Vertical Liouville Foliations on the Big-Tangent Manifold of a Finsler Space

Cristian Ida\textsuperscript{a}, Paul Popescu\textsuperscript{b}

\textsuperscript{a}Department of Mathematics and Computer Science, University Transilvania of Brasov, Brasov 500091, Str. Iuliu Maniu 50, România,
\textsuperscript{b}Department of Applied Mathematics, University of Craiova, Craiova, 200585, Str. Al. Cuza, No. 13, România

Abstract. The present paper unifies some aspects concerning the vertical Liouville distributions on the tangent (cotangent) bundle of a Finsler (Cartan) space in the context of generalized geometry. More exactly, we consider the big-tangent manifold $T_M$ associated to a Finsler space $(M, F)$ and of its $L$-dual which is a Cartan space $(M, K)$ and we define three Liouville distributions on $T_M$ which are integrable. We also find geometric properties of both leaves of Liouville distribution and the vertical distribution in our context.

1. Introduction and Preliminary Notions

1.1. Introduction

The vertical Liouville distribution on the tangent bundle of a (pseudo) Finsler space was defined for the first time in [4] where some aspects of the geometry of the vertical bundle are derived via vertical Liouville distribution. A similar study on the cotangent bundle of a Cartan space can be found in [11]. Also, other significant studies concerning the interrelations between natural foliations defined by Liouville fields on the tangent bundle of a Finsler space and the geometry of the Finsler space itself, as well as similar problems on Cartan spaces are intensively studied in [6] and [2], respectively. See also [11, 14, 19, 20].

As it is well known, in the generalized geometry initiated in [10], the tangent bundle $TM$ of a smooth $n$-dimensional manifold $M$ is replaced by the big-tangent bundle (or Pontryagin bundle) $T_M \oplus T^*M$. On its total space the velocities and moments are considered as independent variables. This idea was proposed and developed in [21, 22] and later was used in the study of Hamiltonian-Jacobi theory for singular Lagrangian systems [13]. The geometry of the total space of the big-tangent bundle, called big-tangent manifold, is intensively studied in [25] and some its applications to mechanical systems can be found in [9].

Using the framework of the geometry on the big-tangent manifold, our aim in this paper is to extend some results concerning the vertical Liouville foliation in the context of generalized geometry. Thus, some aspects concerning the geometry of vertical bundle of the big-tangent bundle of a Finsler space can be obtained via our generalized Liouville distribution. This extension yields new subfoliations of the vertical foliation on the big-tangent bundle and some properties of these subfoliations are studied in the end of the paper. In this sense, we consider the big-tangent manifold $T_M$ associated to a Finsler space $(M, F)$ and of...
its $L$-dual which is a Cartan space $(M, K)$. As usual, we reconsider the vertical Liouville distributions $V_{E_1}$ and $V_{E_2}$ from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [4, 11], for the case of vertical subbundles $V_1$ and $V_2$, respectively, with respect to Liouville vector fields $E_1$ and $E_2$. Next we define the Liouville distribution $V_E$ with respect to the Liouville vector field $E = E_1 + E_2$, we prove that it is integrable (Theorem 2.2) and we study some of its properties (Theorem 2.5 and Proposition 2.6). Also, some links between the vertical Liouville foliations $V_{E_1}$, $V_{E_2}$, and $V_E$, respectively, are established.

On the other hand, it is well known that the vertical Liouville distribution on the tangent (cotangent) bundle of a Finsler (Cartan) space is strongly related with the indicatrix of Finsler (Cartan) space, see [1, 8]. Thus, our generalized vertical Liouville bundle of a Finsler (Cartan) space is strongly related with the indicatrix of Finsler (Cartan) space, see [6, 11].

1.2. Preliminaries and notations

Let $M$ be a $n$-dimensional smooth manifold, and we consider $\pi : TM \to M$ its tangent bundle, $\pi' : T^*M$ its cotangent bundle and $\tau \equiv \pi \oplus \pi' : TM \oplus T^*M \to M$ its big-tangent bundle defined as Whitney sum of the tangent and cotangent bundles of $M$. The total space of the big-tangent bundle, called big-tangent manifold, is a $3n$-dimensional smooth manifold denoted here by $\mathcal{T}M$. Let us briefly recall some elementary notions about the big-tangent manifold $\mathcal{T}M$. For a detailed discussion about its geometry we refer [25].

Let $(U, (x^i))$ be a local chart on $M$. If $\{\frac{\partial}{\partial x^i}\}_i, x \in U$ is a local frame of sections in the tangent bundle over $U$ and $\{dx^i|_x\}, x \in U$ is a local frame of sections in the cotangent bundle over $U$, then by definition of the Whitney sum, $\{\frac{\partial}{\partial x^i}, dx^i|_x\}, x \in U$ is a local frame of sections in the big-tangent bundle $TM \oplus T^*M$ over $U$.

