Topological Properties of a Pair of Relation-Based Approximation Operators

Yan-lan Zhang\textsuperscript{a}, Chang-qing Li\textsuperscript{b}

\textsuperscript{a}College of Computer, Minnan Normal University, Zhangzhou, Fujian 363000, China
\textsuperscript{b}School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, China

Abstract. Rough set theory is an important tool for data mining. Lower and upper approximation operators are two important basic concepts in the rough set theory. The classical Pawlak rough approximation operators are based on equivalence relations and have been extended to relation-based generalized rough approximation operators. This paper presents topological properties of a pair of relation-based generalized rough approximation operators. A topology is induced by the pair of generalized rough approximation operators from an inverse serial relation. Then, connectedness, countability, separation property and Lindelöf property of the topological space are discussed. The results are not only beneficial to obtain more properties of the pair of approximation operators, but also have theoretical and actual significance to general topology.

1. Introduction

Rough set theory was proposed by Pawlak to conceptualize, organize and analyze various types of data in data mining. The rough set method is especially useful for dealing with vagueness and granularity in information systems. It deals with the approximation of an arbitrary subset of a universe by two definable subsets which are referred to as the lower and upper approximations. By using the lower and upper approximations of decision classes, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The lower and upper approximation operators in the Pawlak’s rough set model \cite{15} are induced by equivalence relations or partitions. However, the requirement of an equivalence relation or partition in the Pawlak’s rough set model may limit the applications of the rough set model. Then, many authors have generalized the notion of approximation operators by using more general binary relations \cite{3, 23, 25, 30, 31}, by employing coverings \cite{1, 2, 32, 35}, by utilizing adjoint operators \cite{14}, or by considering the fuzzy environment \cite{5, 12, 29}.

Topology is a branch of mathematics. There exist near connections between topology and rough set theory. Many authors investigated topological structures of rough sets \cite{8–10, 16–22, 27, 28, 31, 33, 35}.
Skowron explored the topic of topology in information systems [22]. Wiweger extended the Pawlak rough sets to topological rough sets [27]. Yao discussed the Pawlak’s rough sets through topological properties of lower and upper approximation operations [31]. Lin and Liu investigated axioms for approximation operators within the framework of topological spaces [10]. Wu and Mi examined topological structure of generalized rough sets in infinite universes of discourse [28]. Polkowski defined the hit-or-miss topology on rough sets and proposed a scheme to approximate mathematical morphology within the general paradigm of soft computing [17, 18]. Kondo presented topological properties of a type of relation-based rough sets [8]. Qin et al. [20], Zhang et al. [33] and Li et al. [9] presented a further investigation of the pair of relation-based approximation operators studied in [8]. Pomykala studied topological properties of two pairs of covering-based rough set approximation operators [19]. Zhu explored a type of covering-based rough sets by topological approach [35]. Zhang et al. presented topological properties of four pairs of relation-based generalized approximation operators [34].

The purpose of this paper is to discuss topological properties of a pair of relation-based generalized approximation operators. In Section 2, we present definitions and properties of the operators. In Section 3, we investigate connectedness, countability, separation property and Lindelöf property of the topological space induced by the operators, and present relationships between the connectedness of topological space and the existence of definable sets in rough sets to show an application of the results on topological structure of the relation-based generalized rough approximation operators.

2. Definitions and Properties of Generalized Approximation Operators

Suppose $U$ is a non-empty set called the universe, and $\mathcal{P}(U)$ is the power set of $U$. For $X \subseteq U$, $\neg X$ is the complement of $X$ in $U$. We do not restrict the universe to be finite.

Let $U$ be a nonempty set and $R$ a binary relation on $U$. For any $(x, y) \in U \times U$, if $(x, y) \in R$, then we say $x$ has relation $R$ with $y$, and denote this relationship as $xRy$. For a binary relation $R$, $\{(y, x) \in R \}$ is denoted by $R^{-1}$. For any $x \in U$, we call the set $\{y \in U \mid xRy\}$ the successor neighborhood of $x$ in $R$ and denote it as $R_s(x)$, and the set $\{y \in U \mid yRx\}$ the predecessor neighborhood of $x$ in $R$ and denote it as $R_p(x)$. Let $R$ be a binary relation on $U$.

