A New Approach to Jacobsthal Quaternions

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Abstract. The Jacobsthal quaternions defined by Szynal-Liana and Wloch [35]. In this paper, we defined some properties of Jacobsthal quaternions. Also, we investigated the relations between the Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers. Furthermore, we gave the Binet formulas and Cassini identities for these quaternions.

1. Introduction

In 1973, the first use of this numbers appears “A Handbook of Integer Sequences” in a paper by Sloane by the title applications of Jacobsthal sequences to curves [1]. Furter, in 1988, Horadam [3] introduced the Jacobsthal and Jacobsthal-Lucas sequences recurrence relation \( \{J_n\} \) and \( \{j_n\} \) are defined by the recurrence relations

\[
J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2}, \quad \text{for } n \geq 2, \quad (1)
\]

\[
j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2}, \quad \text{for } n \geq 2 \quad (2)
\]

respectively.

In 1996, Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and he gave Cassini-like formulas as follows ([3],[4])

\[
J_{n+1}J_{n-1} - J_n^2 = (-1)^n \cdot 2^{n-1}, \quad (3)
\]

\[
j_{n+1}j_{n-1} - j_n^2 = 3^2 \cdot (-1)^{n+1} \cdot 2^{n-1}. \quad (4)
\]

\begin{footnotesize}
\textit{Keywords.} Jacobsthal number, Jacobsthal-Lucas number, Jacobsthal quaternion, Jacobsthal-Lucas quaternion.
\end{footnotesize}
The first eleven terms of Jacobsthal sequence \( \{J_n\} \) are \( 0, 1, 1, 3, 5, 11, 21, 43, 85, 171 \) and \( 341 \). This sequence is given by the formula
\[
J_n = \frac{2^n - (-1)^n}{3}.
\]
(5)

The first eleven terms of Jacobsthal-Lucas sequence \( \{j_n\} \) are \( 2, 1, 5, 7, 17, 31, 65, 127, 257, 511 \) and \( 1025 \). This sequence is given by the formula
\[
j_n = 2^n + (-1)^n.
\]
(6)

Also, for Jacobsthal and Jacobsthal-Lucas numbers the following properties hold [3]:
\[
J_n + j_n = 2J_{n+1},
\]
(7)
\[
j_n = J_{n+1} + 2J_{n-1},
\]
(8)
\[
3J_n + j_n = 2^{n+1},
\]
(9)
\[
j_nJ_m = J_{2n},
\]
(10)
\[
J_mJ_n + J_nj_m = 2J_{m+n},
\]
(11)
\[
J_mJ_n - J_nj_m = (-1)^n 2^{n+1} f_{m-n},
\]
(12)
\[
j_{n+1} + j_n = 3(j_{n+1} + j_n) = 3.2^n,
\]
(13)
\[
j_nJ_{m+1} + 2j_{n-1}J_m = J_{m+n},
\]
(14)
\[
j_n+1 - J_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1},
\]
(15)
\[
j_n+\tau - j_{n-r} = 3(j_{n+\tau} - J_{n-r}) = 2^n - (2^{\tau} - 1),
\]
(16)
\[
j_{n+\tau} + j_{n-r} = 3(j_{n+\tau} + j_{n-r}) + 4(-1)^{n-\tau} = 2^n(2^{\tau} + 1) + 2(-1)^{n-\tau},
\]
(17)

and summation formulas
\[
\sum_{i=2}^{n} J_i = \frac{J_{n+2} - 3}{2},
\]
(18)
\[
\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}.
\]
(19)

Several authors worked on Jacobsthal numbers and polynomials in [5]-[13].

Sum formulas for odd and even Jacobsthal and Jacobsthal-Lucas numbers were given in [8] respectively as follows,
\[
\sum_{i=0}^{n} J_{2i+1} = \frac{2J_{2n+2} + n + 1}{3},
\]
(20)
\[
\sum_{i=0}^{n} j_{2i} = \frac{2j_{2n+1} - n - 2}{3}.
\]
(21)

and
\[
\sum_{i=0}^{n} j_{2i+1} = 2j_{2n+2} - n - 1,
\]
(22)
\[ \sum_{i=0}^{n} j_{2i} = J_{2n+2} + n + 1. \]  

(23)

Identities for Jacobsthal numbers were given in [5] as follows,

\[
\begin{cases}
J_n J_{n+1} + 2J_{n-1}J_n = J_{2n} = J_n J_n, \\
J_n J_{n+1} + 2J_{n-1}J_n = J_{2n+1}, \\
J_{2n+1} = j_{2n+1} + j_{n+1} + j_{n} \quad \text{for } n \geq 1.
\end{cases}
\]  

(24)

Now, we be talked about the history of the quaternions:

The quaternions were first described by Irish mathematician Sir William Rowan Hamilton in 1843, [14]. The description is a kind of extension of complex numbers to higher spatial dimensions. The set of real quaternions, denoted by \( H \), is defined by

\[ H = \{ q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \]  

(25)

where \( i^2 = j^2 = k^2 = -1, \ i j = -j i = k, \ j k = -k j = i, \ k i = -i k = j \).