Every section $(y, p)$ of $\tau$ over $U$ takes the form $(y, p) = y^i\frac{\partial}{\partial x^i} + p dx^i$ and the local coordinates on $\tau^{-1}(U)$ will be defined as the triples $(x^i, y^i, p_i)$, where $i = 1, \ldots, n = \dim M$, $(x^i)$ are local coordinates on $M$, $(y^i)$ are vector coordinates and $(p_i)$ are covector coordinates.

The change rules of these coordinates are:

$$\tilde{x} = x^i(x^i), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \quad \tilde{p}_i = \frac{\partial \tilde{x}^i}{\partial x^j} p_j,$$  \hspace{1cm} (1)

and the local expressions of a vector field $X$ and of a 1-form $\varphi$ on $TM$ are

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial y^i} + \zeta_i \frac{\partial}{\partial p_i} \quad \text{and} \quad \varphi = \alpha dx^i + \beta dy^i + \gamma dp_i.$$  \hspace{1cm} (2)

For the big-tangent manifold $\mathcal{T}M$ we have the following projections

$$\tau : \mathcal{T}M \to M, \quad \tau_1 : \mathcal{T}M \to TM, \quad \tau_2 : \mathcal{T}M \to T^*M$$

on $M$ and on the total spaces of tangent and cotangent bundle, respectively.

As usual, we denote by $V = V(TM)$ the vertical bundle on the big-tangent manifold $\mathcal{T}M$ and it has the decomposition

$$V = V_1 \oplus V_2,$$  \hspace{1cm} (3)

where $V_1 = \tau_{1}^{-1}(V(TM))$, $V_2 = \tau_{2}^{-1}(V(T^*M))$ and have the local frames $\{\frac{\partial}{\partial y^i}\}_i, \{\frac{\partial}{\partial p_i}\}_i$, respectively. The subbundles $V_1, V_2$ are the vertical foliations of $\mathcal{T}M$ by fibers of $\tau_1, \tau_2$, respectively, and $\mathcal{T}M$ has a multi-foliate structure [23, 24]. The Liouville vector fields (or Euler vector fields) are given by

$$\mathcal{E}_1 = y^i \frac{\partial}{\partial y^i} \in \Gamma(V_1), \quad \mathcal{E}_2 = p_i \frac{\partial}{\partial p_i} \in \Gamma(V_2), \quad \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \in \Gamma(V).$$  \hspace{1cm} (4)

In the following we consider that the manifold $M$ is endowed with a Finsler structure $F$, and we present a metric structure on $V$ induced by $F$. According to [3, 5, 17], a function $F : TM \to [0, \infty)$ which satisfies the following conditions:
2.1. Vertical Liouville distributions \( V \) are established. \( V \) of its properties. Also, some links between the vertical Liouville foliations see \([3, 5, 17]\):

Using \( G \) \( V \) of the following holds:

\[
y_i = g_{ij} y^j, \quad y^i = g^{ij} y_j, \quad F^2 = g_{ij} y^i y^j = y_i y^i, \quad C_{ijk} y^k = C_{ikj} y^k = C_{jik} y^k = 0, \quad (5)
\]

where \((g^{ij})\) is the inverse matrix of \((g_{ij})\) and we have put \( y_i = \frac{1}{2} \frac{\partial F^2}{\partial y^i} \), \( C_{ijk} = \frac{1}{4} \frac{\partial g^{ij}}{\partial y^k} \). Also, for a given Finsler structure \( F \) on \( T^0 \) there is a Cartan structure \( K = F^* \) on \( T^0 = T^0 \) - \{zero section\} obtained by Legendre transformation of \( F \) (the \( L \)-duality process, see [15, 16, 18]), that is a function \( K : T^0 \to [0, \infty) \) which has the following properties:

i) \( K \) is \( C^\infty \) on \( T^0 \);

ii) \( K(x, \lambda p) = \lambda K(x, p) \) for all \( \lambda > 0 \);

iii) the \( n \times n \) matrix \((g^{ij})\), where \( g^{ij} = \frac{1}{2} \frac{\partial K^2}{\partial y^i \partial y^j} \), is positive definite at all points of \( T^0 \).

Also \( K(x, p) > 0 \), whenever \( p \neq 0 \). The properties of \( K \) imply that

\[
p^j = g^{ij} p_i, \quad p_i = g_{ij} p^j, \quad K^2 = g^{ij} p_i p_j = p_i p^j, \quad C^{ijk} p_k = C^{ikj} p_k = C^{jik} p_k = 0, \quad (6)
\]

where \((g^{ij})\) is the inverse matrix of \((g^{ij})\) and we have put \( p^j = \frac{1}{2} \frac{\partial K^2}{\partial y^j} \), \( C^{ijk} = -\frac{1}{4} \frac{\partial g^{ij}}{\partial y^k} \).

It is well-known that \( g_{ij} \) determines a metric structure on \( V(T^0) \) and \( g^{ij} \) determines a metric structure on \( V(T^0) \). Similarly, every Finsler structure \( F \) on \( M \) determines a metric structure \( G \) on \( V \) by setting

\[
G(X, Y) = g_{ij}(x, y) X_i^j(x, y, p) Y^j_i(x, y, p) + g^{ij}(x, y) X^i_j(x, y, p) Y_j^i(x, y, p), \quad (7)
\]

for every \( X = X_i^j(x, y, p) \frac{\partial}{\partial y^j} \), \( Y = Y^j_i(x, y, p) \frac{\partial}{\partial y^j} \), \( \frac{\partial}{\partial y^j} \), \( \frac{\partial}{\partial y^j} \in \Gamma(V) \).