If for any $x \in U$, there exists a $y \in U$ such that $yRx$, then $R$ is referred to as an inverse serial relation. In other words, if for any $x \in U$, $x \in R_s(x)$, then $R$ is reflexive.

If for any $x \in U$, $xRx$, then $R$ is referred to as a reflexive relation. In other words, if for any $x \in U$, $x \in R_s(x)$, then $R$ is reflexive.

If for any $x, y \in U$, $xRy \Rightarrow yRx$, then $R$ is referred to as symmetric. In other words, if for any $x, y \in U$, $y \in R_s(x)$, then $R$ is symmetric.

If for any $x, y, z \in U$, $xRy$ and $yRz \Rightarrow xRz$, then $R$ is referred to as transitive. In other words, if for any $x, y \in U$, $y \in R_s(x)$, then $R$ is transitive.

If $R$ is reflexive, symmetric and transitive, then $R$ is referred to as an equivalence relation on $U$.

Clearly, a binary relation $R$ is inverse serial if and only if $\{R_s(x) \mid x \in U\}$ is a cover of $U$. It is easy to see that a reflexive relation is inverse serial, but the converse does not hold. Besides reflexive, symmetric and transitive relations, inverse serial relation is ubiquitous in real life. We give two examples.

Example 2.1. An incomplete information system $S = (U, AT)$ is presented in Table 1, where $U = \{x_1, x_2, \ldots, x_6\}$, $AT = \{a, b\}$ is the conditional attribute set, $a, b$ stand for systolic pressure, diastolic pressure, respectively. $V_a = \{H, N, L\}$, $V_b = \{H, N, L\}$, where $H, N, \text{ and } L$ stand for high, normal and low, respectively. For any $c \in AT$, $c : U \rightarrow V_c$, i.e., $c(x) \in V_c$ for all $x \in U$. 
Continued from Example 2.2. For any applications. Let $R$ be a binary relation on the universe $U$. Then, for any $X \subseteq U$, Proposition 2.5. implies that a binary relation $R^2$ on $U$ by $x R^2 y$ if and only if $c(x) \cap (c(y)) \neq \emptyset$ for all $c \in A (A \subseteq AT)$, which is called weak right negative similarity [14].

In Table 1, we have $R^1_{[a]} = \{(x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_1), (x_3, x_2), (x_3, x_5), (x_3, x_6), (x_4, x_1), (x_4, x_2), (x_4, x_6), (x_5, x_3), (x_6, x_3), (x_6, x_4)\}$, $R^2_{[a]} = \{(x_1, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_6), (x_3, x_1), (x_3, x_2), (x_3, x_4), (x_3, x_6), (x_4, x_1), (x_4, x_2), (x_4, x_3), (x_4, x_5), (x_5, x_1), (x_5, x_2), (x_5, x_4), (x_5, x_6), (x_6, x_1), (x_6, x_2), (x_6, x_4)\}$. Hence $R^1_{[a]}$ is inverse serial and symmetric, and $R^1_{[a]}$ is neither reflexive nor transitive. $R^2_{[a]}$ is inverse serial, and $R^2_{[a]}$ is not reflexive, symmetric or transitive.

Example 2.2. Let $U = \{a, b, c, d, e, f, g\}$ be a small class of students. The class has elected its leader. Define a binary relation on $U$ by:

$x R y$ if and only if $x$ chooses $y$ as the leader.

Suppose that the election result is $R = \{(a, a), (a, c), (b, a), (b, c), (c, b), (c, d), (d, c), (d, f), (e, f), (e, c), (f, a), (f, c), (g, b), (g, g)\}$. Then $R$ is inverse serial, and $R$ is not reflexive, symmetric or transitive.