After the work of Hamilton, in 1849, Cockle introduced the set of split quaternions [15] which can be represented as

\[ H_S = \{ q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \]  

(26)

where \( i^2 = -1, \ j^2 = k^2 = 1, \ i j k = 1 \).

Several authors worked on different quaternions and their generalizations. ([16]-[20],[27]-[31]). In 2013, Akyiğit and et al. [17] defined split Fibonacci quaternions and split Lucas quaternions and obtained some identities for them. Complex split quaternions defined by Kula and Yaylı in 2007, [28]. In 1963, Horadam [21] firstly introduced the \( n \)-th Fibonacci quaternion and generalized Fibonacci quaternions, which can be represented as

\[ H_F = \{ Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \mid F_n, n - th \text{ Fibonacci number} \} \]  

(27)

where \( i^2 = j^2 = k^2 = i j k = -1, \ i j = -j i = k, \ j k = -k j = i, \ k i = -i k = j \)

and \( n \geq 1 \).

In 1969, Iyer ([26],[27]) derived many relations for the Fibonacci quaternions. Also, in 1973, Swamy [30] considered generalized Fibonacci quaternions as a new quaternion as follows:

\[ P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \]  

(28)

where

\[
\begin{align*}
H_n &= H_{n-1} + H_{n-2}, \\
H_1 &= p, \\
H_2 &= p + q, \\
H_n &= (p - q)F_n + qF_{n+1}, \ n \geq 1.
\end{align*}
\]
(See [30] for generalized Fibonacci quaternions). In 1977, Iakin ([24],[25]) introduced higher order quaternions and gave some identities for these quaternions. In 1993, Horadam ([22],[23]) extended into quaternions to the complex Fibonacci numbers defined by Harman [20]. In 2012, Halıcı [18] gave generating functions and Binet’s formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [19] defined complex Fibonacci quaternions as follows

\[ H_{FC} = \{ R_n = C_n + e_1 C_{n+1} + e_2 C_{n+2} + e_3 C_{n+3} \mid C_n = F_n + i F_{n+1}, \, \hat{\jmath}^2 = -1 \} \]  

where

\[
e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1,
\]

\[
e_1 e_2 = -e_2 e_1 = e_3, \, e_2 e_3 = -e_3 e_2 = e_1, \, e_3 e_1 = -e_1 e_3 = e_2, \, n \geq 1.
\]

In 2009, Ata and Yaylı [16] defined dual quaternions with dual numbers\(^1\) coefficient as follows:

\[ H(D) = \{ Q = A + Bi + Cj + Dk \mid A, \, B, \, C, \, D \in \mathbb{D}, \, \hat{\jmath}^2 = k^2 = -1 = i j k \} \]  

In 2014, Nurkan and Güven [29] defined dual Fibonacci quaternions as follows:

\[ H(D) = \{ Q_n = \hat{F}_n + i \hat{F}_{n+1} + j \hat{F}_{n+2} + k \hat{F}_{n+3} \mid \hat{F}_n = F_n + \epsilon F_{n+1}, \, \epsilon^2 = 0, \, \epsilon \neq 0 \}, \]  

where

\[ \hat{\jmath}^2 = \hat{\jmath} = k^2 = i j k = -1, \, i j = -j i = k, \, j k = -k j = i, \, k i = -i k = j \]

\[ n \geq 1 \text{ and } \hat{Q}_n = Q_n + \epsilon Q_{n+1}. \]

Essentially, these quaternions in equations (26) and (27) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. Majernik [32] defined dual quaternions as follows:

\[ H_D = \left\{ Q = a + bi + cj + dk \mid a, \, b, \, c, \, d \in \mathbb{R}, \, \hat{\jmath}^2 = k^2 = i j k = 0, \right\} \]  

For more details on dual quaternions, see [33]. It is clear that \( H(D) \) and \( H_2 \) are different sets. In 2015, Yüce and Torunbalcı Aydın [34] defined dual Fibonacci quaternions as follows:

\[ H_D = \{ Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3} \mid F_n, \, n \text{ th Fibonacci number} \}, \]  

where

\[ \hat{\jmath}^2 = \hat{\jmath} = k^2 = i j k = 0, \, i j = -j i = k j = -k j = i k = -i k = 0. \]