2. Vertical Liouville Foliations on \( TM \)

In this section we reconsider the vertical Liouville distributions \( V_{E_1} \) and \( V_{E_2} \) from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [4, 11], for the case of vertical subbundles \( V_1 \) and \( V_2 \), respectively, with respect to Liouville vector fields \( E_1 \) and \( E_2 \). Next we define the Liouville distribution \( V_E \) with respect to the Liouville vector field \( E = E_1 + E_2 \), we prove that it is integrable and we study some of its properties. Also, some links between the vertical Liouville foliations \( V_{E_1}, V_{E_2} \), and \( V_{E} \), respectively, are established.

2.1. Vertical Liouville distributions \( V_{E_1} \) and \( V_{E_2} \)

Following [4, 11] we define two vertical Liouville distributions on \( TM \) as the complementary orthogonal distributions in \( V_1 \) and \( V_2 \) to the line distributions spanned by the Liouville vector fields \( E_1 \) and \( E_2 \), respectively.

By (4) and (5) we have

\[
G(E_1, E_1) = F^2. \quad (8)
\]

Using \( G \) and \( E_1 \), we define the \( V_1 \)-vertical one form \( \zeta_1 \) by

\[
\zeta_1(X_1) = \frac{1}{F} G(X_1, E_1), \quad \forall X_1 = X_1^j(x, y, p) \frac{\partial}{\partial y^j} \in \Gamma(V_1). \quad (9)
\]
Let us denote by \( \{ E_2 \} \) the line vector bundle over \( TM \) spanned by \( E_1 \) and we define the first vertical Liouville distribution as the complementary orthogonal distribution \( V_{E_1} \) to \( \{ E_1 \} \) in \( V_1 \) with respect to \( G \). Thus, \( V_{E_1} \) is defined by \( \zeta_2 \), that is

\[
\Gamma (V_{E_1}) = \{ X_1 \in \Gamma (V_1) : \zeta_1 (X_1) = 0 \}.
\] (10)

We get that every \( V_1 \)-vertical vector field \( X_1 = X_1^i (x, y, p) \frac{\partial}{\partial y^i} \) can be expressed in the form:

\[
X_1 = P_1 X_1 + \frac{1}{K} \zeta_1 (X_1) E_1,
\] (11)

where \( P_1 \) is the projection morphism of \( V_1 \) on \( V_{E_1} \).

Also, by direct calculus, we get

\[
G(X_1, P_1 Y_1) = G(P_1 X_1, P_1 Y_1) = G(X_1, Y_1) - \zeta_1 (X_1) \zeta_1 (Y_1), \quad \forall X_1, Y_1 \in \Gamma (V_1).
\] (12)

Let us consider \( \{ \theta' \} \) the dual basis of \( \{ \frac{\partial}{\partial y^i} \} \). Then, with respect to the basis \( \{ \theta' \} \) and \( \{ \theta' \otimes \frac{\partial}{\partial y^i} \} \), respectively, \( \zeta_1 \) and \( P_1 \) are locally given by

\[
\zeta_1 = \zeta_1 \theta', \quad P_1 = P_1 \theta' \otimes \frac{\partial}{\partial y^i}, \quad \zeta_1 = \frac{\partial y^j}{\partial y^i}, \quad P_1 = \delta^i_j - \frac{y^j y^j}{K^2},
\] (13)

where \( \delta^i_j \) are the components of the Kronecker delta.

As usual for tangent bundle of a Finsler space (see Theorem 3.1 from [4]), the first vertical Liouville distribution \( V_{E_1} \) is integrable and it defines a foliation on \( TM \), called the first vertical Liouville foliation on the big-tangent manifold \( TM \). Also, some geometric properties of the leaves of vertical foliation \( V_1 \) can be derived via the first vertical Liouville foliation \( V_{E_1} \).

Similarly, by (4) and (6) we have

\[
G(E_2, E_2) = K^2,
\] (14)

and using \( G \) and \( E_2 \), we define the \( V_2 \)-vertical one form \( \zeta_2 \) by

\[
\zeta_2 (X_2) = \frac{1}{K} G(X_2, E_2), \quad \forall X_2 = X_2^i (x, y, p) \frac{\partial}{\partial y^i} \in \Gamma (V_2).
\] (15)

Let us denote by \( \{ E_2 \} \) the line vector bundle over \( TM \) spanned by \( E_2 \) and we define the second vertical Liouville distribution as the complementary orthogonal distribution \( V_{E_2} \) to \( \{ E_2 \} \) in \( V_2 \) with respect to \( G \). Thus, \( V_{E_2} \) is defined by \( \zeta_2 \), that is

\[
\Gamma (V_{E_2}) = \{ X_2 \in \Gamma (V_2) : \zeta_2 (X_2) = 0 \}.
\] (16)

We get that every \( V_2 \)-vertical vector field \( X_2 = X_2^i (x, y, p) \frac{\partial}{\partial y^i} \) can be expressed in the form:

\[
X_2 = P_2 X_2 + \frac{1}{K} \zeta_2 (X_2) E_2,
\] (17)

where \( P_2 \) is the projection morphism of \( V_2 \) on \( V_{E_2} \).