Yao has generalized the Pawlak rough set model by using general binary relations [31], and presented the pair of relation-based generalized approximation operators.

Definition 2.3. (31) Let $R$ be a binary relation on $U$. Define a pair of approximation operators $(\text{apr}^r, \overline{\text{apr}}^r)$ by:

$$\text{apr}^r(X) = \overline{\text{apr}}^r(-X) = \{x | R_s(y) \Rightarrow R_s(y) \subseteq X \} \cup (- \cup \{R_s(x) | x \in U\}),$$

$$\overline{\text{apr}}^r(X) = \cup \{R_s(x) \cap X \neq \emptyset\}.$$

If $R$ is an inverse serial relation, then $\text{apr}^r(X) = \{x | R_s(y) \Rightarrow R_s(y) \subseteq X\}$. It is easy to obtain that the approximation operators $(\text{apr}^r, \overline{\text{apr}}^r)$ are the classical Pawlak approximation operators if $R$ is an equivalence relation. We employ the next example to show that the approximation operators $(\text{apr}^r, \overline{\text{apr}}^r)$ have practical applications.

Example 2.4. Continued from Example 2.2. For any $z \in U$, $\overline{\text{apr}}^r(|z|) = \overline{\text{apr}}^r(|z|) \subseteq \text{apr}^r(X)$. Then, for any $y \in \overline{\text{apr}}^r(|z|) \setminus \{z\}$, there exists an $x \in U$ such that $y \in R_s(x)$ and $R_s(x) \cap (z) \neq \emptyset$. Hence $\{y, z\} \subseteq R_s(x)$, which implies that $x$ chooses $z$ and $y$ in the meantime. Therefore, $y$ is a competitor of $z$.

It is easy to obtain some properties of the pairs of approximation operators $(\text{apr}^r, \overline{\text{apr}}^r)$.

Proposition 2.5. [31] Let $R$ be a binary relation on the universe $U$. Then, for any $X, Y \subseteq U$,

1. $\text{apr}^r(U) = U, \overline{\text{apr}}^r(\emptyset) = \emptyset$,
2. if $R$ is inverse serial, then $\text{apr}^r(X) \subseteq X \subseteq \overline{\text{apr}}^r(X)$,
3. $\text{apr}^r(X \cap Y) = \text{apr}^r(X) \cap \text{apr}^r(Y), \overline{\text{apr}}^r(X \cup Y) = \overline{\text{apr}}^r(X) \cup \overline{\text{apr}}^r(Y)$,
4. $\text{apr}^r(\text{apr}^r(X)) \subseteq \text{apr}^r(X), \overline{\text{apr}}^r(\overline{\text{apr}}^r(X)) \subseteq \overline{\text{apr}}^r(\text{apr}^r(X))$,
5. $\overline{\text{apr}}^r(X) = \cup_{x \in \text{apr}^r(X)} \{x\}$ for all $X \neq \emptyset$,
6. $\text{apr}^r(X) = X \Leftrightarrow \overline{\text{apr}}^r(X) = X$. 

<table>
<thead>
<tr>
<th>Table 1: An information system.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
</tr>
<tr>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
</tr>
<tr>
<td>$x_5$</td>
</tr>
<tr>
<td>$x_6$</td>
</tr>
</tbody>
</table>
The properties (**) of a binary relation $R$ was introduced in [34] as a necessary and sufficient condition for $\text{apr}''$ to be idempotent.

**Definition 2.6.** ([34]) $R$ is said to have property (**) if for any $x, y \in U$, whenever $\{u, v\} \subseteq R_s(x)$ and $\{v, w\} \subseteq R_s(y)$, there exists a $z \in U$ such that $\{u, w\} \subseteq R_s(z)$.

