The $\alpha AB$-, $\beta AB$-, $\gamma AB$- and $NAB$-duals for Sequence Spaces

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Abstract. Let $A = (a_{nk})$ and $B = (b_{nk})$ be two infinite matrices with real entries. The main purpose of this paper is to generalize the multiplier space for introducing the concepts of $\alpha AB$-, $\beta AB$-, $\gamma AB$-duals and $NAB$-duals. Moreover, these duals are investigated for the sequence spaces $X$ and $X(A)$, where $X \in \{c_0, c, l_p\}$ for $1 \leq p \leq \infty$. The other purpose of the present study is to introduce the sequence spaces $X(A, \Delta) = \{x = (x_k) : \sum_{k=1}^{\infty} \left( \sum_{l=1}^{k} a_{nk}x_l - \sum_{l=1}^{k-1} a_{n-1,l}x_l \right) \in X\}$, where $X \in \{l_\infty, c, c_0\}$, and computing the $NAB$-(or Null) duals and $\beta AB$-duals for these spaces.

1. Introduction

Let $\omega$ denote the space of all real-valued sequences. Any vector subspace of $\omega$ is called a sequence space. For $1 \leq p < \infty$, denote by $l_p$ the space of all real sequences $x = (x_n) \in \omega$ such that $\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty$.

For $p = \infty$, $\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$ is interpreted as $\sup_{n \geq 1} |x_n|$. We write $c$ and $c_0$ for the spaces of all convergent and null sequences, respectively. Also, $bs$ and $cs$ are used for the spaces of all bounded and convergent series, respectively. Kizmaz [8] defined the backward difference sequence space $X(\Delta) = \{x = (x_k) : \Delta x \in X\}$, for $X \in \{l_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k-1})_{k=1}^{\infty}$, $x_0 = 0$. Observe that $X(\Delta)$ is a Banach space with the norm $\|x\|_\Delta = \sup_{k \geq 1} |x_k - x_{k-1}|$.

In the summability theory, the $\beta$-dual of a sequence space is very important in connection with inclusion theorems. The idea of dual sequence space was introduced by Köthe and Toeplitz [9], and it is generalized

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to the vector-valued sequence spaces by Maddox [10]. For the sequence spaces $X$ and $Y$, the set $M(X, Y)$ defined by

$M(X, Y) = \{ z = (z_k) \in \omega : (z_k x_k)_{n=1}^{\infty} \in Y \forall x = (x_k) \in X \},$

is called the multiplier space of $X$ and $Y$. With the above notation, the $\alpha$, $\beta$, $\gamma$ and N-duals of a sequence space $X$, which are respectively denoted by $X^\alpha$, $X^\beta$, $X^\gamma$ and $X^N$, are defined by

$X^\alpha = M(X, l_1), \quad X^\beta = M(X, c_s), \quad X^\gamma = M(X, b_s), \quad X^N = M(X, c_0).$

For a sequence space $X$, the matrix domain $X(A)$ of an infinite matrix $A$ is defined by

$X(A) = \{ x = (x_n) \in \omega : A x \in X \},$ (1)

which is a sequence space. The new sequence space $X(A)$ generated by the limitation matrix $A$ from a sequence space $X$ can be the expansion or the contraction and or the overlap of the original space $X$.

In the past, several authors studied Köthe-Toeplitz duals of sequence spaces that are the matrix domains in classical spaces $l_p$, $l_\infty$, $c$ and $c_0$. For instance, some matrix domains of the difference operator was studied in [4]. Domain of backward difference matrix in the space $l_p$ was investigated for $1 \leq p \leq \infty$ by Başar and Altay in [3] and was studied for $0 < p < 1$ by Altay and Başar in [1]. Recently the Köthe-Toeplitz duals were computed for some new sequence spaces by Erfanmanesh and Foroutannia [5], [6] and Foroutannia [7]. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [2].

In this study, the concept of the multiplier space is generalized and the $\alpha AB$, $\beta AB$, $\gamma AB$- and NAB-duals are determined for the classical sequence spaces $l_{\omega_1}$, $c$ and $c_0$. Also the normed sequence space $X(\Delta)$ is extended to semi-normed space $X(\Delta, \Lambda)$, where $X \in \{l_{\omega_1}, c, c_0\}$. We consider some topological properties of this space and derive inclusion relations concerning with its. Moreover, we compute the NAB-(or Null) duals for the space $X(\Delta, \Lambda)$. The results are generalizations of some results of Malkowsky and Rakocevic [11], Kizmaz [8] and Erfanmanesh and Foroutannia [5].

2. The Generalized Multiplier Space and its Köthe-Toeplitz Duals and Null Duals

In this section, we introduce the generalization of multiplier space and present the new generalizations of Köthe-Toeplitz duals and Null duals of sequence spaces. Furthermore, we obtain these duals for the sequence spaces $l_{\omega_1}$, $c$ and $c_0$. Throughout this paper, let $I$ be the identity matrix.

**Definition 2.1.** Suppose that $A = (a_{n,k})$ and $B = (b_{n,k})$ are two infinite matrices with real entries such that $\sum_{k=1}^{\infty} a_{n,k} z_k < \infty$ for all $x = (x_k) \in X$ and $n = 1, 2, \ldots$. For the sequence spaces $X$ and $Y$, the set $M_{A,B}(X, Y)$ defined by

$M_{A,B}(X, Y) = \{ z \in \omega : \sum_{k=1}^{\infty} b_{n,k} z_k < \infty, \forall n \text{ and } \left( \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k \right)_{n=1}^{\infty} \in Y, \forall x \in X \},$

is called the generalized multiplier space of $X$ and $Y$.