Similarly, by direct calculus, we get

\[
G(X_2, P_2 Y_2) = G(P_2 X_2, P_2 Y_2) = G(X_2, Y_2) - \zeta_2 (X_2) \zeta_2 (Y_2), \quad \forall X_2, Y_2 \in \Gamma (V_2).
\] (18)

Let us consider \( \{ \kappa \} \) the dual basis of \( \{ \frac{\partial}{\partial y^i} \} \). Then, with respect to the basis \( \{ \kappa \} \) and \( \{ \kappa \otimes \frac{\partial}{\partial y^i} \} \), respectively, \( \zeta_2 \) and \( P_2 \) are locally given by

\[
\zeta_2 = \zeta_2 \kappa, \quad P_2 = P_2 \kappa \otimes \frac{\partial}{\partial y^i}, \quad \zeta_2 = \frac{\partial y^j}{\partial y^i}, \quad P_2 = \delta^i_j - \frac{y^j y^j}{K^2}.
\] (19)

As usual for cotangent bundle of a Cartan space (see Theorem 2.1 from [11]), the second vertical Liouville distribution \( V_{E_2} \) is integrable and it defines a foliation on \( TM \), called the second vertical Liouville foliation on the big-tangent manifold \( TM \). Also, some geometric properties of the leaves of vertical foliation \( V_2 \) can be derived via the second vertical Liouville foliation \( V_{E_2} \).
2.2. Vertical Liouville distribution $V_E$

In this subsection we unify the concepts presented in the previous subsection and we define a vertical Liouville distribution on $\mathcal{T}M$ as the complementary orthogonal distribution in $V$ to the line distribution spanned by the Liouville vector field $\mathcal{E} = E_1 + E_2$. We prove that this distribution is an integrable one, and also, we find some geometric properties of both leaves of Liouville distribution and the vertical distribution on the big-tangent manifold $\mathcal{T}M$. Finally, some links between the vertical Liouville foliations $V_{E_1}$, $V_{E_2}$ and $V_E$, respectively, are established.

By (4), (5) and (6) we have

$$G(\mathcal{E}, \mathcal{E}) = F^2 + K^2.$$ (20)

Now, by means of $G$ and $\mathcal{E}$, we define the vertical one form $\zeta$ by

$$\zeta(X) = \frac{1}{\sqrt{F^2 + K^2}} G(X, \mathcal{E}), \quad \forall X = X_1(x, y, p) \frac{\partial}{\partial y} + X_2(x, y, p) \frac{\partial}{\partial p} \in \Gamma(V).$$ (21)

Let us denote by $\{\mathcal{E}\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}$ and we define the vertical Liouville distribution as the complementary orthogonal distribution $V_E$ to $\{\mathcal{E}\}$ in $V$ with respect to $G$. Thus, $V_E$ is defined by $\zeta$, that is

$$\Gamma(V_E) = \{X \in \Gamma(V) : \zeta(X) = 0\}.$$ (22)

We get that every vertical vector field $X = X_1(x, y, p) \frac{\partial}{\partial y} + X_2(x, y, p) \frac{\partial}{\partial p}$ can be expressed in the form:

$$X = PX + \frac{1}{\sqrt{F^2 + K^2}} \zeta(X)\mathcal{E},$$ (23)

where $P$ is the projection morphism of $V$ on $V_E$.

Also, by direct calculus, we get

$$G(X, PY) = G(PX, PY) = G(X, Y) - \zeta(X)\zeta(Y), \quad \forall X, Y \in \Gamma(V).$$ (24)

With respect to the basis $\{\theta^i, k_i\}$ and $\{\theta^i \otimes \frac{\partial}{\partial y}, \theta^i \otimes \frac{\partial}{\partial p}, k_j \otimes \frac{\partial}{\partial y}, k_j \otimes \frac{\partial}{\partial p}\}$, respectively, $\zeta$ and $P$ are locally given by

$$\zeta = \zeta_i \theta^i + \zeta^i k_i, \quad P = P^i_j \theta^i \otimes \frac{\partial}{\partial y} + P^i_j k_j \otimes \frac{\partial}{\partial p} + P^3_j \theta^i \otimes \frac{\partial}{\partial p} + P^4_j k_j \otimes \frac{\partial}{\partial y},$$ (25)

where their local components are expressed by

$$\zeta_i = \frac{y_i}{\sqrt{F^2 + K^2}}, \quad \zeta^i = \frac{p_i}{\sqrt{F^2 + K^2}},$$ (26)

$$P^1_j = \delta^j_i - \frac{y_j y^i}{F^2 + K^2}, \quad P^2_j = \delta^j_i - \frac{p_j p^i}{F^2 + K^2}, \quad P^3_j = -\frac{y_j p^i}{F^2 + K^2}, \quad P^4_j = -\frac{p_j p^i}{F^2 + K^2}.$$ (27)