**Theorem 2.7.** ([34]) Let $R$ be a binary relation on the universe $U$. Then the following are equivalent:

1. $R$ satisfies (**),
2. $\text{apr}''(\text{apr}''(X)) = \text{apr}''(X)$ for all $X \subseteq U$,
3. $\text{apr}''(\text{apr}''(X)) = \text{apr}''(X)$ for all $X \subseteq U$.

### 3. Topological Properties of the Generalized Upper (Lower) Approximation Operators

Topology is a theory with many applications not only in almost all branches of mathematics, but also in many real-life applications. Binary relation on a set is a simple mathematical model to which many real-life data can be connected. There exist many results on the relationships between topological spaces and binary relations.

**Definition 3.1.** ([4, 6]) Let $U$ be a non-empty set and $cI : P(U) \to P(U)$. For any $X, Y \subseteq U$, consider the following axioms:

1. $cI(\emptyset) = \emptyset$,
2. $X \subseteq cI(X)$,
3. $cI(X \cup Y) = cI(X) \cup cI(Y)$,
4. $cI(cI(X)) = cI(X)$,
5. $cI(X) = \bigcup_{x \in X} cI(x)$.

If $cI$ satisfies (1)–(3), then $cI$ is called a closure operator, and $(U, cI)$ is called a closure space [4]. If $cI$ satisfies (1)–(4), that is, $cI$ satisfies Kuratowski closure axiom, then $cI$ is called a topological closure operator [6]. If a closure operator $cI$ satisfies (5), then $cI$ is called a quasi-discrete closure operator [4].

In fact, in a closure space $(U, cI)$, it is easy to prove that $cI(X) = \{x \in U | cI(x) = X\}$ is a topology. Similarly, the topological interior operator can be defined by corresponding axioms.

**Proposition 3.2.** ([34]) Let $R$ be a binary relation on $U$. Then the following are equivalent:

1. $cI(\text{apr}'' = \{X \subseteq U | \text{apr}''(X) = X\}$ is a topology,
2. $R$ is inverse serial,
3. $\text{apr}''$ is a closure operator,
4. $\text{apr}''$ is a quasi-discrete closure operator.

To present a necessary and sufficiency condition for $\text{apr}''$ (appr'') being a topological closure (interior) operator, we define a binary relation $R'$ from the binary relation $R$ by:

$xR'y$ if and only if there exists a $z \in U$ such that $\{x, y\} \subseteq R_s(z)$.
Proposition 3.3. ([34]) Let $R$ be a binary relation on $U$. Then the following are equivalent:

1. $R'$ is an equivalence relation,
2. $R$ is inverse serial and satisfies $(**)$,
3. $apr^r$ is a topological interior operator,
4. $apr^s$ is a topological closure operator.

By Proposition 3.3, the inverse seriality and the property $(**)$ are necessary and sufficiency conditions for $apr^r$ ($apr^s$) to be a topological closure (interior) operator. In order to present more properties of the topological space $(U, \tau(\overline{apr^r}))$, we define another binary relation.

Definition 3.4. Let $R$ be a binary relation on $U$. For any $a, b, \cdots, n$ and $b_1, b_2, \cdots, b_{n-1}$ of $U$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. In this case, we write $x \tilde{R} y$.

Proposition 3.5. Let $R$ be a binary relation on $U$.

1. If $R$ is inverse serial, then $\tilde{R}$ is reflexive.
2. $\tilde{R}$ is symmetric.
3. $\tilde{R}$ is transitive.

Proof. (1) For any $x \in U$, since $R$ is inverse serial, there exists a $y \in U$ such that $x \in R_s(y)$. Then $xR^{-1}y$ and $yRx$. Hence $x\tilde{R}x$. It means that $\tilde{R}$ is reflexive.

(2) For any $x, y, z \in U$, if $x\tilde{R}y$ and $y\tilde{R}z$, then there exist $a, b, \cdots, n$ and $b_1, b_2, \cdots, b_{n-1}$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Then we have $y = b_nR^{-1}a_n, a_nRb_{n-1}, b_{n-1}R^{-1}a_{n-1}, \cdots, b_1R^{-1}a_2, b_2Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Thus, we have $x \tilde{R} y$.