The Lucas sequence \( (L_n) \) and \( D_n^L \) which is the \( n \text{ th} \) term of the dual Lucas quaternion sequence \( (D_n^L) \) are defined by the following recurrence relations:

\[
\begin{align*}
L_{n+2} &= L_{n+1} + L_n, \quad \forall n \geq 0 \\
L_0 &= 2, \, L_1 &= 1
\end{align*}
\]

and

\[ D_n^L = L_n + i L_{n+1} + j L_{n+2} + k L_{n+3}, \]

\(^1\)Dual number: \( A = a + \epsilon b, \, a, b \in \mathbb{R}, \, \epsilon^2 = 0, \, \epsilon \neq 0. \)
In this paper, we will give the Jacobsthal quaternions as follows

\[ JQ_n = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}, \]
\[ JLQ_n = j_n + i J_{n+1} + j J_{n+2} + k J_{n+3}. \]

In [35], using (7)-(17) relations between Jacobsthal and Jacobsthal-Lucas numbers are given as follows

\[ JQ_{n+1} + JQ_n = 2^n(1 + 2i + 4j + 8k), \]
\[ JQ_{n+1} - JQ_n = \frac{1}{3}[2^n(1 + 2i + 4j + 8k) + 2(-1)^n(1 - i + j - k)], \]
\[ JQ_{n+r} + JQ_{n-r} = \frac{1}{2}[2^n(2^{2r} + 1)(1 + 2i + 4j + 8k) - 2(-1)^n(1 - i + j - k)], \]
\[ JQ_{n+r} - JQ_{n-r} = \frac{1}{3}[2^n(2^{2r} - 1)(1 + 2i + 4j + 8k)], \]
\[ N(JQ_n) = JQ_n \overline{JQ_n} = \frac{1}{9}[85.2^n + 10.2^n(-1)^n + 4], \]
\[ JLQ_{n+1} + JLQ_n = 3.2^n(1 + 2i + 4j + 8k), \]
\[ JLQ_{n+1} - JLQ_n = 2^n(1 + 2i + 4j + 8k) - 2(-1)^n(1 - i + j - k), \]
\[ JLQ_{n+r} + JLQ_{n-r} = 2^n(2^{2r} + 1)(1 + 2i + 4j + 8k) + 2(-1)^n(1 - i + j - k), \]
\[ JLQ_{n+r} - JLQ_{n-r} = [2^n(2^{2r} - 1)(1 + 2i + 4j + 8k)], \]
\[ N(JLQ_n) = 85.2^{2n} + 10.2^n(-1)^n + 4, \]
\[ JQ_n + JLQ_n = 2JQ_{n+1}. \]

In this paper, we will give the Jacobsthal quaternions as follows

\[ Q_l = \{ JQ_n = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3} \mid J_n, \text{nth Jacobsthal number} \} \]
\[ \tilde{r}^2 = \tilde{p}^2 = k^2 = i j k = -1, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j \]

and \( n \geq 1 \). The scaler and the vector part of the Jacobsthal quaternion \( JQ_n \) are denoted by

\[ S_{Q_n} = J_n \quad \text{and} \quad V_{Q_n} = i J_{n+1} + j J_{n+2} + k J_{n+3}. \]

Let \( JQ_n \) and \( JR_n \) be two Jacobsthal quaternions such that

\[ JQ_n = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3} \]

and

\[ JR_n = K_n + i K_{n+1} + j K_{n+2} + k K_{n+3} \]
where $K_n$ is the $n$-th Jacobsthal number.

Then, the addition, subtraction and multiplication of the Jacobsthal quaternions are the same as for real quaternions.

The conjugate of the Jacobsthal quaternion $JQ_n$ is denoted by $\overline{JQ_n}$ and it is

$$\overline{JQ_n} = J_n - iJ_{n+1} - jJ_{n+2} - kJ_{n+3}.$$  \hfill (54)

The norm of $JQ_n$ is defined as

$$|JQ_n| = JQ_n \overline{JQ_n} = J_n^2 + J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2.$$  \hfill (55)