**Remark 2.1.** We have the following relations between $\zeta$, $P$, $\zeta_1$, $\zeta_2$, $P_1$ and $P_2$:

$$\zeta(X) = \frac{F}{\sqrt{F^2 + K^2}} \zeta_1(X_1) + \frac{K}{\sqrt{F^2 + K^2}} \zeta_2(X_2),$$ (28)

$$P(X) = P_1(X_1) + P_2(X_2) + \frac{1}{F^2 + K^2} \left( \frac{\zeta_1(X_1)}{F^2} - \frac{\zeta_2(X_2)}{K} \right) (K^2 E_1 - F^2 E_2),$$ (29)

for every vertical vector field $X = X_1 + X_2 = X_1(x, y, p) \frac{\partial}{\partial y} + X_2(x, y, p) \frac{\partial}{\partial p}$. 

Theorem 2.2. The vertical Liouville distribution $V_E$ is integrable and it defines a foliation on $\mathcal{T}M$, called vertical Liouville foliation on the big-tangent manifold $\mathcal{T}M$.

Proof. Follows using an argument similar to that used in [4]. Let $X,Y \in \Gamma(V_E)$. As $V$ is an integrable distribution on $\mathcal{T}M$, it is sufficient to prove that $[X,Y]$ has no component with respect to $E$.

It is easy to see that a vertical vector field $X = X_i(x,y,p) \frac{\partial}{\partial y^i} + X_j^l(x,y,p) \frac{\partial}{\partial p^l}$ is in $\Gamma(V_E)$ if and only if

$$g_{ij}(x,y)X_i^j y^j + g^{ij}(x,p)X_j^i p^j = 0.$$  

(30)

Differentiating (30) with respect to $y^k$ we get

$$\frac{\partial g_{ij}}{\partial y^k} X_i^j y^j + g_{ik} X_i^j + g_{ij} \frac{\partial X^l_k}{\partial y^j} y^j + g^{ij} p^k \frac{\partial X^l_k}{\partial y^j} = 0, \forall k = 1,\ldots,n$$

(31)

and taking into account the relation $\frac{\partial y^j}{\partial y^i} y^i = 0$ (see (5)), one gets

$$g_{ik} X_i^j + g_{ij} \frac{\partial X^l_k}{\partial y^j} + g^{ij} p^k \frac{\partial X^l_k}{\partial y^j} = 0, \forall k = 1,\ldots,n.$$  

(32)

Similarly, differentiating (30) with respect to $p_k$ we get

$$g_{ij} y^j \frac{\partial X^l_k}{\partial p_k} + g^{ij} X^l_k + g_{ij} p^j \frac{\partial X^l_k}{\partial p_k} = 0, \forall k = 1,\ldots,n$$

(33)

and taking into account the relation $\frac{\partial y^j}{\partial p_k} p^k = 0$ (see (6)), one gets

$$g^{ij} X^l_k + g_{ij} y^j \frac{\partial X^l_k}{\partial p_k} + g^{ij} p^j \frac{\partial X^l_k}{\partial p_k} = 0, \forall k = 1,\ldots,n.$$  

(34)

Let $X = X_i^j(x,y,p) \frac{\partial}{\partial y^i} + X_j^l(x,y,p) \frac{\partial}{\partial p^l}$, $Y = Y_i^j(x,y,p) \frac{\partial}{\partial y^i} + Y_j^l(x,y,p) \frac{\partial}{\partial p^l} \in \Gamma(V)$. Then, by direct calculations using (32) and (34), we have

$$G([X,Y], E) = g_{ij} y^j \left( X_i^j \frac{\partial Y^l_k}{\partial y^i} - Y_i^j \frac{\partial X^l_k}{\partial y^i} \right) + g^{ij} p^j \left( X_i^j \frac{\partial Y^l_k}{\partial y^i} - Y_i^j \frac{\partial X^l_k}{\partial y^i} \right)$$

$$+ g_{ik} y^j \frac{\partial X^l_k}{\partial y^j} - g^{ij} p^j \frac{\partial X^l_k}{\partial y^j} + g_{ik} y^j \frac{\partial Y^l_k}{\partial y^j} - g^{ij} p^j \frac{\partial Y^l_k}{\partial y^j}$$

$$= -g_{ij} y^j X_i^j + g_{ij} y^j Y_i^j - g^{ij} Y^l_k X_j^l + g^{ij} Y^l_k Y_j^l$$

$$= 0$$

which completes the proof. □

Remark 2.3. The proof of Theorem 2.2 can be also obtained using an argument similar to [7]. More exactly, if we consider $P(\frac{\partial}{\partial y^j}) = P_j^i \frac{\partial}{\partial y^i} + P_{ij} \frac{\partial}{\partial p^j}$ and $P(\frac{\partial}{\partial p^j}) = P_j^i \frac{\partial}{\partial y^i} + P_{ij} \frac{\partial}{\partial p^j}$, by direct calculus we obtain

$$P(\frac{\partial}{\partial y^j})(\sqrt{F^2 + K^2}) = P(\frac{\partial}{\partial p^j})(\sqrt{F^2 + K^2}) = 0.$$  

