Coefficient Bounds for Certain Subclasses of Close-To-Convex Functions of Complex Order

Serap Bulut

Abstract. In this paper, we determine the coefficient bounds for functions in certain subclasses of close-to-convex functions of complex order, which are introduced here by means of a certain non-homogeneous Cauchy-Euler-type differential equation of order \( m \). Relevant connections of some of the results obtained with those in earlier works are also provided.

1. Introduction, Definitions and Preliminaries

Let \( \mathbb{R} = (-\infty, \infty) \) be the set of real numbers, \( \mathbb{C} = \mathbb{C}^* \cup \{0\} \) be the set of complex numbers,

\[
\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}
\]

be the set of positive integers and

\[
\mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}.
\]

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk

\[
\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.
\]

Recently Xu \textit{et al.} \cite{12} introduced the subclasses \( S_{\phi}(\lambda, \gamma) \) and \( \mathcal{K}_{\phi}(\lambda, \gamma, m; u) \) of analytic functions of complex order \( \gamma \in \mathbb{C}^* \), and obtained the coefficient bounds for the Taylor-Maclaurin coefficients for functions in each of these new subclasses \( S_{\phi}(\lambda, \gamma) \) and \( \mathcal{K}_{\phi}(\lambda, \gamma, m; u) \) of complex order \( \gamma \in \mathbb{C}^* \), which is given by Definitions 1.1 and 1.2 below.
Definition 1.1. (see [12]) Let $\varphi : \mathbb{U} \to \mathbb{C}$ be a convex function such that

$$\varphi(0) = 1 \quad \text{and} \quad \Re(\varphi(z)) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}_\varphi(\lambda, \gamma)$ the class of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{\gamma} \left( z \left[ (1 - \lambda) f(z) + \lambda z f'(z) \right] - 1 \right) \in \varphi(U) \quad (z \in \mathbb{U}),$$

where $0 \leq \lambda \leq 1; \gamma \in \mathbb{C}$.

Definition 1.2. (see [12]) A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_\varphi(\lambda, \gamma, m; u)$ if it satisfies the following non-homogenous Cauchy–Euler differential equation:

$$z^n \frac{d^m w}{dz^m} + \left( \frac{m}{1} \right) (u + m - 1) z \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \left( \frac{m}{m} \right) \prod_{j=0}^{m-1} (u + j) = h(z) \prod_{j=0}^{m-1} (u + j + 1)$$

$$\left( w = f(z) \in \mathcal{A}; h \in \mathcal{S}_\varphi(\lambda, \gamma); m \in \mathbb{N}^*; u \in \mathbb{R} \setminus (-\infty, -1) \right).$$

Making use of Definitions 1.1 and 1.2, Xu et al. [12] proved the following coefficient bounds for the Taylor-Maclaurin coefficients for functions in the subclasses $\mathcal{S}_\varphi(\lambda, \gamma)$ and $\mathcal{K}_\varphi(\lambda, \gamma, m; u)$ of analytic functions of complex order $\gamma \in \mathbb{C}$.

Theorem 1.3. (see [12]) Let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{S}_\varphi(\lambda, \gamma)$, then

$$|a_n| \leq \frac{\sum_{k=0}^{n-2} \left( k + \varphi'(0) \right) \cdot |\gamma|}{(n - 1)! \left[ 1 + \lambda (n - 1) \right]} \quad (n \in \mathbb{N}^*).$$

Theorem 1.4. (see [12]) Let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{K}_\varphi(\lambda, \gamma, m; u)$, then

$$|a_n| \leq \frac{\sum_{k=0}^{n-2} \left( k + \varphi'(0) \right) \cdot |\gamma| \prod_{j=0}^{m-1} (u + j + 1)}{(n - 1)! \left[ 1 + \lambda (n - 1) \right] \prod_{j=0}^{m-1} (u + j + n)}$$

$$(0 \leq \lambda \leq 1; \gamma \in \mathbb{C}; u \in \mathbb{R} \setminus (-\infty, -1));$$

Here, in our present sequel to some of the aforecited works (especially [12]), we first introduce the following subclasses of analytic functions of complex order $\gamma \in \mathbb{C}$.