2. The Properties of the Jacobsthal Quaternions

**Theorem 2.1.** Let $J_n$ and $JQ_n$ be the $n$-th terms of the Jacobsthal sequence $(J_n)$ and the Jacobsthal quaternion sequence $(JQ_n)$, respectively. In this case, for $n \geq 1$ we can give the following relations:

$$JQ_n + \overline{JQ_n} = 2J_n.$$  \hfill (56)

$$JQ_n^2 = 2J_n JQ_n - JQ_n \overline{JQ_n}.$$  \hfill (57)

$$JQ_{n+1} + 2JQ_n = JQ_{n+2}.$$  \hfill (58)

$$JQ_n - iJQ_{n+1} - jJQ_{n+2} - kJQ_{n+3} = J_n + J_{n+2} + J_{n+4} + J_{n+6}.$$  \hfill (59)

$$JQ_n JQ_m + 2JQ_{n+1} JQ_{m-1} = 2JQ_{n+m-1} - J_{n+m-1} - J_{n+m+1} - J_{n+m+3} - J_{n+m+5}.$$  \hfill (60)

**Proof.** (56): From (1.52) and (1.54) proof can easily be done.

(57): By (1.52) and (1.55)

$$JQ_n^2 = J_n^2 - J_{n+1}^2 - J_{n+2}^2 - J_{n+3}^2 + 2J_n (iJ_{n+1} + jJ_{n+2} + kJ_{n+3})$$

$$= 2J_n (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) - (J_n^2 + J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2)$$

$$= 2J_n JQ_n - JQ_n \overline{JQ_n}.$$

(58): By the equations (1.52) and

$$JQ_{n+1} = J_n + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}$$

we get,

$$JQ_{n+1} + 2JQ_n = (J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}) + 2(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3})$$

$$= (J_{n+1} + 2J_n) + i(J_{n+2} + 2J_{n+1}) + j(J_{n+3} + 2J_{n+2}) + k(J_{n+4} + 2J_{n+3})$$

$$= J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5}$$

$$= JQ_{n+2}.$$

(59): By using (1.52) and conditions (1.50) we get

$$JQ_n - iJQ_{n+1} - jJQ_{n+2} - kJQ_{n+3} = (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3})$$

$$- i(J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4})$$

$$- j(J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5})$$

$$- k(J_{n+3} + iJ_{n+4} + jJ_{n+5} + kJ_{n+6})$$

$$= J_n + J_{n+2} + J_{n+4} + J_{n+6}.$$
(60): By using (1.52), we get

\[ JQ_n JQ_m = J_n J_m - J_{n+1} J_{m+1} - J_{n+2} J_{m+2} - J_{n+3} J_{m+3} + i (J_n J_{m+1} + J_{n+1} J_m + J_{n+2} J_{m+3} - J_{n+3} J_{m+2}) + j (J_n J_{m+2} - J_{n+1} J_{m+3} + J_{n+2} J_m + J_{n+3} J_{m+1}) + k (J_n J_{m+3} + J_{n+1} J_{m+2} - J_{n+2} J_{m+1} + J_{n+3} J_m) \]  

(62)

\[ 2 JQ_{n-1} JQ_{m-1} = 2 (J_{n-1} J_{m-1} - J_n J_{m-1} - J_{n+1} J_{m+1} - J_{n+2} J_{m+2}) + 2 i (J_{n-1} J_m + J_n J_{m-1} + J_{n+1} J_{m+2} - J_{n+2} J_{m+1}) + 2 j (J_{n-1} J_{m+1} - J_n J_{m+2} + J_{n+1} J_m + J_{n+2} J_{m+1}) + 2 k (J_{n-1} J_{m+2} + J_n J_{m+1} - J_{n+1} J_m + J_{n+2} J_{m-1}) \]  

(63)

Finally, adding equations (62) and (63) side by side and using (24), we obtain

\[ JQ_n JQ_m + 2 JQ_{n-1} JQ_{m-1} = (J_n + J_m - J_{n+1} J_{m+1} - J_{n+2} J_{m+2}) + i (2 J_{n+m}) + j (2 J_{n+m+1}) + k (2 J_{n+m+2}) - (J_{n+m-1} + J_{n+m+1} + J_{n+m+3} + J_{n+m+5}) = 2 JQ_{n+m-1} - J_{n+m-1} - J_{n+m+1} - J_{n+m+3} - J_{n+m+5}. \]

\[ \square \]

**Theorem 2.2.** Let \( JQ_n \) be the Jacobsthal quaternion and \( JLQ_n \) be Jacobsthal-Lucas quaternion. The following relations are satisfied

\[ JQ_{n+1} + 2 JQ_{n-1} = JLQ_n, \]
\[ 2 JQ_{n+1} - JQ_n = JLQ_n. \]  

(64)