The $\alpha AB$, $\beta AB$, $\gamma AB$- and NAB-duals of a sequence space $X$, which are respectively denoted by $X^{\alpha AB}$, $X^{\beta AB}$, $X^{\gamma AB}$ and $X^{NAB}$, are defined by

$X^{\alpha AB} = M_{A,B}(X, l_1), \quad X^{\beta AB} = M_{A,B}(X, c_s), \quad X^{\gamma AB} = M_{A,B}(X, b_s), \quad X^{NAB} = M_{A,B}(X, c_0).$

It should be noted that in the special case $A = B = I$, we have $M_{A,B}(X, Y) = M(X, Y)$. So

$X^{\alpha AB} = X^\alpha, \quad X^{\beta AB} = X^\beta, \quad X^{\gamma AB} = X^\gamma, \quad X^{NAB} = X^N.$

Let $E = (E_n)$ and $F = (F_n)$ be two partitions of finite subsets of the positive integers such that

$max E_n < min E_{n+1}, \quad max F_n < min F_{n+1},$
for $n = 1, 2, \ldots$. If the infinite matrices $A = (a_{n,k})$ and $B = (b_{n,k})$ are defined by

$$a_{n,k} = \begin{cases} 1 & \text{if } k \in E_n \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

and

$$b_{n,k} = \begin{cases} 1 & \text{if } k \in F_n \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

then $M_{A,B}(X, Y) = M_{E,F}(X, Y)$ and the new multiplier space $M_{A,B}(X, Y)$ is a generalization of the multiplier space $M_{E,F}(X, Y)$ introduced in [5].

**Lemma 2.2.** Let $X, Y, Z \subset \omega$ and $\{X_\delta: \delta \in I\}$ be any collection of subsets of $\omega$, then

(i) $X \subset Z$ implies $M_{A,B}(Z, Y) \subset M_{A,B}(X, Y)$,

(ii) $Y \subset Z$ implies $M_{A,B}(X, Y) \subset M_{A,B}(X, Z)$,

(iii) $X \subset M_{A,B}(M_{A,B}(X, Y), Y)$,

(iv) $M_{A,B}(X, Y) = M_{A,B}(M_{A,B}(X, Y), Y), Y$,

(v) $M_{A,B}(\bigcup_{\delta \in I} X_\delta, Y) = \bigcap_{\delta \in I} M_{A,B}(X_\delta, Y)$.

**Proof.** Parts (i) and (ii) are obvious, by using the definition of generalized multiplier space.

(iii) Let $x \in X$. We have $(\sum_{k=1}^{\infty} a_{n,k} x_k)^{\infty}_{k=1} \in Y$ for all $z \in M_{A,B}(X, Y)$, and consequently $x \in M_{A,B}(M_{A,B}(X, Y), Y)$.

(iv) By applying (iii) with $X$ replaced by $M_{A,B}(X, Y)$, we deduce that

$$M_{A,B}(X, Y) \subset M_{A,B}(M_{A,B}(X, Y), Y).$$

Conversely, due to (iii), we have $X \subset M_{A,B}(M_{A,B}(X, Y), Y)$. So

$$M_{A,B}(M_{A,B}(M_{A,B}(X, Y), Y), Y) \subset M_{A,B}(X, Y),$$

by part (i).

(v) First, $X_\delta \subset \bigcup_{\delta \in I} X_\delta$ for all $\delta \in I$ implies

$$M_{A,B}(\bigcup_{\delta \in I} X_\delta, Y) \subset \bigcap_{\delta \in I} M_{A,B}(X_\delta, Y),$$

by part (i). Conversely, if $a \in \bigcap_{\delta \in I} M_{A,B}(X_\delta, Y)$, then $z \in M_{A,B}(X_\delta, Y)$ for all $\delta \in I$. So

$$\left(\sum_{k=1}^{\infty} b_{n,k} x_k \sum_{k=1}^{\infty} a_{n,k} x_k\right)^{\infty}_{n=1} \in Y,$$

for all $\delta \in I$ and for all $x \in X_\delta$. This implies $(\sum_{k=1}^{\infty} b_{n,k} x_k \sum_{k=1}^{\infty} a_{n,k} x_k)^{\infty}_{n=1} \in Y$ for all $x \in \bigcup_{\delta \in I} X_\delta$, hence $z \in M_{A,B}(\bigcup_{\delta \in I} X_\delta, Y)$. Thus $\bigcap_{\delta \in I} M_{A,B}(X_\delta, Y) \subset M_{A,B}(\bigcup_{\delta \in I} X_\delta, Y)$. \hfill \Box

**Remark 2.3.** If $A = B = I$, we have Lemma 1.25 from [11].

**Remark 2.4.** If two matrices $A$ and $B$ are defined by (2) and (3), then we obtain Lemma 2.1 from [5].

If $\dagger$ denotes either of the symbols $\alpha, \beta, \gamma$ or $N$, from now on we will use the following notation

$$(X^{\dagger AB})^{\dagger AB} = X^{\dagger AB}.$$
Corollary 2.5. Let \( X, Y \subseteq \omega \) and \( \{ X_\delta : \delta \in I \} \) be any collection of subsets of \( \omega \), also \( + \) denotes either of the symbols \( \alpha, \beta, \gamma \), or \( N \), then

(i) \( X^{AB} \subseteq X^{BA} \subseteq X^{\gamma AB} \subseteq \omega \); in particular, \( X^{\gamma AB} \) is a sequence space.
(ii) \( X \subseteq Z \) implies \( Z^{AB} \subseteq X^{\gamma AB} \).
(iii) \( X \subseteq X^{ABA} \).
(iv) \( X^{AA} \subseteq X^{AAA} \).
(v) \( (\bigcup_{\delta \in I} X_\delta)^{AB} \subseteq \bigcap_{\delta \in I} X_\delta^{AB} \).

Remark 2.6. If \( A = B = I \), we have Corollary 2.16 from [11].

Remark 2.7. If two matrices \( A \) and \( B \) are defined by (2) and (3), then we obtain Corollary 2.1 from [5].