(3) For any $x, y, z \in U$, if $x\tilde{R}y$ and $y\tilde{R}z$, then there exist $a, b, \cdots, n$ and $b_1, b_2, \cdots, b_{n-1}$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Then we have $y = b_nR^{-1}a_n, a_nRb_{n-1}, b_{n-1}R^{-1}a_{n-1}, \cdots, b_1R^{-1}a_2, b_2Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Thus, $x \tilde{R} y$.

From Proposition 3.5, we can obtain that if $R$ is inverse serial, then $\tilde{R}$ is an equivalence relationship, i.e., $U/\tilde{R} \cong (R_s(x) | x \in U)$ is a partition of $U$ and $[x]_{\tilde{R}} = R_s(x)$ is an equivalence class.

Proposition 3.6. If $R$ is inverse serial, then for any $x \in U$

1. $apr^r([x]_{\tilde{R}}) = [x]_{\tilde{R}}$.
2. $[x]_{\tilde{R}}$ is an open and closed subset of $(U, \tau(\overline{apr^r}))$.

Proof. (1) According to Proposition 2.5(2), we obtain $apr^r([x]_{\tilde{R}}) \subseteq [x]_{\tilde{R}}$. For any $y \in [x]_{\tilde{R}}$, there exists $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of $U$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. For any $z \in U$ with $y \in R_s(z)$, we have $R_s(z) \subseteq [x]_{\tilde{R}}$. Indeed, for any $u \in R_s(z)$, we get $yR_{\tilde{R}}z$ and $zRu$. Then, there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}, y$ of $U$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n, a_nRy, yR_{\tilde{R}}z$ and $zRu$. It implies that $u \in [x]_{\tilde{R}}$. Hence we have $y \in apr^r([x]_{\tilde{R}})$. Therefore, $[x]_{\tilde{R}} \subseteq apr^r([x]_{\tilde{R}})$.

(2) By (1), $[x]_{\tilde{R}}$ is an open set. Then, according to Proposition 2.5(6), we deduce that $[x]_{\tilde{R}}$ is a closed set.

Proposition 3.7. If $R$ is inverse serial, then

1. for any $X \in \tau(\overline{apr^r})$ and $x \in X$, $[x]_{\tilde{R}} \subseteq X$,
2. $[x]_{\tilde{R}}$ is an open neighborhood base of $x \in U$. 

Proof. (1) For any $x \in [x]_R$, there exist $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$ of $U$ such that $x = b_0 R^{-1} a_1, a_1 R b_1, b_1 R^{-1} a_2, a_2 R b_2, \ldots, b_{n-1} R^{-1} a_n$ and $a_n R y = b_n$. Then, $x \in \cap \{ v \in w \text{ if and only if } v \in x \}$.

(2) By Proposition 3.6(2) and (1), it is easy to see that $[[x]_R]$ is an open neighborhood base of $x$. □

Proposition 3.8. If $R$ is inverse serial, then for any $x \in U$,

(1) $[x] = [x]_R$.

(2) $[x]_R$ is a connected component that contains $x$.

Proof. (1) According to Proposition 2.5(6), $X \subseteq U$ is an open set in $(U, \tau(\overline{apr}^n))$, if and only if $X$ is a closed set. Then, by Proposition 3.7, we have

$$[x] = \cap \{ B \mid x \in B, \text{ and } B \text{ is closed} \} = \cap \{ B \mid x \in B, \text{ and } B \text{ is open} \} = [x]_R.$$  

(2) Assume that $C_x$ is a connected component containing $x$. Then $C_x$ is closed. It follows that $C_x$ is open. By Proposition 3.7, we get that $[x]_R \subseteq C_x$. Then $[x]_R = C_x$. Otherwise, $[x]_R \neq C_x$. Hence $[x]_R$ is an open and closed proper subset of $C_x$, which contradicts that $C_x$ is a connected component. Therefore, $[x]_R$ is a connected component that contains $x$. □

Proposition 3.9. If $R$ is inverse serial, then $(U, \tau(\overline{apr}^n))$ is a locally connected space.

