EP elements and $\ast$–Strongly Regular Rings

Hua Yao*, Junchao Wei*

*School of Mathematics, Yangzhou University, Yangzhou, 225002, P. R. China

Abstract. Let $R$ be a ring with involution $\ast$. An element $a \in R$ is called $\ast$–strongly regular if there exists a projection $p$ of $R$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p$ is invertible, and $R$ is said to be $\ast$–strongly regular if every element of $R$ is $\ast$–strongly regular. We discuss the relations among strongly regular rings, $\ast$–strongly regular rings, regular rings and $\ast$–regular rings. Also, we show that an element $a$ of a $\ast$–ring $R$ is $\ast$–strongly regular if and only if $a$ is EP. We finally give some characterizations of EP elements.

1. Introduction

In this article, all rings are associative with identity unless otherwise stated, and modules will be unitary modules. Let $R$ be a ring, write $E(R), N(R), U(R), J(R)$ and $Z(R)$ to denote the set of all idempotents, the set of all nilpotents, the set of units, the Jacobson radical and the center of $R$, respectively.

Rings in which every element is the product of a unit and an idempotent which commute are said to be strongly regular, and have been studied by many authors. According to Koliha and Patricio [11], the commutant and double commutant of an element $a \in R$ are defined by $comm(a) = \{x \in R | xa = ax\}$ and $comm^2(a) = \{x \in R | xy = yx \text{ for all } y \in comm(a)\}$. It is known that a ring $R$ is strongly regular if and only if for each $a \in R$, there exists an idempotent $p \in comm(a)$ such that $a + p \in U(R)$ and $ap = 0$.

Let $R$ be a ring and write $R^{nil} = \{a \in R | 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. Recall that an element $a \in R$ is called polar (quasipolar) provided that there exists an idempotent $p \in R$ such that $p \in comm^2(a), a + p \in U(R)$ and $ap \in N(R)$ ($ap \in R^{nil}$), the idempotent $p$ is unique, we denote it by $a^\pi$, which is called a spectral idempotent of $a$. A ring $R$ is polar [7] (quasipolar [18]) in the case that every element in $R$ is polar (quasipolar). [5, Theorem 2.4] shows that a ring $R$ is strongly regular if and only if $R$ is a quasipolar ring and $R^{nil} = \{0\}$.

Following [3], an element $a$ of a ring $R$ is called group invertible if there is $a^g \in R$ such that $aa^g a = a, a^g a a^g = a^g, a a^g = a^g a$.

Denote by $R^f$ the set of all group invertible elements of $R$. Clearly, a ring $R$ is strongly regular if and only if $R = R^f$.

An involution $a \mapsto a^\ast$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^\ast)^\ast = a, (a + b)^\ast = a^\ast + b^\ast, (ab)^\ast = b^\ast a^\ast.$$
A ring $R$ with an involution $\ast$ is called $\ast$-ring. An element $a^\dagger$ in a $\ast$-ring $R$ is called the Moore-Penrose inverse (or MP-inverse) of $a$. If $a$, then

$$aa^\dagger a = a, a^\dagger = a^\dagger, a = (a^\dagger)^\dagger, a^\dagger = (a^\dagger)^\dagger.$$ 

In this case, we call $a$ is MP-invertible in $R$. The set of all MP-invertible elements of $R$ is denoted by $R^\dagger$. An involution $\ast$ of $R$ is called proper if $x^\ast x = 0$ implies $x = 0$ for all $x \in R$. Following [1], a $\ast$-ring $R$ is $\ast$-regular if and only if for each $R$ is regular and the involution is proper.

An idempotent $p$ of a $\ast$-ring $R$ is called projection if $p = p^\dagger$. Denote by $PE(R)$ the set of all projection elements of $R$. Clearly, $PE(R) \subseteq E(R)$. It is known that an idempotent $e$ in a $\ast$-ring $R$ is projection if and only if $e = e^\dagger e$ if and only if $Re = Re^\dagger$. [6, Lemma 2.1] shows that a $\ast$-ring $R$ is $\ast$-regular if and only if for each $a \in R$, there exists $p \in PE(R)$ such that $aR = pR$.