Below, we determine the generalized multiplier space for some sequence spaces. For this purpose, we recall the following theorem from [11]. Let \( X \) and \( Y \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real numbers \( a_{nk} \), where \( n, k \in \mathbb{N} = \{1, 2, \cdots \} \). We say that \( A \) defines a matrix mapping from \( X \) into \( Y \), and we denote it by \( A : X \rightarrow Y \), if for every sequence \( x \in X \) the sequence \( Ax = \{(Ax)_n\}_{n=1}^{\infty} \) exists and is in \( Y \), where \((Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k \) for \( n = 1, 2, \cdots \). By \((X, Y)\), we denote the class of all infinite matrices \( A \) such that \( A : X \rightarrow Y \). We consider the conditions

\[
\sup_n \left( \sum_{k=1}^{\infty} |a_{nk}| \right) < \infty, \quad (4)
\]

\[
limit_{n \to \infty} a_{nk} = 0 \quad (k = 1, 2, \cdots), \quad (5)
\]

\[
limit_{n \to \infty} a_{nk} = l_k \quad \text{for some } l_i \in \mathbb{R} \quad (i = 1, 2, \cdots), \quad (6)
\]

\[
\sum_{k=1}^{\infty} a_{nk} = l \quad \text{for some } l \in \mathbb{R}. \quad (7)
\]

With the notation of (1), the spaces \( l_\omega(A), c(A) \) and \( c_0(A) \) contain all of the sequences \( x = (x_n) \) that \( Ax = \{(Ax)_n\} \) are the bounded, convergent and null sequences, respectively.

Theorem 2.8. ([11], Theorem 1.36) We have

(i) \( A \in (l_\omega, l_\omega) \) if and only if the condition (4) holds, in this case \( l_\omega \subseteq l_\omega(A) \);
(ii) \( A \in (c_0, c_0) \) if and only if the conditions (4) and (5) hold, in this case \( c_0 \subseteq c_0(A) \);
(iii) \( A \in (c, c) \) if and only if the conditions (4), (6) and (7) hold, in this case \( c \subseteq c(A) \);
(iv) \( A \in (c_0, c) \) if and only if the conditions (4) and (6) hold, in this case \( c_0 \subseteq c(A) \).

Theorem 2.9. Let \( A \) be an invertible matrix. We have the following statements.

(i) \( M_{A,B}(c_0, X) = l_\omega(B) \), where \( X \subseteq \{l_\omega, c, c_0\} \) and \( A \) satisfies the conditions (4) and (5);
(ii) \( M_{A,B}(l_\omega, X) = c_0(B) \), where \( X \subseteq \{c, c_0\} \) and \( A \) satisfies the condition (4);
(iii) If in addition \( \sum_{k=1}^{\infty} a_{nk} = R \) for all \( n \), then \( M_{A,B}(c, c) = l_\omega(B) \) and \( A \) satisfies the conditions (4), (6) and (7).

Proof. (i) Since \( c_0 \subseteq c \subseteq l_\omega \), by applying Lemma 2.2(ii), we have

\[
M_{A,B}(c_0, c_0) \subseteq M_{A,B}(c_0, c) \subseteq M_{A,B}(c_0, l_\omega).
\]

So it is sufficient to verify \( l_\omega(B) \subseteq M_{A,B}(c_0, c_0) \) and \( M_{A,B}(c_0, l_\omega) \subseteq l_\omega(B) \). Suppose that \( z \in l_\omega(B) \) and \( x \in c_0 \). Due to Theorem 2.8(ii) we have \( x \in c_0(A) \), so

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{\infty} b_{nk}z_k \sum_{k=1}^{\infty} a_{nk}x_k \right) = 0,
\]

this means that \( z \in M_{A,B}(c_0, c_0) \). Thus \( l_\omega(B) \subseteq M_{A,B}(c_0, c_0) \).
Now we assume $z \notin l_\infty(B)$. Then there is a subsequence $\left(\sum_{k=1}^{\infty} b_{n,k}z_k\right)^{\infty}_{n=1}$ of the sequence $\left(\sum_{k=1}^{\infty} b_{n,k}z_k\right)^{\infty}_{k=1}$ such that $|\sum_{k=1}^{\infty} b_{n,k}z_k| > \beta$ for $j = 1, 2, \cdots$. Since $A$ is an invertible matrix, there exists a sequence $x = (x_k)$ such that

$$\sum_{k=1}^{\infty} a_{n,k}x_k = \frac{(-1)^j}{\sum_{k=1}^{\infty} b_{n,k}z_k},$$

for all $j$. Hence

$$\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)^{\infty}_{n=1} \notin \bigcap_{n=1}^{\infty} A,$$

this shows that $M_{A,B}(c_0, l_\infty) \subset l_\infty(B)$.

(ii) We have

$$M_{A,B}(l_\infty, c_0) \subset M_{A,B}(l_\infty, c),$$

by applying Lemma 2.2(ii). It is sufficient to prove $c_0(B) \subset M_{A,B}(l_\infty, c_0)$ and $M_{A,B}(l_\infty, c) \subset c_0(B)$. Suppose that $z \in c_0(B)$. By Theorem 2.8, we have lim$_{n \rightarrow \infty}$ $\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right) = 0$ for all $x \in l_\infty$, that is $z \in M_{A,B}(l_\infty, c_0)$. Thus $c_0(B) \subset M_{A,B}(l_\infty, c_0)$.

Now we assume $z \notin c_0(B)$. Then there is a real number as $b > 0$ and a subsequence $\left(\sum_{k=1}^{\infty} b_{n,k}z_k\right)^{\infty}_{n=1}$ of the sequence $\left(\sum_{k=1}^{\infty} b_{n,k}z_k\right)^{\infty}_{k=1}$ such that $|\sum_{k=1}^{\infty} b_{n,k}z_k| > b$ for all $j = 1, 2, \cdots$. We define the sequence $x$ as in part (ii). We have $x \in l_\infty$ and

$$\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)^{\infty}_{n=1} \notin c,$$

which implies $z \notin M_{A,B}(l_\infty, c)$. This shows that $M_{A,B}(l_\infty, c) \subset c_0(B)$.