(35)

Now, since $V = V_E \oplus \{E\}$ is integrable, the Lie brackets of vector fields from $V_E$ are given by

$$\left[ P(\frac{\partial}{\partial y^j}), P(\frac{\partial}{\partial y^j}) \right] = A_{ij} P(\frac{\partial}{\partial y^i}) + B_{ijk} P(\frac{\partial}{\partial p^j}) + C_{ij} E,$$  

(36)
we denote by $L$. In order to obtain this interplay, we consider a leaf of formulas (36), (37) and (38) to the function $\sqrt{F^2 + K^2}$ and using (35), we obtain $C_{ij} \sqrt{F^2 + K^2} = F^i \sqrt{F^2 + K^2} = L^i \sqrt{F^2 + K^2} = 0$. This implies that $C_{ij} = F^i = L^i = 0$, and then the vertical Liouville distribution $V_E$ is integrable.

As usual, the Theorem 2.2, we may say that the geometry of the leaves of vertical foliation $V$ should be derived from the geometry of the leaves of vertical Liouville foliation $V_L$ and of integral curves of $E$. In order to obtain this interplay, we consider a leaf $L_V$ of $V$ given locally by $x^i = a^i$, $i = 1, \ldots, n$, where the $a^i$’s are constants. Then, $g_{ij}(a, y)$ and $g^{ij}(a, p)$ are the components of a Riemannian metric $G_{L_V}$ on $L_V$. If we denote by $V$ the Levi-Civita connection on $L_V$ with respect to $G_{L_V}$ then its local expression is

$$\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C_{ij}^k(a, y) \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^i} = 0, \quad \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \quad \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = C_{ij}^k(a, p) \frac{\partial}{\partial y^k},$$

where $C_{ij}^k(a, y) = \frac{1}{2} g^{k\ell}(a, y) \frac{\partial g_{ij}(a, y)}{\partial \gamma^\ell}$ and $C_{ij}^k(a, p) = -\frac{1}{2} g^{k\ell}(a, p) \frac{\partial g^{ij}(a, p)}{\partial \gamma^\ell}$.

Contracting $C_{ij}^k(a, y)$ by $y^j$ and $C_{ij}^k(a, p)$ by $p_j$, respectively, we deduce

$$C_{ij}^k(a, y) y^j = 0, \quad C_{ij}^k(a, p) p_j = 0.$$

In the following lemma we obtain the covariant derivatives with respect to $V$ of $E$, $\zeta$ and $P$, respectively.

**Lemma 2.4.** On any leaf $L_V$ of $V$, we have

$$\nabla_X \left( \frac{E}{\sqrt{F^2 + K^2}} \right) = \frac{PX}{\sqrt{F^2 + K^2}},$$

$$\nabla_X \zeta = \frac{1}{\sqrt{F^2 + K^2}} G_{L_V}(PX, PY),$$

and

$$\nabla_X P = -\frac{1}{\sqrt{F^2 + K^2}} \left[ G_{L_V}(PX, PY) E + \sqrt{F^2 + K^2} \zeta(Y) PX \right],$$

for any $X, Y \in \Gamma(TL_V)$.

**Proof.** We take $X = X^i(a, y, p) \frac{\partial}{\partial y^i} + X^i_{ij}(a, y, p) \frac{\partial}{\partial y^j}, \quad Y = Y^i(a, y, p) \frac{\partial}{\partial y^i} + Y^i_{ij}(a, y, p) \frac{\partial}{\partial y^j} \in \Gamma(TL_V)$ and the relation (41) follows by:

$$\nabla_X \left( \frac{E}{\sqrt{F^2 + K^2}} \right) = \frac{X^i_{ij}}{\sqrt{F^2 + K^2}} \left[ \delta^j_{i} - \frac{y^j y_i}{F^2 + K^2} \frac{\partial}{\partial y^j} - \frac{p_j y_i}{F^2 + K^2} \frac{\partial}{\partial p_j} \right]$$

$$+ \frac{X^i_{ij}}{\sqrt{F^2 + K^2}} \left[ \delta^j_{i} - \frac{p_j p_i}{F^2 + K^2} \frac{\partial}{\partial p_j} - \frac{y^j p_i}{F^2 + K^2} \frac{\partial}{\partial y^j} \right]$$

$$= \frac{1}{\sqrt{F^2 + K^2}} \left[ X^i_{ij} \frac{1}{p_j} \frac{\partial}{\partial y^j} + X^i_{ij} \frac{3}{p_j} \frac{\partial}{\partial p_j} + X^i_{ij} \frac{4}{p_i} \frac{\partial}{\partial y^i} + X^i_{ij} \frac{2}{p_i} \frac{\partial}{\partial p_j} \right]$$

$$= \frac{PX}{\sqrt{F^2 + K^2}}.$$
For the relation (42) we have
\[
(V_X \zeta) Y = X(\zeta(Y)) - \zeta(V_X Y)
\]
\[
= X_1^i Y_i^j \partial \zeta_i^j / \partial y^i + X_1^i Y_i^j \partial \zeta_i^j / \partial p_i + X_2^i Y_i^j \partial \zeta_i^j / \partial p_i
\]
\[
= \frac{X_1^i Y_i^j}{\sqrt{F^2 + K^2}} \left( g_{ij} - \frac{y_i p_j}{F^2 + K^2} \right) - \frac{X_2^i Y_i^j}{\sqrt{F^2 + K^2}} \left( g_{ij} - \frac{p_i p_j}{F^2 + K^2} \right).
\]