Definition 1.5. Let $\varphi : \mathbb{U} \to \mathbb{C}$ be a convex function such that

$$\varphi(0) = 1 \quad \text{and} \quad \Re(\varphi(z)) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}_\varphi(\delta, \gamma, \tau)$ the class of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{\gamma} \left( z \left[ (1 - \lambda) g(z) + \lambda z g'(z) \right] - 1 \right) \in \varphi(U) \quad (z \in \mathbb{U}),$$

where $g \in \mathcal{S}_\varphi(\delta, \tau); 0 \leq \lambda, \delta \leq 1; \gamma, \tau \in \mathbb{C}$.
Definition 1.6. A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, m; u) \) if it satisfies the following non-homogenous Cauchy-Euler differential equation of order \( m \):

\[
\frac{d^m w}{dz^m} + \left( \frac{m}{1} \right) (u + m - 1) \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \left( \frac{m}{m} \right) \frac{d^{m-1} w}{dz^{m-1}} (u + 1) = h(z) \prod_{j=0}^{m-1} (u + j + 1)
\]

\( (w = f(z) \in \mathcal{A}; h \in \mathcal{SQ}_\varphi (\lambda, \gamma, \delta, \tau); m \in \mathbb{N}^*; u \in \mathbb{R}\backslash (-\infty, -1)) \).

Remark 1. There are many choices of the function \( \varphi \) which would provide interesting subclasses of analytic functions of complex order \( \gamma \in \mathbb{C} \). In particular,

(i) if we let

\[
\varphi (z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}),
\]

then it is easy to verify that \( \varphi \) is a convex function in \( \mathbb{U} \) and satisfies the hypotheses of Definition 1.5. Therefore we obtain the new classes

\[
\mathcal{SQ}_\varphi (\lambda, \gamma, \delta, \tau) = \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, A, B) \quad \text{and} \quad \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, m; u) = \mathcal{DK}_\varphi (\lambda, \gamma, \delta, \tau, A, B, m; u).
\]

For \( \delta = \lambda \) and \( \tau = 1 \), these classes introduced and studied by Ul-Haq et al. [10].

(ii) if we let

\[
\varphi (z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1; z \in \mathbb{U}),
\]

then we obtain the new classes

\[
\mathcal{SQ}_\varphi (\lambda, \gamma, \delta, \tau) = \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, \beta) \quad \text{and} \quad \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, m; u) = \mathcal{BK}_\varphi (\lambda, \gamma, \delta, \tau, \beta; u).
\]

For \( \delta = \lambda \), \( \tau = 1 \) and \( m = 2 \), these classes are introduced and studied by Ul-Haq et al. [9].

In this paper, by using the subordination principle between analytic functions, we obtain coefficient bounds for the Taylor-Maclaurin coefficients for functions in the substantially more general function classes \( \mathcal{SQ}_\varphi (\lambda, \gamma, \delta, \tau) \) and \( \mathcal{KQ}_\varphi (\lambda, \gamma, \delta, \tau, m; u) \) of analytic functions of complex order \( \gamma \in \mathbb{C} \), which we have introduced here.

Our results presented here would generalize and improve the corresponding results obtained earlier by (for example) Altıntas et al. [1], Nasr and Aouf [4], Robertson [5], Srivastava et al. [7] and Ul-Haq et al. [9, 10], (see also [2, 3, 8, 11]).

In our investigation, we shall make use of the principle of subordination between analytic functions, which is explained in Definition 1.7 below.

Definition 1.7. For two functions \( f \) and \( g \), analytic in \( \mathbb{U} \), we say that the function \( f \) is subordinate to \( g \) in \( \mathbb{U} \), and write

\[
f (z) < g (z) \quad (z \in \mathbb{U}),
\]

if there exists a Schwarz function \( \omega \), analytic in \( \mathbb{U} \), with

\[
\omega (0) = 0 \quad \text{and} \quad |\omega (z)| < 1 \quad (z \in \mathbb{U})
\]

such that

\[
f (z) = g (\omega (z)) \quad (z \in \mathbb{U}).
\]

Indeed, it is known that

\[
f (z) < g (z) \quad (z \in \mathbb{U}) \Rightarrow f (0) = g (0) \quad \text{and} \quad f (\mathbb{U}) \subset g (\mathbb{U}).
\]

Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence

\[
f (z) < g (z) \quad (z \in \mathbb{U}) \Leftrightarrow f (0) = g (0) \quad \text{and} \quad f (\mathbb{U}) \subset g (\mathbb{U}).
\]
2. Main Results and their Demonstration

In order to prove our main results (Theorems 2.2 and 2.3 below), we first recall the following lemma due to Rogosinski [6].