(iii) Suppose that $z \in c(B)$. By applying Theorem 2.8(iii), we deduce that lim$_{n \rightarrow \infty}$ $\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)$ exists for all $x \in c$. So $z \in M_{A,B}(c, c)$ and $c(B) \subset M_{A,B}(c, c)$.

Conversely we assume $z \notin c(B)$. We define the sequence $x$ by $x = (\frac{1}{n}, \frac{1}{n}, \cdots)$. It is obvious that $x \in c$ and $\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)^{\infty}_{n=1} = (\sum_{k=1}^{\infty} b_{n,k}z_k)^{\infty}_{k=1} \notin c$. So $z \notin M_{A,B}(c, c)$, this shows $M_{A,B}(c, c) \subset c(B)$. \qed

Remark 2.10. If $A = B = I$, we have Example 1.28 from [11].

Remark 2.11. If two matrices $A$ and $B$ are defined by (2) and (3), then we obtain Theorem 2.2 from [5].

Corollary 2.12. Suppose that sup$_n \sum_{k=1}^{\infty} |a_{n,k}| < \infty$, we have $c_0^{NAB} = l_\infty(B)$ and $l_\infty^{NAB} = c_0(B)$.

Proof. The desired result follows from Theorem 2.9. \qed

Theorem 2.13. If matrix $A$ satisfies the conditions in Theorem 2.9, then we have the following statements.

(i) $M_{A,B}(c_0(A), X) = l_\infty(B)$, where $X \in [l_\infty, c, c_0]$. In particular $(c_0(A))^{NAB} = l_\infty(B)$.

(ii) $M_{A,B}(l_\infty(A), X) = c_0(B)$, where $X \in [c, c_0]$. In particular $(l_\infty(A))^{NAB} = c_0(B)$.

(iii) If in addition $\sum_{k=1}^{\infty} a_{n,k} = R$ for all $n$, then $M_{A,B}(c(A), c) = c(B)$.

Proof. We only prove the part (i), the other parts are proved similarly. Since $c_0 \subset c_0(A)$, according to Corollary 2.5(ii) and Theorem 2.9 we obtain

$$M_{A,B}(c_0(A), X) \subset M_{A,B}(c_0, X) = l_\infty(B).$$

The inclusion $l_\infty(B) \subset M_{A,B}(c_0(A), X)$ is gained by the relation (8). \qed

In the following, we obtain the $\alpha AB$, $\beta AB$- and $\gamma AB$-duals for the sequence spaces $l_\infty$, $c$ and $c_0$.
**Theorem 2.14.** Suppose that \( A \) is an invertible matrix that satisfies the condition (4), and \( \star \) denote either of the symbols \( \alpha, \beta \) or \( \gamma \). We have
\[
c^\star_0 = c^\star_1 = l^\star_0 = l^\star_1.
\]
In particular for \( B = I \),
\[
c^\star_0 = c^\star_1 = l^\star_0 = l^\star_1.
\]

**Remark 2.15.** If \( A = B = I \) and \( \star \) denote either of the symbols \( \alpha, \beta \) or \( \gamma \), we have
\[
c^\star_0 = c^\star_1 = l^\star_0 = l^\star_1,
\]
hence Theorem 1.29 from [11] is resulted.

**Remark 2.16.** If two matrices \( A \) and \( B \) are defined by (2) and (3), then we obtain Theorem 2.3 from [5].

In the next theorem, we examine the \( \alpha AB \), \( \beta AB \) and \( \gamma AB \)-duals for the sequence spaces \( l^\alpha_0(A) \), \( c^\alpha(A) \) and \( c_0(A) \).

**Theorem 2.17.** Let \( A \) be a matrix which satisfies the conditions in Theorem 2.8. If \( \star \) denote either of the symbols \( \alpha, \beta \) or \( \gamma \), then
\[
(c_0(A))^\star_1 = (c(A))^\star_1 = (l^\alpha_0(A))^\star_1 = l^\alpha_1(B).
\]

**Proof.** We only prove the statement for the case \( \star = \beta \), the other case prove similarly. Obviously
\[
(l^\alpha_0(A))^\beta_1 \subset (c(A))^\beta_1 \subset (c_0(A))^\beta_1,
\]
by Corollary 2.5(ii). So it is sufficient to verify \((c_0(A))^\beta_1 \subset l^\beta_1(B)\) and \(l^\beta_1(B) \subset (l^\alpha_0(A))^\beta_1\). By applying Corollary 2.5(ii) and Theorem 2.14, we deduce that \((c_0(A))^\beta_1 \subset c_0^\beta_1 \subset l^\beta_1(B)\). The other inclusion will gain by the relation (9). \(\square\)
Theorem 2.18. Suppose that $A$ is an invertible matrix. If $1 < p < \infty$ and $q = p/(p-1)$, then $(l_p(A))^{\ell AB} = l_q(B)$. Moreover for $p = 1$, we have $(l_1(A))^{\ell AB} = l_\infty(B)$.

Proof. We only prove the statement for the case $1 < p < \infty$, the case $p = 1$ will prove similarly. Let $z \in l_q(B)$ be given. By Hölder’s inequality, we have

$$
\left| \sum_{k=1}^n \left( \left| \sum_{j=1}^n b_{k,j}z_j \right| \right) \right| \leq \left( \sum_{k=1}^n \left( \left| \sum_{j=1}^n b_{k,j}z_j \right|^p \right) \right)^{1/p} \left( \sum_{j=1}^n \left| a_{k,j}x_j \right|^q \right)^{1/q},
$$

for all $x \in l_p(A)$. This shows $z \in (l_p(A))^{\ell AB}$ and hence $l_q(B) \subset (l_p(A))^{\ell AB}$.

