\]

Hence \( f^{-1}(v) \) is fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \). \( \square \)

**Corollary 5.7.** Let \( f \) be a \( \Gamma \)-ring homomorphism from \( R \) onto \( S \). \( f \) induces a one-to-one inclusion preserving correspondence between fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideals of \( S \) in such a way that if \( \mu \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \) constant on \( \text{Ker} f \), then \( f(\mu) \) is the corresponding fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \), and if \( v \) is a fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( S \), then \( f^{-1}(v) \) is the corresponding fuzzy \( K \)-2-absorbing primary \( \Gamma \)-ideal of \( R \).

**Remark 5.8.** The following table summarizes findings of fuzzy 2-absorbing primary \( \Gamma \)-ideals.

| \( f \). primary \( \Gamma \)-i. | \( \rightarrow \) | f. str. 2 – abs. primary \( \Gamma \)-i. | \( \downarrow \) | \( f. \) p. \( \Gamma \)-i. | \( \rightarrow \) | f. 2 – abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. 2 – \) abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. w.c. 2 – \) abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. w.c. 2 – \) abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. K \)-\( \Gamma \)-i. | \( \rightarrow \) | f. \( K \)-2 – abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. K \)-2 – abs. \( \Gamma \)-i. | \( \downarrow \) | \( f. K \)-\( \Gamma \)-i. | \( \rightarrow \) | f. \( K \)-primary \( \Gamma \)-i. |
6. Conclusion

In this paper, the theoretical point of view of fuzzy 2-absorbing primary $\Gamma$-ideals in a $\Gamma$-ring was discussed. The work was focused on fuzzy 2-absorbing primary $\Gamma$-ideals, fuzzy weakly completely 2-absorbing primary $\Gamma$-ideals and fuzzy K-2-absorbing primary $\Gamma$-ideals of a $\Gamma$-ring and their properties were investigated. Finally, we have given a diagram in which transition between definitions of fuzzy 2-absorbing $\Gamma$-ideals of a $\Gamma$-ring are presented. These concepts are basic structures for improvement of fuzzy primary $\Gamma$-ideals in a $\Gamma$-ring. In the future, one could investigate intuitionistic fuzzy 2-absorbing primary $\Gamma$-ideals and intuitionistic fuzzy weakly completely 2-absorbing primary $\Gamma$-ideals in a $\Gamma$-ring.

References