On the other hand we have
\[
G_{L_v}(P, X, Y) = G_{L_v}(X, Y, \zeta(X)) \zeta(Y)
\]
\[
= X_1^i Y_i^j g_{ij} + X_2^i Y_i^j g_{ij} = \frac{(X_1^i y_i + X_2^i p_j)(Y_1^i y_j + Y_2^i p_j)}{F^2 + K^2}
\]
and the relation (42) follows easily.

The relation (43) follows using (23), (41) and (42).

**Theorem 2.5.** Let \((M, F)\) be a n-dimensional Finsler space and \(L_{V_L}, L_{V_E}\) and \(y\) be a leaf of \(V\), a leaf of \(V_E\) that lies in \(L_V\), and an integral curve of \(\frac{E}{\sqrt{F^2 + K^2}}\), respectively. Then the following assertions are valid:

i) \(y\) is a geodesic of \(L_V\) with respect to \(V\).

ii) \(L_{V_E}\) is totally umbilical immersed in \(L_V\).

iii) \(L_{V_E}\) lies in the generalized indicatrix \(I_e = \{ (y, p) \in T_y M^0 \oplus T_y M^0 : F^2(a, y) + K^2(a, p) = 1 \}\) and has constant mean curvature equal to \(-1\).

**Proof.** Replace \(X\) by \(\frac{E}{\sqrt{F^2 + K^2}}\) in (41) and we obtain i). Taking into account that \(\frac{E}{\sqrt{F^2 + K^2}}\) is the unit normal vector field of \(L_{V_E}\), the second fundamental form \(B\) of \(L_{V_E}\) as a hypersurface of \(L_V\) is given by
\[
B(X, Y) = \frac{1}{\sqrt{F^2 + K^2}} G_{L_v}(V_X Y, E), \forall X, Y \in \Gamma(TL_{V_E}).
\]

On the other hand, by using (41) and taking into account that \(G_{L_v}\) is parallel with respect to \(V\), we deduce that
\[
G_{L_v}(V_X Y, E) = -G_{L_v}(X, Y), \forall X, Y \in \Gamma(TL_{V_E}).
\]

Hence,
\[
B(X, Y) = -\frac{1}{\sqrt{F^2 + K^2}} G_{L_v}(X, Y), \forall X, Y \in \Gamma(TL_{V_E}),
\]
that is, \(L_{V_E}\) is totally umbilical immersed in \(L_V\). Now, we have
\[
\frac{g_{ij} y^j}{\sqrt{F^2 + K^2}} + \frac{g_{ij} p_i}{\sqrt{F^2 + K^2}} = \frac{\partial \sqrt{F^2 + K^2}}{\partial y^i} + \frac{\partial \sqrt{F^2 + K^2}}{\partial p_j}
\]
(47)
which says that \(\frac{E}{\sqrt{F^2 + K^2}}\) is a unit normal vector field for both \(L_{V_E}\) and the component \(l_a\). Thus, \(L_{V_E}\) lies in \(l_a\) and \(F^2(a, y) + K^2(a, p) = 1\) at any point \((y, p) \in L_{V_E}\). Then (46) becomes
\[
B(X, Y) = -G_{L_v}(X, Y), \forall X, Y \in \Gamma(TL_{V_E})
\]
(48)
which implies that

\[
\frac{1}{2^n - 1} \sum_{i=1}^{2n-1} e_i B(E_i, E_i) = -1, 
\]

where \( [E_i] \) is an orthonormal frame field on \( L_V \) of signature \( [e_i] \). Hence, the mean curvature of \( L_V \) is \(-1\) which completes the proof. \( \square \)

**Proposition 2.6.** Let \((M, F)\) be a \( n \)-dimensional Finsler space and \( L \) be a leaf of the vertical foliation \( V \). Then the sectional curvature of any nondegenerate plane section on \( L \) which contain the vertical Liouville vector field \( E \) is equal to zero.

**Proof.** Denote by \( R_{L_V} \) the curvature tensor field of \( V \) on \( L_V \). Then, by using (41) and (43), we obtain

\[
R_{L_V}(X, E)E = - \left( 1 - \frac{E(\sqrt{F^2 + K^2})}{\sqrt{F^2 + K^2}} \right) PX
\]

for every vector field \( X \) on \( L_V \). Now, taking into account \( E(\sqrt{F^2 + K^2}) = \sqrt{F^2 + K^2} \), the sectional curvature of a plane section \( \{X, E\} \) vanishes on \( L_V \). \( \square \)

**Remark 2.7.** Let \((M, F)\) be a \( n \)-dimensional Finsler space. Then there exist no leaves of \( V \) which are positively or negatively curved.