Now, let $z \in (l_p(A))^{\ell AB}$ be given. We consider the linear functional $f_n : l_p(A) \to \mathbb{R}$ defined by

$$
f_n(x) = \sum_{k=1}^n \left( \left| \sum_{j=1}^n b_{k,j}z_j \right| \right) \left( \sum_{j=1}^n \left| a_{k,j}x_j \right| \right)^{1/q},
$$

for $n = 1, 2, \ldots$. Similar to (10), we obtain

$$
|f_n(x)| \leq \left( \sum_{k=1}^n \left| \sum_{j=1}^n b_{k,j}z_j \right|^q \right)^{1/q} \left( \sum_{j=1}^n \left| a_{k,j}x_j \right|^q \right)^{1/p},
$$

for every $x \in l_p(A)$. So the linear functional $f_n$ is bounded and

$$
\|f_n\| \leq \left( \sum_{k=1}^n \left| \sum_{j=1}^n b_{k,j}z_j \right|^q \right)^{1/q},
$$

for all $n$. We now prove reverse of the above inequality. Since $A$ is invertible, we define the sequence $x = (x_k)$ such that

$$
\sum_{j=1}^n a_{k,j}x_j = \left( \text{sgn} \sum_{j=1}^n b_{k,j}z_j \right) \left( \sum_{j=1}^n b_{k,j}z_j \right)^{1/p-1},
$$

for $1 \leq k \leq n$, and put the remaining elements zero. Obviously $x \in l_p(A)$, so

$$
\|f_n\| \geq \frac{|f_n(x)|}{\|x\|_p} = \frac{\sum_{k=1}^n \left| \sum_{j=1}^n b_{k,j}z_j \right|^q \left( \sum_{j=1}^n \left| a_{k,j}x_j \right| \right)^{1/q}}{\left( \sum_{k=1}^n \left| \sum_{j=1}^n b_{k,j}z_j \right|^q \right)^{1/q}} = \left( \sum_{k=1}^n \left| \sum_{j=1}^n b_{k,j}z_j \right|^q \right)^{1/q},
$$

for $n = 1, 2, \ldots$. Since $z \in l_p(A)^{\ell AB}$, the map $f_z : l_p(A) \to \mathbb{R}$ defined by

$$
f_z(x) = \sum_{k=1}^n \left( \left| \sum_{j=1}^n b_{k,j}z_j \right| \right) x_k \qquad (x \in l_p(A)),
$$

is well-defined and linear, and also the sequence $(f_n)$ is pointwise convergent to $f_z$. By using the Banach-Steinhaus theorem, it can be shown that $\|f_z\| \leq \sup_n \|f_n\| < \infty$, so $\left( \sum_{k=1}^\infty \left| \sum_{j=1}^\infty b_{k,j}z_j \right|^q \right)^{1/q} < \infty$ and $z \in l_q(B)$. This establishes the proof of theorem. □

Remark 2.19. If $A = B = I$ and $1 < p < \infty$ and $q = p/(p-1)$. Then we have $\ell_p^I = l_p$. Moreover for $p = 1$, $\ell_1^I = l_\infty$.

Definition 2.20. A subset $X$ of $\omega$ is said to be $A$-normal if $y \in X$ and $|\sum_{k=1}^\infty a_{n,k}y_k| \leq |\sum_{k=1}^\infty a_{n,k}x_k|$ for $n = 1, 2, \ldots$, together imply $x \in X$. In the special case that $A = I$, the set $X$ is called normal.
Example 2.21. The sequence spaces $c_0$ and $l_\infty$ are normal, but they are not $A$-normal. Since if $x = (1, -1, 2, -2, \cdots)$, $y = (1, \frac{1}{2}, \cdots)$ and the matrix $A = (a_{n,k})$ is defined by

$$a_{n,k} = \begin{cases} 1 & \text{if } k \in \{2n - 1, 2n\} \\ 0 & \text{otherwise} \end{cases}$$

We have $|\sum_{k=1}^\infty a_{n,k}x_k| \leq |\sum_{k=1}^\infty a_{n,k}y_k|$ and $y \in c_0, l_\infty$, while $x \notin c_0, l_\infty$.

Example 2.22. The sequence spaces $c_0(A)$ and $l_\infty(A)$ are $A$-normal, but they are not normal. Because, if $x = (1, 1, 2, 2, \cdots)$ and $y = (1, -1, 2, -2, \cdots)$ and $A$ is the matrix as in Example 2.21, then it is obvious that $|x| \leq |y|$, $y \in c_0(A)$ and $y \in l_\infty(A)$, while $x \notin c_0(A)$ and $x \notin l_\infty(A)$.

Example 2.23. The sequence spaces $c$ and $c(A)$ are neither $A$-normal nor normal.

Theorem 2.24. Suppose that $A$ is an invertible matrix and $X$ is a $A$-normal subset of $\mathbb{w}$. We have

$$X^{\omega AB} = X^{\delta AB} = X^{\gamma AB}.$$ 

Proof. Obviously $X^{\omega AB} \subseteq X^{\delta AB} \subseteq X^{\gamma AB}$, by Corollary 2.5(i). To prove the statement, it is sufficient to verify $X^{\gamma AB} \subseteq X^{\omega AB}$. Let $z \in X^{\gamma AB}$ and $x \in X$ be given. Since $A$ is invertible, we define the sequence $y$ such that

$$\sum_{k=1}^\infty a_{n,k}y_k = \left( \operatorname{sgn} \sum_{k=1}^\infty b_{n,k}z_k \right) \left| \sum_{k=1}^\infty a_{n,k}x_k \right|,$$

for $n = 1, 2, \cdots$. It is clear $\left| \sum_{k=1}^\infty a_{n,k}y_k \right| \leq \left| \sum_{k=1}^\infty a_{n,k}x_k \right|$ for all $n$. Consequently $y \in X$, since $X$ is $A$-normal. So

$$\sup_n \left| \sum_{k=1}^n \sum_{k=1}^\infty b_{n,k}z_k \sum_{k=1}^\infty a_{n,k}x_k \right| < \infty.$$ 

Furthermore, by the definition of the sequence $y$, $\sum_{k=1}^\infty b_{n,k}z_k \sum_{k=1}^\infty a_{n,k}x_k < \infty$. Since $x \in X$ was arbitrary, $z \in X^{\omega AB}$. This finishes the proof of the theorem. \qed

Remark 2.25. If $A = B = I$ and $X$ be a normal subset of $\mathbb{w}$, we have

$$X^\alpha = X^\beta = X^\gamma,$$

hence Remark 1.27 from [11] is gained.