Following [11], a $\ast$-ring $R$ is $\ast$-regular if and only if $R = R^\dagger$. Due to [9], a $\ast$-ring $R$ is said to satisfy the $k$-term star-cancellation law (or $SC_k$) if

$$a_1^\dagger a_1 + \cdots + a_k^\dagger a_k = 0 \implies a_1 = \cdots = a_k = 0.$$ 

[10] shows that the $2 \times 2$ matrix ring $M_2(R)$ over a $\ast$-ring $R$ is $\ast$-regular if and only if $R$ is regular and satisfies $SC_2$.

Due to [8], an element $a$ of a $\ast$-ring $R$ is said to be $EP$ if $a \in R^\dagger \cap R^\ast$ and $a^\dagger = a^\ast$. In [14], many characterizations of $EP$ elements are given.

The $EP$ matrices and $EP$ linear operators on Banach or Hilbert spaces have been investigated by many authors. This article is motivated by the papers [6, 14]. In this paper, we shall first give some new characterizations of $EP$ elements. Next, we introduce $\ast$-strongly regular elements and $\ast$-strongly regular rings. We investigate the characterizations of $\ast$-strongly regular rings. Finally, we discuss $\ast$-exchange rings. With the help of $\ast$-exchange rings, we give some characterizations of $\ast$-strongly regular rings.

2. Some Characterizations of $EP$ elements

Let $R$ be a $\ast$-ring and $a \in R^\dagger$. Then by [14, Theorem 1.1], one knows that $a^\ast = a^\dagger a^\ast = a^\ast a^\dagger$. Hence we have the following proposition.

**Proposition 2.1.** Let $R$ be a $\ast$-ring and $a \in R$. Then $a$ is an $EP$ element if and only if $a \in R^\dagger$ and $Ra = Ra^\dagger$.

**Proof.** Suppose that $a$ is $EP$. Then $a \in R^\dagger \cap R^\ast$ and $a^\dagger = a^\ast$, it follows that $Ra = Ra^\dagger a = Ra^\dagger = Ra^\ast = Ra^\dagger = Ra^\ast$.

Conversely, assume that $a \in R^\dagger$ and $Ra = Ra^\dagger$. Then $Ra = Ra^\dagger = R(aa^\dagger)^\dagger = R(a^\dagger)^\dagger a = Ra^\dagger \subseteq Ra^\ast = Ra^\ast a^\dagger \subseteq Ra^\ast = Ra^\ast$. It follows that $Ra = Ra^\dagger$. By [13, Theorem 3.1], one knows that $a$ is $EP$. \(\square\)

Similar to the proof of Proposition 2.1, we have the following corollary.

**Corollary 2.2.** Let $R$ be a $\ast$-ring and $a \in R$. Then $a$ is an $EP$ element if and only if $a \in R^\dagger$ and $aR = a^\dagger R$.

It is known that for a $\ast$-ring $R$, $a \in R$ is $EP$ if and only if $a^\dagger$ is $EP$. Hence we can obtain the following corollary.

**Corollary 2.3.** Let $R$ be a $\ast$-ring and $a \in R$. Then $a$ is an $EP$ element if and only if $a \in R^\dagger$ and $Ra^\ast = R(a^\dagger)^\ast$.

**Proof.** Suppose that $a$ is $EP$. Then Proposition 2.1 and [13, Theorem 3.1] imply $Ra^\ast = Ra = Ra^\dagger$. Note that $a^\dagger$ is $EP$. Then [13, Theorem 3.1] gives $Ra^\dagger = R(a^\dagger)^\dagger$. Hence $Ra^\ast = R(a^\dagger)^\ast$.