**Proof.** From equations (52) and (8), it follows that

\[ JQ_{n+1} + 2 JQ_{n-1} = (J_{n+1} + i J_{n+2} + j J_{n+3} + k J_{n+4}) + 2 (J_{n-1} + i J_n + j J_{n+1} + k J_{n+2}) \]
\[ = (J_{n+1} + 2 J_n) + i (J_{n+2} + 2 J_n) + j (J_{n+3} + 2 J_{n+1}) + k (J_{n+4} + 2 J_{n+2}) \]
\[ = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3} = JLQ_n. \]

and

\[ 2 JQ_{n+1} - JQ_n = 2 (J_{n+1} + i J_{n+2} + j J_{n+3} + k J_{n+4}) - (J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}) \]
\[ = (2 J_{n+1} - J_n) + i (J_{n+2} - J_{n+1}) + j (2 J_{n+3} - J_{n+2}) + k (2 J_{n+4} - J_{n+3}) \]
\[ = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3} = JLQ_n. \]

where we used (8) and \( 2 J_{n+1} - J_n = j_n \). \[ \square \]

**Theorem 2.3.** Let \( JQ_n \) be the Jacobsthal quaternion and \( \overline{JQ}_n \) be conjugate of \( JQ_n \). Then, we can give the following relations between these quaternions:

\[ JQ_n^2 = JQ_n (2 J_n - JQ_n), \]  

(65)

\[ JQ_n JQ_n + 2 JQ_{n-1} JQ_{n-1} = J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}, \]  

(66)

\[ JQ_n^2 + 2 JQ_{n-1}^2 = 2 JQ_{2n-1} - (J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}) = 2 JQ_{2n-1} - JQ_n JQ_n - 2 JQ_{n-1} JQ_{n-1}. \]  

(67)
Proof. (65): By using (52) and (55) we get

\[ JQ^2_n = (J^2_n - J_{n+1}^2 - J_{n+2}^2 - J_{n+3}^2) + 2i((J_nJ_{n+1}) + 2J_nJ_{n+2} + 2k(J_nJ_{n+3})) \]
\[ = 2J_n(J_n + iJ_{n+1} + J_{n+2} + kJ_{n+3}) - (J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2) \]
\[ = 2J_nJQ_n - JQ_n\overline{JQ_n} = JQ_n(2J_n - \overline{JQ_n}). \]

(66): By using (55) and (68) we get

\[ JQ_n\overline{JQ_n} + 2JQ_{n-1}\overline{JQ_{n-1}} = (J^2_n + J^2_{n-1} + J^2_{n+2} + J^2_{n+3} + (J^2_{n+2} + J^2_{n+3} + J^2_{n+4}) + (J^2_{n+2} + J^2_{n+3} + J^2_{n+4}) \]
\[ = J_{2n-1}J_{2n-2} + J_{2n-1}J_{2n-3} + J_{2n-3}J_{2n-4} \]
\[ = 2J_{2n-1}J_{2n-2} + J_{2n-3}J_{2n-4} \]
\[ = 2J_{2n-1}J_{2n-2} - J_{2n-3}J_{2n-4} \]
\[ = 2JQ_{2n-1} - JQ_n\overline{JQ_n} - 2JQ_{n-1}\overline{JQ_{n-1}} \]

where we used relations (24). □

Theorem 2.4. Let \( JQ_n \) be the \( n \)-th term of the Jacobsthal quaternion sequence. Then, we have the following identities

\[ \sum_{s=1}^{n} JQ_s = \frac{1}{2}[JQ_{n+2} - JQ_2]. \]  \hspace{1cm} (69)

\[ \sum_{s=0}^{n} JQ_{n+s} = \frac{1}{2}[JQ_{n+p+2} - JQ_{n+1}], \] \hspace{1cm} (70)

\[ \sum_{s=1}^{n} JQ_{2s-1} = \frac{2}{3}JQ_{2n} + \frac{1}{3}[n(2JQ_2 - JQ_3) - 2JQ_0], \] \hspace{1cm} (71)

\[ \sum_{s=1}^{n} JQ_{2s} = \frac{2}{3}JQ_{2n+1} - \frac{1}{3}[n(2JQ_2 - JQ_3) - 2JQ_1]. \] \hspace{1cm} (72)