Remark 2.26. If two matrices $A$ and $B$ are defined by (2) and (3), then we obtain Theorem 2.4 from [5].

3. The Difference Sequence Space $X(A, \Delta)$

Suppose that $A = (a_{n,k})$ is an infinite matrix with real entries. For every sequence space $X$, we define the generalized difference sequence space $X(A, \Delta)$ as follows:

$$X(A, \Delta) = \left\{ x = (x_k) : \left( \sum_{k=1}^\infty (a_{n,k} - a_{n-1,k}) x_k \right)_{n=1}^\infty \in X \right\},$$

where $X \in \{l_\infty, c, c_0\}$. The seminorm $|\cdot|_{A,\Delta}$ on $X(A, \Delta)$ is defined by

$$|x|_{A,\Delta} = \sup_n \left| \sum_{k=1}^\infty (a_{n,k} - a_{n-1,k}) x_k \right|.$$  

(11)
It should be noted that the function $||| \cdot |||_{A, \Delta}$ cannot be the norm. Since if $x = (1, -1, 0, 0, \cdots)$ and $A = (a_{n,k})$ is defined by,

$$a_{n,k} = \begin{cases} 1 & \text{if } k \in \{2n - 1, 2n\} \\ 0 & \text{otherwise}, \end{cases}$$

then $||| x |||_{A, \Delta} = 0$ while $x \neq 0$. It is also significant that in the special case $A = I$, we have $X(A, \Delta) = X(\Delta)$ and $||| x |||_{A, \Delta} = ||| x |||_{\Delta}$.

If the infinite matrix $\Delta = (\delta_{n,k})$ is defined by

$$\delta_{n,k} = \begin{cases} 1 & \text{if } k = n \\ -1 & \text{if } k = n - 1 \\ 0 & \text{otherwise}, \end{cases}$$

with the notation of (1), we can redefine the spaces $l_\infty(A, \Delta)$, $c(A, \Delta)$ and $c_0(A, \Delta)$ as follows:

$$l_\infty(A, \Delta) = (l_\infty)_{\Delta A}, \quad c(A, \Delta) = (c)_{\Delta A}, \quad c_0(A, \Delta) = (c_0)_{\Delta A}.$$ 

The purpose of this section is to consider some properties of the sequence spaces $X(A, \Delta)$ and is to derive some inclusion relations related to them. We also characterize $NAB$-duals and $\beta AB$-duals of $X(A, \Delta)$ where $X \in \{l_\infty, c, c_0\}$.

Now, we may begin with the following theorem which is essential in the study.

**Theorem 3.1.** The sequence spaces $X(A, \Delta)$ for $X \in \{l_\infty, c, c_0\}$ are complete semi-normed linear spaces with respect to the semi-norm defined by (11).

**Proof.** This is a routine verification and so we omit the details. □

It can easily be checked that the absolute property does not hold on the space $X(A, \Delta)$, that is $||| x |||_{A, \Delta} \neq ||| x |||_{\Delta A}$ for at least one sequence in this space which says that $X(A, \Delta)$ is the sequence space of non-absolute type, where $|x| = (|x|_{\Delta})$.

**Theorem 3.2.** Let $A = (a_{n,k})$ be an invertible matrix. The space $X(A, \Delta)$ is linearly isomorphic to the space $X(\Delta)$, for $X \in \{l_\infty, c, c_0\}$.

**Proof.** Consider the map

$$T : X(A, \Delta) \rightarrow X(\Delta)$$

$$x \mapsto \left( \sum_{k=1}^{\infty} a_{n,k}x_k \right)_{n=1}^{\infty},$$

obviously the map $T$ is linear, surjective and injective. □

In the following, we derive some inclusion relations concerning with the spaces $X$, $X(A)$, $X(\Delta)$ and $X(A, \Delta)$ where $X \in \{l_\infty, c, c_0\}$.

**Theorem 3.3.** We have the following inclusions.

(i) If the condition (4) holds, then $l_\infty \subset l_\infty(A, \Delta)$.

(ii) If the conditions (4) and (5) hold, then $c_0 \subset c_0(A, \Delta)$.

(iii) If the conditions (4), (6) and (7) hold, then $c \subset c(A, \Delta)$.

(iv) We have $X(A) \subset X(A, \Delta)$ where $X \in \{l_\infty, c, c_0\}$.

**Proof.** The parts (i), (ii) and (iii) obtain by applying Theorem 2.8.

(iv) Put $A = I$ in parts (i), (ii) and (iii), it can conclude that $X \subset X(\Delta)$. Let $x \in X(A)$ be given. We deduce that $\left( \sum_{k=1}^{\infty} a_{n,k}x_k \right)_{n=1}^{\infty} \in X$ so $\left( \sum_{k=1}^{\infty} a_{n,k}x_k \right)_{n=1}^{\infty} \in X(\Delta)$. Hence $x \in X(A, \Delta)$ and $X(A) \subset X(A, \Delta)$. □
Below, we compute $NAB$-dual of the difference sequence spaces $X(A, \Delta)$ where $X \in \{l_\infty, c, c_0\}$. In order to do this, we first give a preliminary lemma.

**Lemma 3.4.** (i) If $x \in l_\infty(A) \Delta$ then $\sup_{k} \frac{|a_k|}{\Delta_k} < \infty$.

(ii) If $x \in c(A) \Delta$ then $\frac{\Delta_k}{\Delta} \to \xi (k \to \infty)$ where $\Delta_k \to \xi (k \to \infty)$.

(iii) If $x \in c_0(A) \Delta$ then $\frac{\Delta_k}{\Delta} \to 0 (k \to \infty)$.