Finally, let us study certain relations between the vertical Liouville foliations \( V_{E_1}, V_{E_2} \) and \( V_E \), respectively.

We notice that we have the following decompositions of the vertical distribution:

\[
V = V_{E_1} \oplus V_{E_2} \oplus \{E_1\} \oplus \{E_2\} \quad \text{and} \quad V = V_E \oplus \{E\}. \quad (51)
\]

Taking into account that \( \left[ P^i_j, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right] = 0 \) and \( [E_1, E_2] = 0 \) we get that both distributions \( V_{E_1} \oplus V_{E_2} \) and \( \{E_1\} \oplus \{E_2\} \) are integrable. Evidently, \( \{E\} \subset \{E_1\} \oplus \{E_2\} \) and by (28) we have also \( V_{E_1} \oplus V_{E_2} \subset V_E \). Thus, we have the following vertical subfoliations on \( TM \):

\[
\{E\} \subset \{E_1\} \oplus \{E_2\} \subset V, \quad V_{E_1} \oplus V_{E_2} \subset V_E \subset V. \quad (52)
\]

The relations (51) says that \( \{E\} \) and \( V_{E_1} \oplus V_{E_2} \) have the same orthogonal complement in \( \{E_1\} \oplus \{E_2\} \) and in \( V_E \), respectively. It is a line distribution \( \{E'\} \), where \( E' = K^2 E_1 - F^2 E_2 \), see (29) (or by direct calculations in \( G(\alpha_1 E_1 + \alpha_2 E_2, E) = 0 \) it results \( \alpha_1 = K^2 \) and \( \alpha_2 = -F^2 \)). Thus

\[
\{E_1\} \oplus \{E_2\} = \{E\} \oplus \{E'\}, \quad V_E = V_{E_1} \oplus V_{E_2} \oplus \{E'\}. \quad (53)
\]

**Proposition 2.8.** The leaves of the foliation \( \{E_1\} \oplus \{E_2\} \) are totally geodesic submanifolds of the leaves of vertical foliation \( V \).

**Proof.** Follows easily taking into account that \( V_{E_1} E_1 = E_1, \quad V_{E_2} E_2 = V_{E_2} E_1 = 0, \quad V_E E_2 = E_2 \). \( \square \)

Also by direct calculus we obtain \( \nabla_E E' = -K^2 F^2 E + (K^2 - F^2) E' \notin \Gamma(\{E'\}) \), which leads to

**Proposition 2.9.** If \( \gamma \) is an integral curve of \( E' \) then it is not a geodesic of a leaf of vertical foliation \( V \).

A natural question is if between the foliations \( V_{E_1} \oplus V_{E_2} \) and \( V_E \) exists certain relations. Although the leaves of \( V_{E_1} \) are totally umbilical submanifolds of the leaves of \( V_1 \), the leaves of \( V_{E_2} \) are totally umbilical submanifolds of the leaves of \( V_2 \) and the leaves of \( V_E \) are totally umbilical submanifolds of the leaves of \( V \), we have...
Theorem 2.10. The leaves of $V_{E_1} \oplus V_{E_2}$ are not totally umbilical submanifolds of the leaves of $V_E$.

Proof. Taking into account that $E'_{FK}$ is the unit normal vector field of $L_{V_{E_1} \oplus V_{E_2}}$, the second fundamental form $B'$ of $L_{V_{E_1} \oplus V_{E_2}}$ as hypersurface of $L_{V_E}$ is given by

$$B'(X', Y') = \frac{1}{FK \sqrt{I^2 + K^2}} G_{L_E}(V_X Y', E'), \forall X', Y' \in \Gamma(TL_{V_{E_1} \oplus V_{E_2}}).$$

(54)

Taking into account that $G_{L_E}$ is parallel with respect to $V$, we deduce that

$$G_{L_E}(V_X Y', E') = -G_{L_E}(Y', V_X E'), \forall X', Y' \in \Gamma(TL_{V_{E_1} \oplus V_{E_2}}).$$

(55)

Now, let us take $X' = P_1(X_1) + P_2(X_2)$ and $Y' = P_1(Y_1) + P_2(Y_2)$ for every $X_1, Y_1 \in \Gamma(V_1)$ and $X_2, Y_2 \in \Gamma(V_2)$. Then by direct calculus we get

$$V_X E' = K^2 P_1(X_1) - F^2 P_2(X_2).$$

(56)

Thus the relation (54) becomes

$$B'(X', Y') = \frac{-1}{FK \sqrt{I^2 + K^2}} G_{L_E}(K^2 P_1(X_1) - F^2 P_2(X_2), Y') \neq \lambda G_{L_E}(X', Y'),$$

(57)

that is, $L_{V_{E_1} \oplus V_{E_2}}$ is not totally umbilical immersed in $L_{V_E}$. 

Acknowledgement. The authors are grateful to the referee for useful suggestions and comments.

References