Proof. (69): we get

\[ \sum_{s=1}^{n} JQ_s = \sum_{s=1}^{n} J_s + i \sum_{s=1}^{n} J_{s+1} + j \sum_{s=1}^{n} J_{s+2} + k \sum_{s=1}^{n} J_{s+3} \]
\[ = \frac{1}{2}[(J_{n+2} - 1) + i(J_{n+3} - 3) + j(J_{n+4} - 5) + k(J_{n+5} - 11)] \]
\[ = \frac{1}{2}[(J_{n+2} - J_2) + i(J_{n+3} - J_3) + j(J_{n+4} - J_4) + k(J_{n+5} - J_5)] \]
\[ = \frac{1}{2}[(J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5} - (J_2 + iJ_3 + jJ_4 + kJ_5)] \]
\[ = \frac{1}{2}[JQ_{n+2} - JQ_2]. \]
Hence, we can write
\[
\sum_{s=0}^{n} JQ_{n+s} = (J_{n} + \ldots + J_{n+p}) + i(J_{n+1} + \ldots + J_{n+p+1}) + j(J_{n+2} + \ldots + J_{n+p+2}) + k(J_{n+3} + \ldots + J_{n+p+3})
\]
\[
= \frac{1}{2}[(J_{n+p+2} - J_{n+1}) + i (J_{n+p+3} - J_{n+2}) + j (J_{n+p+4} - J_{n+3}) + k (J_{n+p+5} - J_{n+4})]
\]
\[
= \frac{1}{2}[J_{n+p+2} + i J_{n+p+3} + j J_{n+p+4} + k J_{n+p+5} - (J_{n+1} + i J_{n+2} + j J_{n+3} + k J_{n+4})]
\]
\[
= \frac{1}{2}[JQ_{n+p+2} - JQ_{n+1}].
\]

Using (20) and (21), we get
\[
\sum_{s=1}^{n} JQ_{2s-1} = (J_{1} + J_{3} + \ldots + J_{2n-1}) + i(J_{2} + J_{4} + \ldots + J_{2n}) + j(J_{5} + J_{5} + \ldots + J_{2n+1}) + k(J_{4} + J_{6} + \ldots + J_{2n+2})
\]
\[
= \left[\frac{2}{3}J_{2n} + i J_{2n+1} + j J_{2n+2} + k J_{2n+3}\right] + \frac{1}{3}[n(1 - i + j - k) - 2(i + j + 3k)]
\]
\[
= \frac{2}{3} JQ_{2n} + \frac{1}{3} [n(2 JQ_{2} - JQ_{3}) - 2 JQ_{0}].
\]

Using (20) and (21), we obtain
\[
\sum_{s=1}^{n} JQ_{2s} = (J_{2} + J_{4} + \ldots + J_{2n}) + i(J_{3} + J_{5} + \ldots + J_{2n+1}) + j(J_{4} + J_{6} + \ldots + J_{2n+2}) + k(J_{5} + J_{7} + \ldots + J_{2n+3})
\]
\[
= \left[\frac{2}{3}J_{2n+1} - n - 2 + i \left(\frac{2}{3}J_{2n+2} + n - 2\right) + j \left(\frac{2}{3}J_{2n+3} - n - 6\right) + k \left(\frac{2}{3}J_{2n+4} + n - 10\right)\right]
\]
\[
= \frac{2}{3}[J_{2n+1} + i J_{2n+2} + j J_{2n+3} + k J_{2n+4}] + \frac{1}{3}[-n(1 - i + j - k) - 2(1 + i + 3j + 5k)]
\]
\[
= \frac{2}{3} JQ_{2n+1} + \frac{1}{3} [-n(2 JQ_{2} - JQ_{3}) - 2 JQ_{1}].
\]

Theorem 2.5. Let $\overline{JQ_n}$ be the conjugate of the Jacobsthal quaternions $JQ_n = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}$ and $\overline{JLQ_n}$ be the conjugate of the Jacobsthal-Lucas quaternions $JLQ_n = J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}$. Then
\[
JLQ_n \overline{JQ_n} - \overline{JLQ_n} JQ_n = (-1)^n 2^n (4i + 4j + 12k),
\]
\[
JLQ_n \overline{JQ_n} + \overline{JLQ_n} JQ_n = 2 [(J_{2n} + J_{2n+2} + J_{2n+4} + J_{2n+6}) + (-1)^n 2^n (-8i - 4j - 4k)],
\]
\[
JLQ_n JQ_n - \overline{JLQ_n} \overline{JQ_n} = 2 [(J_{2n} - J_{2n+2} - J_{2n+4} - J_{2n+6}) + (-1)^n 2^n (8i - 4j - 4k)].
\]

Proof.
(73): Using the relations (12), (36) and (37), we get
The characteristic equation of recurrence relations and \( \alpha \) The roots of this equation are \( \alpha \) and \( \beta \). Using the relations (12), (36) and (37), we find \( (74) \): Using the relations (12), (36) and (37), follows \( (75) \): Using the relations (12), (36) and (37), we find

\[
JLQ_n \overline{JLQ}_n = (j_n + i j_{n+1} + j j_{n+2} + k j_{n+3}) (j_n - i j_{n+1} - j j_{n+2} - k j_{n+3})
\]

\[
\alpha = 3 + 6i + 12j + 24k, \quad \beta = 3 - 3i + 3j - 3k.
\]

\[
JQ_{n+2} = JQ_{n+1} + 2JQ_n \quad \text{and} \quad JLQ_{n+2} = JLQ_{n+1} + 2JLQ_n \quad \text{is} \quad i^2 - t - 2 = 0.
\]