Proof. For any $x \in U$ and $A \in \tau(\overline{apr}^n)$ with $x \in A$, we have $[x]_R \subseteq A$. By Proposition 3.8, $[x]_R$ is a connected set. Then $(U, \tau(\overline{apr}^n))$ is a locally connected space. □

Theorem 3.10. If $R$ is an inverse serial relation on $U$, then $(U, \tau(\overline{apr}^n))$ is connected if and only if $x \overline{R} y$ for all $x, y \in U$.

Proof. “⇒”. Suppose $U = X \cup Y$ and $X \cap Y = \emptyset$. Let $x \in X$ and $y \in Y$. By the assumption, we have $x \overline{R} y$. Then, there exist $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$ of $U$ such that $x = b_0 R^{-1} a_1, a_1 R b_1, b_1 R^{-1} a_2, a_2 R b_2, \ldots, b_{n-1} R^{-1} a_n$ and $a_n R y = b_n$. Since $x = b_1 \in X$ and $y = b_n \in Y$, there exists a $b_i$ for which $b_i \in X$. Then $b_{i+1} \in Y$. By Definition 7, $a_{i+1} R b_i$ and $b_i R^{-1} a_{i+1}$. It follows that $b_{i+1} \subseteq R \cap Y \neq \emptyset$, and $b_i \subseteq R \cap X \neq \emptyset$. Then $b_i \in \overline{apr}^n(X)$ and $b_i \not\in \overline{apr}^n(Y)$. Hence $\overline{apr}^n(X) \neq X$ and $\overline{apr}^n(Y) \neq Y$, which implies that $X$ and $Y$ are not closed. Thus, $U$ is not the union of two disjoint closed sets, that is, $U$ is connected.

“⇐”. Suppose that there exist $x, y \in U$ such that $x \overline{R} y$ does not hold. Let $X = \{ z \in U \mid x \overline{R} z \}$.

Then $X$ is closed. In fact, for any $u \in \overline{apr}^n(X)$, there exists a $v \in U$ such that $u = R(v)$ and $R(v) \cap X \neq \emptyset$. Let $w \in R(v) \cap X$. Then $w R^{-1} v$ and $v R w$. Since $w \in X$, we get $u \in X$. Hence $\overline{apr}^n(X) \subseteq X$. It follows that $\overline{apr}^n(X) = X$, which implies that $X$ is closed.

We are going to prove that $-X$ is closed. If not, $\overline{apr}^n(-X) \not\subseteq -X$. Then there exists a $u \in U$ such that $u \in \overline{apr}^n(-X)$ and $u \not\in -X$. Hence there exists a $v \in U$ such that $u \in R(v)$ and $R(v) \cap (-X) \neq \emptyset$. Let $w \in R(v) \cap (-X)$. It follows that $u R^{-1} v$ and $v R w$. Since $u \in X$, we have $w \in X$, which contradicts the fact $w \not\in X$.

Since $x \in X$ and $y \not\in X$, $U$ is the union of two disjoint non-empty closed sets, that is, $U$ is not connected. □

Definition 3.11. ([6]) Let $(U, \tau)$ be a topological space. If $A \subseteq U$ is open in $U$ if and only if $A$ is closed in $U$, then $(U, \tau)$ is called a pseudo-discrete space. If the intersection of arbitrarily many open sets in $U$ is still open, then $\tau$ is called an Alexandrov topology, and $(U, \tau)$ is said to be an Alexandrov space.