Lemma 2.1. Let the function $g$ given by

$$g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. Also let the function $f$ given by

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \quad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. If

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

then

$$|a_k| \leq |b_1| \quad (k \in \mathbb{N}).$$

We now state and prove each of our main results given by Theorems 2.2 and 2.3 below.

Theorem 2.2. Let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{SQ}_\phi (\lambda, \gamma, \delta, \tau)$, then

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} |k + |\phi'(0)|| \cdot |\tau|}{n! \cdot [1 + \delta (n-1)]} \left(1 + \sum_{j=1}^{n-2} \frac{1 + \lambda (n-j-1) \prod_{k=0}^{n-j-2} |k + |\phi'(0)|| \cdot |\tau|}{(n-j-1)! \cdot [1 + \delta (n-j-1)]} \right) \quad (n \in \mathbb{N}^*),$$

$$\left(\phi \in \mathcal{S}_\phi (\delta, \tau) ; 0 \leq \lambda, \delta \leq 1; \gamma, \tau \in \mathbb{C}^* \right).$$

Proof. Let the function $f \in \mathcal{SQ}_\phi (\lambda, \gamma, \delta, \tau)$ be of the form (1). Therefore, there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_\phi (\delta, \tau) \quad (\tau \in \mathbb{C}^*)$$

so that

$$1 + \frac{1}{\gamma} \left(\frac{z [(1-\lambda) f(z) + \lambda z f'(z)]' - 1}{(1-\lambda) g(z) + \lambda z g'(z)} \right) \in \phi (\mathbb{U}).$$

Note that by Theorem 1.3, we have

$$|b_n| \leq \frac{\prod_{k=0}^{n-2} |k + |\phi'(0)|| \cdot |\tau|}{(n-1)! \cdot [1 + \delta (n-1)]} \quad (n \in \mathbb{N}^*).$$
Let

\[ F(z) = (1 - \lambda) f(z) + \lambda z f'(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad A_n = [1 + \lambda (n - 1)] a_n \]  

(5)

\[ G(z) = (1 - \lambda) g(z) + \lambda z g'(z) = z + \sum_{n=2}^{\infty} B_n z^n, \quad B_n = [1 + \lambda (n - 1)] b_n. \]  

(6)

Then (3) is of the form

\[ 1 + \frac{1}{\gamma} \left( \frac{zF'(z)}{G(z)} - 1 \right) \in \varphi(U). \]  

(7)

Let us define the function \( p(z) \) by

\[ p(z) = 1 + \frac{1}{\gamma} \left( \frac{zF'(z)}{G(z)} - 1 \right) \quad (z \in \mathbb{U}). \]  

(8)

Therefore, we deduce that

\[ p(0) = \varphi(0) = 1 \text{ and } p(z) \in \varphi(U) \quad (z \in \mathbb{U}). \]

So we have

\[ p(z) < \varphi(z) \quad (z \in \mathbb{U}). \]

Hence, by Lemma 2.1, we obtain

\[ \left| \frac{p^{(m)}(0)}{m!} \right| = |c_m| \leq |\varphi'(0)| \quad (m \in \mathbb{N}), \]  

(9)

where

\[ p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}). \]  

(10)

Also from (8), we find

\[ zF'(z) - G(z) = \gamma (p(z) - 1) G(z). \]  

(11)

Since \( A_1 = B_1 = 1 \), in view of (11), we obtain

\[ nA_n - B_n = \gamma \left( c_{n-1} + c_{n-2} B_2 + \cdots + c_1 B_{n-1} \right) = \gamma \left( c_{n-1} + \sum_{j=1}^{n-2} c_j B_{n-j} \right) \quad (n \in \mathbb{N}'). \]  

(12)