The roots of this equation are \( \alpha = 2 \) and \( \beta = -1 \) where \( \alpha + \beta = 1, \quad \alpha - \beta = 3, \quad \alpha \beta = -2. \)
Using recurrence relation and initial values $JQ_0 = (0, 1, 1, 3)$, $JQ_1 = (1, 1, 3, 5)$ the Binet’s formula for $JQ_n$ we get
$$JQ_n = A \alpha^n + B \beta^n = \frac{1}{3} \left[ (1 + 2i + 4j + 8k) 2^n - (1 - i + j - k)(-1)^n \right],$$
where $A = \frac{JQ_1 - JQ_0 \beta}{\alpha - \beta}$, $B = \frac{\alpha JQ_0 - JQ_1}{\alpha - \beta}$ and $\alpha = 1 + 2i + 4j + 8k$, $\beta = 1 - i + j - k$.

Similarly, the Binet’s formula for $JLQ_n$ is obtained as follows:
$$JLQ_n = \left[ (3 + 6i + 12j + 24k) 2^n + (3 - 3i + 3j - 3k) (-1)^n \right]$$
where $\alpha = 3 + 6i + 12j + 24k$, $\beta = 3 - 3i + 3j - 3k$
respectively. □

**Theorem 2.7. (Cassini Identity).** Let $JQ_n$ and $JLQ_n$ be the $n$-th terms of the Jacobsthal quaternion sequence $(JQ_n)$ and the Jacobsthal-Lucas quaternion sequence $(JLQ_n)$, respectively. For $n \geq 1$, the Cassini identities for $JQ_n$ and $JLQ_n$ are as follows:
$$\begin{align*}
JQ_{n-1}JQ_{n+1} - JQ_n^2 &= (-1)^n 2^{n-1} (7 + 5i + 7j + 5k). \\
JLQ_{n-1}JLQ_{n+1} - JLQ_n^2 &= (-2)^{n-1} 3^2 (7 + 5i + 7j + 5k).
\end{align*}
$$

Proof. For the proof of (78) and (79), we will use relations of Jacobsthal number and Jacobsthal-Lucas number [5, 6] as follows:
$$\begin{align*}
J_mJ_{n-1} - J_{n-1}J_m &= (-1)^n 2^{n-1} J_{m-n} \\
J_mJ_{n-1} - J_{n-1}J_m &= (-2)^{n-1} 3^2 J_{m-n}
\end{align*}
$$

(78): Using the relations (3) and (80), we get
$$\begin{align*}
JQ_{n-1}JQ_{n+1} - JQ_n^2 &= (J_{n-1} + i J_n + j J_{n+1} + k J_{n+2})(J_{n+1} + i J_{n+2} + j J_{n+3} + k J_{n+4}) \\
&\quad - (J_n + i J_{n+1} + j J_{n+2} + k J_{n+3})(J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}) \\
&= [(J_{n-1}J_{n+1} - J_n^2) - (J_{n+1}J_{n+2} - J_{n+1}^2) - (J_{n+2}J_{n+3} - J_{n+2}^2) - (J_{n+3}J_{n+4} - J_{n+3}^2)] \\
&\quad + i[(J_{n-1}J_{n+1} - J_n^2) - (J_{n+1}J_{n+2} - J_{n+1}^2) - (J_{n+2}J_{n+3} - J_{n+2}^2) - (J_{n+3}J_{n+4} - J_{n+3}^2)] \\
&\quad + j[(J_{n+1}J_{n+2} - J_{n-1}J_n) - (J_{n+1}J_{n+2} - J_{n+2}^2) + J_{n+1}^2] \\
&\quad + k[(J_{n+2}J_{n+3} - J_{n-1}J_n) - (J_{n+2}J_{n+3} - J_{n+2}^2) + J_{n+2}^2] \\
&= (-1)^n 2^{n-1} (7 + 5i + 7j + 5k).
\end{align*}
$$