**Proof.** The proof is trivial and so is omitted. $\square$

**Theorem 3.5.** Define the set $d_1$ as follows:

$$d_1 = \left\{ z = (z_k) : \left( \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \right)_{n=1}^{\infty} \in c_0 \right\},$$

then

$$c^{NAB}(A, \Delta) = l_\infty^{NAB}(A, \Delta) = d_1.$$ 

**Proof.** We first show that $c^{NAB}(A, \Delta) = d_1$. Suppose that $z \in c^{NAB}(A, \Delta)$, we have

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k = 0,$$

for all $x \in c(A, \Delta)$. Since $A$ is invertible, we can choose the sequence such that $\sum_{k=1}^{\infty} a_{nk} x_k = n$ for all $n$, so $x \in c(A, \Delta)$ and hence $\lim_{n \to \infty} n \sum_{k=1}^{\infty} b_{nk} z_k = 0$. Thus $c^{NAB}(A, \Delta) \subseteq d_1$. Now let $z \in d_1$. Since $\sum_{k=1}^{\infty} a_{nk} x_k = n$, by previous lemma $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = \xi$, where $\xi = \lim_{n \to \infty} \sum_{k=1}^{\infty} (a_{nk} - a_{n-1,k}) x_k$. Hence

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \sum_{k=1}^{\infty} a_{nk} x_k = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \frac{\sum_{k=1}^{\infty} a_{nk} x_k}{n} = 0,$$

therefore $z \in c^{NAB}(A, \Delta)$ and $d_1 \subseteq c^{NAB}(A, \Delta)$.

Below, we prove that $l_\infty^{NAB}(A, \Delta) = d_1$. It is clear that $c(A, \Delta) \subseteq l_\infty(A, \Delta)$, so $l_\infty^{NAB}(A, \Delta) \subseteq c^{NAB}(A, \Delta) = d_1$. Now let $z \in d_1$ and $x \in l_\infty(A, \Delta)$. We have $\sum_{k=1}^{\infty} a_{nk} x_k = n$, and $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = \xi$ by Lemma 3.4. So

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \sum_{k=1}^{\infty} a_{nk} x_k = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \frac{\sum_{k=1}^{\infty} a_{nk} x_k}{n} = 0.$$

This implies that $z \in l_\infty^{NAB}(A, \Delta)$. $\square$

**Remark 3.6.** If $A = B = I$, we have $c^N(\Delta) = l_\infty^N(\Delta) = \{ z = (z_k) : (ka_k) \in c_0 \}$, [8].

**Remark 3.7.** If two matrices $A$ and $B$ are defined by (2) and (3), then we obtain Theorem 3.4 from [5].

**Theorem 3.8.** Let $A = (a_{nk})$ be an invertible matrix. We define the set $d_2$ as follows:

$$d_2 = \left\{ z = (z_k) : \left( \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} z_k \right)_{n=1}^{\infty} \in l_\infty \right\},$$

then $c^{NAB}(A, \Delta) = d_2$. 
Proof. Suppose that \( z \in d_2 \). Since \( \left( \sum_{k=1}^{\infty} a_{n,k} x_k \right)_{n=1}^{\infty} \in c_0(\Lambda) \) for all \( x \in c_0(\Lambda, \Lambda) \), we have \( \lim_{n \to \infty} \frac{\sum_{k=1}^{\infty} a_{n,k} x_k}{n} = 0 \), by Lemma 3.4. So

\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k = \lim_{n \to \infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} \frac{a_{n,k} x_k}{n} = 0,
\]

this implies that \( z \in c_0^{NAB}(\Lambda, \Lambda) \).

Now let \( z \in c_0^{NAB}(\Lambda, \Lambda) \) and \( x \in c_0(\Lambda, \Lambda) \) be given. By Theorem 3.2, there exists one and only one \( y = (y_k) \in c_0 \) such that \( \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_{j=1}^{n} y_j \). So

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k y_j = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k = 0,
\]

for all \( y = (y_k) \in c_0 \). If we define the matrix \( D = (d_{n,j})_{n=1}^{\infty} \) by

\[
d_{n,j} = \begin{cases} \sum_{k=1}^{\infty} b_{n,k} z_k & \text{for } 1 \leq j \leq n \\ 0 & \text{for } j > n, \end{cases}
\]

then \( \lim_{n \to \infty} \sum_{j=1}^{\infty} d_{n,j} y_j = 0 \) for all \( y \in c_0 \). So \( D = (d_{i,j}) \in (c_0, c_0) \) and

\[
\sup_{n} \left| \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_{n} \left| \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_{n} \left| \sum_{j=1}^{n} d_{n,j} \right| < \infty,
\]

by Theorem 2.8(ii). This completes the proof of the theorem. \( \square \)

\[
\textbf{Remark 3.9.} \quad \text{If } A = B = I, \text{ we have } c_0^N(\Lambda) = \{ z = (z_k) : (k a_n) \in l_1(\omega) \}, \text{ hence Lemma 2 from [8] is resulted.}
\]

\[
\textbf{Remark 3.10.} \quad \text{If two matrices } A \text{ and } B \text{ are defined by (2) and (3), then we obtain Theorem 3.6 from [5].}
\]

In order to investigate the \( \beta AB \)-dual of the difference sequence space \( c_0^N(\Lambda) \), we need the following lemma.

\[
\textbf{Lemma 3.11.} \quad \text{(8), Lemma 1} \quad \text{Let } (z_k) \in l_1 \text{ and if } \lim_{k \to \infty} |z_k x_k| = L \text{ exists for an } x \in c_0(\Lambda), \text{ then } L = 0.
\]