Proposition 3.12. If $R$ is an inverse serial relation on $U$, then

(1) $(U, \tau(\overline{apr}^n))$ is pseudo-discrete,

(2) $(U, \tau(\overline{apr}^n))$ is an Alexandrov space.
Proof. (1) It is easy to prove that $\operatorname{apr}''(X) = X \Leftrightarrow \operatorname{apr}''(X) = X$ for all $X \subseteq U$. Then $(U, \tau(\operatorname{apr}''))$ is quasi-discrete.

(2) Since each open set in $U$ is closed, the intersection of arbitrarily many open sets in $U$ is still open. Hence $(U, \tau(\operatorname{apr}''))$ is an Alexandrov space. □

If $R$ is inverse serial, by Proposition 3.12, for any $X \subseteq U$, $X \subseteq U$ is definable by $\operatorname{apr}''$ and $\overline{\operatorname{apr}''}$

$\Leftrightarrow \operatorname{apr}''(X) = X = \overline{\operatorname{apr}''(X)}$  

$\Leftrightarrow X$ is an open and closed set in $(U, \tau(\operatorname{apr}''))$  

Then the family of all definable subsets of $U$ is $\tau(\operatorname{apr}'')$. On the other hand,

$(U, \tau(\overline{\operatorname{apr}'}))$ is not connected

$\Leftrightarrow (U, \tau(\overline{\operatorname{apr}'}))$ has non-empty open and closed proper subsets

$\Leftrightarrow (U, \tau(\overline{\operatorname{apr}'}))$ has other definable sets besides $\emptyset$ and $U$.

$(U, \tau(\overline{\operatorname{apr}'}))$ is connected

$\Leftrightarrow (U, \tau(\overline{\operatorname{apr}'}))$ do not have non-empty open and closed proper subsets

$\Leftrightarrow$ the definable sets by $\operatorname{apr}''$ and $\overline{\operatorname{apr}''}$ are no other than $\emptyset$ and $U$.

Hence we can note that $(U, \tau(\overline{\operatorname{apr}'}))$ is connected, if and only if the definable sets by $\operatorname{apr}''$ and $\overline{\operatorname{apr}''}$ are no other than $\emptyset$ and $U$. $(U, \tau(\overline{\operatorname{apr}'}))$ is not connected, if and only if $(U, \tau(\overline{\operatorname{apr}'}))$ has other definable sets besides $\emptyset$ and $U$. Thus, there exist relationships between the connectedness of topological spaces and the existence of definable sets in approximation spaces.

Proposition 3.13. Let $R$ be an inverse serial relation on $U$. Then

(1) $(U, \tau(\overline{\operatorname{apr}'}))$ is a first countable space.

(2) $(U, \tau(\overline{\operatorname{apr}'}))$ is a locally separable space.

Proof. (1) By Proposition 3.7(2), we have that $[x]_{\overline{R}}$ is an open neighborhood base of $x$. Then $(U, \tau(\overline{\operatorname{apr}'}))$ is first countable.

(2) According to Proposition 3.8(1), $\{x\}$ is a dense subset of $[x]_{\overline{R}}$, then $[x]_{\overline{R}}$ is separable. Hence, by Proposition 3.7(1), each neighborhood of $x$ has separable subset $[x]_{\overline{R}}$. It implies that $(U, \tau(\overline{\operatorname{apr}'}))$ is locally separable. □

Proposition 3.14. Let $R$ be an inverse serial relation on $U$. Then

(1) $(U, \tau(\overline{\operatorname{apr}'}))$ is a regular space,

(2) $(U, \tau(\overline{\operatorname{apr}'}))$ is a normal space.

Proof. (1) For any $x \in U$ and closed set $B$ with $x \notin B$, by Proposition 3.12, we have that $B$ is open. Then there exist two disjoint open sets $U \setminus B$ and $B$ such that $x \in U \setminus B$ and $B \subseteq B$. Hence $(U, \tau(\overline{\operatorname{apr}'}))$ is regular.