(79): Using the relations (4) and (81), we obtain
$$\begin{align*}
JLQ_{n-1}JLQ_{n+1} - JLQ_n^2 &= (J_{n-1} + i J_n + j J_{n+1} + k J_{n+2})(J_{n+1} + i J_{n+2} + j J_{n+3} + k J_{n+4}) \\
&\quad - (J_n + i J_{n+1} + j J_{n+2} + k J_{n+3})(J_n + i J_{n+1} + j J_{n+2} + k J_{n+3}) \\
&= [(J_{n-1}J_{n+1} - J_n^2) - (J_{n+1}J_{n+2} - J_{n+1}^2) - (J_{n+2}J_{n+3} - J_{n+2}^2) - (J_{n+3}J_{n+4} - J_{n+3}^2)] \\
&\quad + i[(J_{n-1}J_{n+1} - J_n^2) - (J_{n+1}J_{n+2} - J_{n+1}^2) - (J_{n+2}J_{n+3} - J_{n+2}^2) - (J_{n+3}J_{n+4} - J_{n+3}^2)] \\
&\quad + j[(J_{n+1}J_{n+2} - J_{n-1}J_n) - (J_{n+1}J_{n+2} - J_{n+2}^2) + J_{n+1}^2] \\
&\quad + k[(J_{n+2}J_{n+3} - J_{n-1}J_n) - (J_{n+2}J_{n+3} - J_{n+2}^2) + J_{n+2}^2] \\
&= (-2)^{n-1} 3^2 (7 + 5i + 7j + 5k).
\end{align*}
$$

□
We will give an example in which we check in a particular case the Cassini identity for the Jacobsthal quaternions.

**Example 1.** Let \( \mathcal{JQ}_1, \mathcal{JQ}_2, \mathcal{JQ}_3 \) and \( \mathcal{JQ}_4 \) be the Jacobsthal quaternions such that

\[
\begin{align*}
\mathcal{JQ}_1 &= 1 + i + 3j + 5k, \\
\mathcal{JQ}_2 &= 1 + 3i + 5j + 11k, \\
\mathcal{JQ}_3 &= 3 + 5i + 11j + 21k, \\
\mathcal{JQ}_4 &= 5 + 11i + 21j + 43k.
\end{align*}
\]

In this case,

\[
\begin{align*}
\mathcal{JQ}_1 \mathcal{JQ}_3 - \mathcal{JQ}_2^2 &= (1 + i + 3j + 5k)(3 + 5i + 11j + 21k) - (1 + 3i + 5j + 11k)^2 \\
&= (-140 + 16i + 24j + 32k) - (-154 + 6i + 10j + 22k) \\
&= (14 + 10i + 14j + 10k) \\
&= (-1)^2(7 + 5i + 7j + 5k)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{JQ}_2 \mathcal{JQ}_4 - \mathcal{JQ}_3^2 &= (1 + 3i + 5j + 11k)(5 + 11i + 21j + 43k) - (3 + 5i + 11j + 21k)^2 \\
&= (-606 + 10i + 38j + 106k) - (-578 + 30i + 66j + 126k) \\
&= (-28 - 20i - 28j - 20k) \\
&= (-1)^3(7 + 5i + 7j + 5k).
\end{align*}
\]

**Example 2.** Let \( \mathcal{JLQ}_1, \mathcal{JLQ}_2, \mathcal{JLQ}_3 \) and \( \mathcal{JLQ}_4 \) be the Jacobsthal-Lucas quaternions such that

\[
\begin{align*}
\mathcal{JLQ}_1 &= 1 + 5i + 7j + 17k, \\
\mathcal{JLQ}_2 &= 5 + 7i + 17j + 31k, \\
\mathcal{JLQ}_3 &= 7 + 17i + 31j + 65k, \\
\mathcal{JLQ}_4 &= 17 + 31i + 65j + 127k.
\end{align*}
\]

In this case,

\[
\begin{align*}
\mathcal{JLQ}_1 \mathcal{JLQ}_3 - \mathcal{JLQ}_2^2 &= (1 + 5i + 7j + 17k)(7 + 17i + 31j + 65k) - (5 + 7i + 17j + 31k)^2 \\
&= (-1400 - 20i + 44j + 220k) - (-1274 + 70i + 170j + 310k) \\
&= (-126 - 90i + 126j + 90k) \\
&= (-2)3^2(7 + 5i + 7j + 5k)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{JLQ}_2 \mathcal{JLQ}_4 - \mathcal{JLQ}_3^2 &= (5 + 7i + 17j + 31k)(17 + 31i + 65j + 127k) - (7 + 17i + 31j + 65k)^2 \\
&= (-5174 + 418i + 686j + 1090k) - (-5426 + 238i + 434j + 910k) \\
&= (252 - 1120i + 252j - 910k) \\
&= (-2)^23^2(7 + 5i + 7j + 5k).
\end{align*}
\]
References