Now we get from (4), (5), (6), (9) and (12),

\[ |a_n| \leq \frac{\prod_{k=0}^{n-2} \left[ k + |\varphi'(0)| \cdot |\tau| \right]}{n! \left[ 1 + \delta (n - 1) \right]} + \frac{|\varphi'(0)| \cdot |\gamma|}{n \left[ 1 + \lambda (n - 1) \right]} \left( 1 + \sum_{j=1}^{n-2} \frac{[1 + \lambda (n - j - 1)] \prod_{k=0}^{n-j-2} \left[ k + |\varphi'(0)| \cdot |\tau| \right]}{(n - j - 1)! \left[ 1 + \delta (n - j - 1) \right]} \right) \quad (n \in \mathbb{N}'). \]

This evidently completes the proof of Theorem 2.2. \( \Box \)
Theorem 2.3. Let the function \( f \in A \) be defined by (1). If \( f \in KQ_\phi(\lambda, \gamma, \delta, \tau, m; u) \), then

\[
|a_n| \leq \begin{cases} 
\prod_{k=0}^{n-2} [k + |\tau| (A - B)] \\
\frac{n!}{n! [1 + \delta (n - 1)]} 
\end{cases} 
+ \frac{n-2}{n [1 + \lambda (n - 1)]} \left( 1 + \sum_{j=1}^{n-2} \frac{1 + \lambda (n - j - 1)}{(n - j - 1)! [1 + \delta (n - j - 1)]} \right)
\]

\[
m - 1 \prod_{j=0}^{m-1} (u + j + 1) \prod_{j=0}^{m-1} (u + j + n) (n \in \mathbb{N}^*),
\]

(0 ≤ \( \lambda, \delta \leq 1 \); \( \gamma, \tau \in \mathbb{C}^* \); \( m \in \mathbb{N}^* \); \( n \in \mathbb{R} \setminus (-\infty, -1] \)).

Proof. Let the function \( f \in A \) be given by (1). Also let

\[
h(z) = z + \sum_{n=2}^{\infty} h_n z^n \in SQ_\phi(\lambda, \gamma, \delta, \tau).
\]

We then deduce from Definition 1.6 that

\[
a_n = \frac{\prod_{j=0}^{m-1} (u + j + 1)}{\prod_{j=0}^{m-1} (u + j + n)} h_n (n \in \mathbb{N}^*, u \in \mathbb{R} \setminus (-\infty, -1]).
\]

Thus, by using Theorem 2.2 in conjunction with the above equality, we have assertion (13) of Theorem 2.3. This completes the proof of Theorem 2.3. \( \blacksquare \)

3. Corollaries and consequences

In this section, we apply our main results (Theorems 2.2 and 2.3) in order to deduce each of the following corollaries and consequences.

Setting

\[
\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; \ z \in \mathbb{U}),
\]

in Theorems 2.2 and 2.3, we get Corollaries 3.1 and 3.2, respectively.

Corollary 3.1. Let the function \( f \in A \) be defined by (1). If \( f \in KQ_\phi(\lambda, \gamma, \delta, \tau, A, B) \), then

\[
|a_n| \leq \begin{cases} 
\prod_{k=0}^{n-2} [k + |\tau| (A - B)] \\
\frac{n!}{n! [1 + \delta (n - 1)]} 
\end{cases} 
+ \frac{n-2}{n [1 + \lambda (n - 1)]} \left( 1 + \sum_{j=1}^{n-2} \frac{1 + \lambda (n - j - 1)}{(n - j - 1)! [1 + \delta (n - j - 1)]} \right) \left( n \in \mathbb{N}^* \right),
\]
\( g \in S_\psi (\delta, \tau); \ 0 \leq \lambda, \delta \leq 1; \ \gamma, \tau \in \mathbb{C}^*; \ -1 \leq B < A \leq 1 \).