For the next result, we introduce the sequence \( (R_k) \) given by

\[
R_k = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} b_{n,j} z_j.
\]

\[
\textbf{Theorem 3.12.} \quad \text{Let } A = (a_{n,k}) \text{ be an invertible matrix. If}
\]

\[
d_3 = \{ z = (z_k) \in l_1(B) : (R_k) \in l_1(\Lambda) \},
\]

then we have \( c_0^{\beta AB}(\Lambda, \Lambda) = d_3 \)

\[
\text{Proof.} \quad \text{Suppose that } z \in d_3 \text{ and } x \in c_0(\Lambda, \Lambda), \text{ by using Abel’s summation formula we have}
\]

\[
\sum_{n=1}^{m} \left( \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k \right)
\]

\[
= \sum_{n=1}^{m} \left( \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{j,k} z_k + \sum_{k=1}^{\infty} a_{n,k} x_k - \sum_{k=1}^{\infty} a_{n+1,k} x_k \right)
\]

\[
+ \sum_{n=1}^{m} \left( \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{m+1,k} x_k \right)
\]

\[
= \sum_{n=1}^{m} \left( R_n - R_{n+1} \right) \left( \sum_{k=1}^{\infty} a_{n,k} x_k - \sum_{k=1}^{\infty} a_{n+1,k} x_k \right)
\]

\[
+ \sum_{n=1}^{m} \sum_{k=1}^{\infty} a_{n,k} x_k \sum_{k=1}^{\infty} \left( R_n - R_{n+1} \right) a_{m+1,k} x_k
\]

\[
= \sum_{n=1}^{m} \left( \sum_{k=1}^{\infty} a_{n,k} x_k \sum_{k=1}^{\infty} d_{n-k,1} x_k \right) - R_{m+1} \sum_{k=1}^{\infty} a_{m+1,k} x_k.
\]
This implies that \( \sum_{k=1}^{\infty} (\sum_{l=1}^{\infty} b_{n,l}x_k) = \sum_{k=1}^{\infty} a_{n,k}x_k \) is convergent, so \( z \in c_0^{AB}(A, \Delta) \).

Conversely let \( z \in c_0^{AB}(A, \Delta) \), we show that \( z \in d_3 \). Obviously \( z \in l_1(B) \). Suppose that \( z \not\in l_1(B) \), we can choose an index sequence \( (n_v) \) in \( \mathbb{N} \) with

\[
n_0 = 1 \quad \text{and} \quad \sum_{n=n_v-1}^{n_v} \sum_{k=1}^{\infty} b_{n,k}z_k > v \quad (v \in \mathbb{N}),
\]

since \( A \) is an invertible matrix, we may find \( x = (x_k) \in c_0(A) \subset c_0(A, \Delta) \) such that

\[
\sum_{k=1}^{\infty} a_{n,k}x_k = \frac{1}{v} \sum_{n=n_v-1}^{n_v} \sum_{k=1}^{\infty} b_{n,k}z_k \quad (n_v-1 \leq n < n_v \quad \text{and} \quad v \in \mathbb{N}),
\]

hence

\[
\sum_{n=n_v-1}^{n_v} \sum_{k=1}^{\infty} b_{n,k}z_k = \frac{1}{v} \sum_{n=n_v-1}^{n_v} \sum_{k=1}^{\infty} b_{n,k}z_k > 1 \quad (v \in \mathbb{N}),
\]

therefore \( (\sum_{k=1}^{\infty} b_{n,k}z_k, \sum_{k=1}^{\infty} a_{n,k}x_k) \not\in cs \) and \( z \not\in c_0^{AB}(A, \Delta) \).

Let \( x \in c_0(A, \Delta) \). Since \( A \) is invertible, by Theorem 3.2 there exist \( y = (y_k) \in c_0 \) such that \( \sum_{k=1}^{\infty} a_{n,k}x_k = \sum_{k=1}^{n} y_k \), then by Abel’s summation formula

\[
\sum_{n=1}^{m} R_n y_n = \sum_{n=1}^{m} (R_n - R_{n+1}) \left( \sum_{j=1}^{n} y_j \right) + \sum_{n=1}^{m} R_{n+1} y_n
\]

\[
= \sum_{n=1}^{m} \left( \sum_{j=1}^{n} y_j \right) \left( \sum_{k=1}^{\infty} b_{n,k}z_k \right) + \sum_{n=1}^{m} R_{n+1} y_n.
\]

So

\[
\sum_{n=1}^{m} \left( \sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k \right) = \sum_{n=1}^{m} (R_n - R_{n+1}) y_n = \sum_{n=1}^{m} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} b_{n,k}z_k \right) y_n.
\]

(13)

Now we define the matrix \( D = (d_{n,k}) \) by

\[
d_{n,k} = \begin{cases} \sum_{i=k}^{\infty} \sum_{j=1}^{\infty} b_{i,j}z_j & \text{for } 1 \leq k \leq n \\ 0 & \text{for } k > n, \end{cases}
\]

Since \( \lim_{n \to \infty} \sum_{k=1}^{\infty} d_{n,k} y_k = \lim_{n \to \infty} \sum_{k=1}^{n} d_{n,k} y_k \) exists for all \( y \in c_0 \) by (13), then \( D = (d_{n,k}) \in (c_0, c) \). This implies that

\[
\sup_{n} \sum_{k=1}^{\infty} |d_{n,k}| = \sup_{n} \sum_{k=1}^{n} \left| \sum_{i=k}^{\infty} \sum_{j=1}^{\infty} b_{i,j}z_j \right| < \infty,
\]

by Theorem 2.8(iv). Thus we conclude \( \sum_{k=1}^{\infty} |R_k| < \infty \). Furthermore (12) implies that \( \lim_{n \to \infty} R_{n+1} \sum_{k=1}^{\infty} d_{n+1,k} x_k \) exists for each \( x \in c_0(A, \Delta) \). So by Lemma 3.11 we have \( (R_n) \in c_0^N(\Delta) \), which completes the proof.

Remark 3.13. If \( A = B = I \), we have \( c_0^N(\Delta) = \{ z = (z_k) \in l_1 : (R_k) \in l_1 \cap c_0^N(\Delta) \} \) where \( R_k = \sum_{i=k}^{\infty} z_i \), hence Lemma 3 from [8] is resulted.
References