(2) For each pair $A, B$ of disjoint closed subsets of $U$, by Proposition 3.12, we have that $A$ and $B$ are open sets. Then there exist disjoint open sets $A$ and $B$ such that $A \subseteq A$ and $B \subseteq B$. It follows that $(U, \tau(\overline{\operatorname{apr}'}))$ is normal. □

Proposition 3.15. Let $R$ be an inverse serial relation on $U$. Then the following are equivalent:

(1) $U/\overline{R}$ is countable,

(2) $(U, \tau(\overline{\operatorname{apr}'}))$ is second-countable,

(3) $(U, \tau(\overline{\operatorname{apr}'}))$ is separable,

(4) $(U, \tau(\overline{\operatorname{apr}'}))$ is a Lindelöf space.

Proof. (1)⇒(2). By Proposition 3.7, $[x]_{\overline{R}} \forall x \in U$ is a base of $(U, \tau(\overline{\operatorname{apr}'}))$. Since $U/\overline{R} = \{[x]_{\overline{R}} \forall x \in U\}$ is countable, we obtain that $(U, \tau(\overline{\operatorname{apr}'}))$ is second-countable.

(2)⇒(3). It is clear.

(3)⇒(4). Let $C$ be an open cover of $U$, and $D$ be a countable dense subset of $U$. For any $x \in D$, there exists $K_x \in C$ such that $x \in K_x$. Let $C_0 = \{K_x \forall x \in D\}$. Then $C_0$ is countable. Now we prove that $C_0$ is a cover
of \(U\). For any \(y \in U\), there exists \(K \in C\) such that \(y \in K\). Hence \([y]\) \(\subseteq K\). Since \(D\) is a dense subset of \(U\), we get \([y]\) \(\cap D \neq \emptyset\). Let \(x \in [y]\) \(\cap D\). Then there exists a \(K_1 \in C_0\) such that \(x \in K_1\). It follows that \([x]\) \(\subseteq K_1\). Since \(x \in [y]\) and \(R\) is an equivalence relation, we obtain \([x] = [y]_R\). Then \(y \in [y]_R = [x]_R \subseteq K_1\). Therefore, we can conclude that \((U, \tau(\bar{\alpha}))\) is a Lindelöf space.

(4)\(\Rightarrow\)(1). \(U/\bar{R} = ([x]_R \mid x \in U)\) is an open cover of \(U\). Since \(\bar{R}\) is an equivalence relation, \(U/\bar{R}\) is the only subcover of \(U/\bar{R}\). Since \((U, \tau(\bar{\alpha}))\) is a Lindelöf space, we obtain that \(U/\bar{R}\) is countable. \(\square\)

**Example 3.16.** Continued from Example 2.2. We have

\[
\tau(\bar{\alpha}) = \{\emptyset, \{a, c\}, \{e, f\}, \{b, a, g\}, \{a, c, e, f\}, \{a, c, b, d, g\}, \{c, f, b, d, g\}, U\},
\]

\[
\bar{R} = \{(a, a), (a, c), (c, c), (c, a), (b, b), (b, d), (d, b), (d, d), (g, b), (g, g), (g, d), (d, g), (e, c), (f, f),
\]

\[
eq (e, f), (f, f)\}.
\]

Then \(\bar{R}\) is an equivalence relation, and

\[
U/\bar{R} = \{\{a, c\}, \{e, f\}, \{b, d, g\}\}.
\]

We obtain that \((U, \tau(\bar{\alpha}))\) is quasi-discrete, regular, normal and non-connected. \(\tau(\bar{\alpha})\) is the family of definable sets by \(\bar{\alpha}\) and \(\bar{\alpha}\).

**4. Conclusion**

In this paper, we have investigated topological properties of a pair of relation-based generalized approximation operators. We have discussed connectedness, countability, separation property and Lindelöf property of the topological space induced by the approximation operators. We have described relationships between the connectedness of topological space and the existence of definable sets in rough sets to show an application of the discussion of the topological structure of the relation-based generalized rough approximation operators.

**References**