**Corollary 3.2.** Let the function \( f \in \mathcal{A} \) be defined by (1). If \( f \in \mathcal{DK} (\lambda, \gamma, \delta, \tau, A, B, m; u) \), then

\[
|a_n| \leq \begin{cases} 
\frac{n-2}{n!} \prod_{k=0}^{n-2} \left[ k + |\tau| (A - B) \right] \\
\frac{2}{n!} \prod_{k=0}^{n-2} \left[ k + 2 |\tau| (1 - \beta) \right]
\end{cases}
\]

\[ + \frac{|\gamma| (A - B)}{n \left[ 1 + \lambda (n - 1) \right]} \left( 1 + \sum_{j=1}^{n-2} \frac{[1 + \lambda (n - j - 1)] \prod_{k=0}^{n-j-2} [k + |\tau| (A - B)]}{(n - j - 1)! \left[ 1 + \delta (n - j - 1) \right]} \right) \left( \frac{m-1}{\prod_{j=0}^{m-1} (u + j + n)} \right) (n \in \mathbb{N}^*),
\]

\((0 \leq \lambda, \delta \leq 1; \ \gamma, \tau \in \mathbb{C}^*; \ -1 \leq B < A \leq 1; \ m \in \mathbb{N}^*; \ u \in \mathbb{R} \setminus (-\infty, -1]).

**Remark 2.** It is easy to see that

\[ k + |\tau| (A - B) \leq k + \frac{2 |\tau| (A - B)}{1 - B} \quad (k \in \mathbb{N}^*, \ -1 \leq B < A \leq 1, \ \tau \in \mathbb{C}^*),
\]

which would obviously yield significant improvements over [10, Theorems 1 and 2], with \( \delta = \lambda \) and \( \tau = 1 \) in Corollaries 3.1 and 3.2, respectively.

Setting

\( \varphi (z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1; \ z \in \mathbb{U}), \)

in Theorems 2.2 and 2.3, we get Corollaries 3.3 and 3.4, respectively.

**Corollary 3.3.** Let the function \( f \in \mathcal{A} \) be defined by (1). If \( f \in \mathcal{KQ} (\lambda, \gamma, \delta, \tau, \beta) \), then

\[
|a_n| \leq \begin{cases} 
\frac{n-2}{n!} \prod_{k=0}^{n-2} \left[ k + |\tau| (1 - \beta) \right] \\
\frac{2}{n!} \prod_{k=0}^{n-2} \left[ k + 2 |\tau| (1 - \beta) \right]
\end{cases}
\]

\[ + \frac{2 |\gamma| (1 - \beta)}{n \left[ 1 + \lambda (n - 1) \right]} \left( 1 + \sum_{j=1}^{n-2} \frac{[1 + \lambda (n - j - 1)] \prod_{k=0}^{n-j-2} [k + 2 |\tau| (1 - \beta)]}{(n - j - 1)! \left[ 1 + \delta (n - j - 1) \right]} \right) \left( \frac{m-1}{\prod_{j=0}^{m-1} (u + j + n)} \right) (n \in \mathbb{N}^*),
\]

\((g \in S_\psi (\delta, \tau); \ 0 \leq \lambda, \delta \leq 1; \ \gamma, \tau \in \mathbb{C}^*; \ 0 \leq \beta < 1).
Corollary 3.4. Let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{BK}(\lambda, \gamma, \delta, \tau, \beta; u)$, then

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} [k + 2|\tau|(1 - \beta)] & \frac{n!}{n!} \left[1 + \delta (n - 1)\right] \\ \frac{2|\gamma|(1 - \beta)}{n \left[1 + \lambda (n - 1)\right]} \left(1 + \sum_{j=1}^{n-2} \frac{[1 + \lambda (n - j - 1)] \prod_{k=0}^{n-j-2} [k + 2|\tau|(1 - \beta)]}{(n - j - 1)! \left[1 + \delta (n - j - 1)\right]}\right) \\ \frac{m-1}{m-1} \prod_{j=0}^{m-1} (u + j + 1) \prod_{j=0}^{m-1} (u + j + n) \right) & (n \in \mathbb{N}^*) \end{cases},$$

$(0 \leq \lambda, \delta \leq 1; \gamma, \tau \in \mathbb{C}^*; 0 \leq \beta < 1; m \in \mathbb{N}^*; u \in \mathbb{R} \setminus (-\infty, -1])$.

Remark 3. Taking $\delta = \lambda$, $\tau = 1$ and $m = 2$ in Corollaries 3.3 and 3.4, we have [9, Theorems 1 and 2], respectively.

References