Conversely, assume that $Ra^\ast = R(a^\dagger)^\ast$. Then $aR = a^\dagger R$, by Corollary 2.2, one gets $a$ is $EP$. \(\square\)

**Theorem 2.4.** Let $R$ be a $\ast$-ring and $a \in R$. Then the following conditions are equivalent:

1. $a$ is $EP$;
2. $a \in R^\dagger$ and $Ra = R(a^\dagger)^n$ for each $n \geq 2$;
3. $a \in R^\dagger$ and $Ra = R(a^\dagger)^n$ for some $n \geq 2$. 
Proof. (1) \implies (2) Since $a$ is a EP, by Proposition 2.1, we have $a \in R^i$ and $Ra = Ra^i$. Noting that $Ra^i = Rad = Ra^i a^i = R(a^i)^i$, repeating the process, one obtains that $Ra = R(a^i)^n$ for each $n \geq 2$.

(2) \implies (3) It is trivial.

(3) \implies (1) Since $Ra = R(a^i)^n$ for some $n \geq 2$, $Ra \subseteq Ra^i = Ra^i$. Note that $Ra^i = Rad$. Then $Ra^i = R(a^i)^{n+1} \subseteq R(a^i)^n = Ra$, it follows that $Ra \subseteq Ra^i = Ra^i \subseteq Ra$. Hence $Ra = Ra^i = Ra^i$, this implies that $a$ is EP.

Let $R$ be a $\ast$--ring and $a \in R$. Then it is easy to show that $a \in R^i$ and $aa^i = 0$ imply $a = 0$. Also, $a \in R^i \cap R^f$ is EP if and only if $aa^i = a^i a$. Hence we have the following theorem.

**Theorem 2.5.** Let $R$ be a $\ast$--ring and $a \in R$. Then the following conditions are equivalent:

1. $a$ is EP;
2. $a \in R^i \cap R^f$ and $a^2 a = a^2 a$;
3. $a \in R^i \cap R^f$ and $a^2 a = aa a$;
4. $a \in R^i \cap R^f$ and $a^2 a = a^{-1} a^i a$ for some $n \geq 2$.

Proof. (1) \implies (2) It is trivial.

(2) \implies (3) Suppose that $a^2 a = a^2 a$. Then $aa = a^2 a = a^2 a = a^2 a a = a^2 a = a^2 a = a^2 a$, so $a = (a^2 a a) = a^2 a$. Hence $aa = aa = a^2 a a = a^2 a = a^2 a$.

(3) \implies (4) Suppose that $a^2 a = a^2 a a$. Then $aa = a^2 a = a^2 a a = a^2 a a = a^2 a a$, so $a = a^2 a a = a^2 a a$. Hence $aa = a^2 a a = a^2 a a$.

(4) \implies (1) Assume that $a^2 a = a^2 a a$. Then $aa = a^2 a = a^2 a a = a^2 a a = a^2 a a$, so $a = a^2 a a = a^2 a a$. Hence $a = EP$.

**Remark:** The condition (4) of Theorem 2.5 exists in [12, Theorem 2.1(xii)] for $m = n - 1$ and $n = 1$.

**Theorem 2.6.** Let $R$ be a $\ast$--ring and $a \in R$. Then the following conditions are equivalent:

1. $a$ is EP;
2. $a \in R^i \cap R^f$ and $a^2 a = a + a^i$;
3. $a \in R^i \cap R^f$ and $a^2 a = a + a^i$;
4. $a \in R^i \cap R^f$ and $a^2 a = 2a^i a$;
5. $a \in R^i \cap R^f$ and $a^2 a = 2a^i a$.

Proof. (1) \implies (i, i = 2, 3, 4, 5, 6) They are trivial.

(2) \implies (1) From the assumption $a^2 a = a + a^i$, we get $a^2 a = a + a^i$. So, $a^2 a = a + a^i$, it follows that $a = EP$.

(3) \implies (1) By the equality $a^2 a^i = a + a^i$, we get $a^2 a^i + a^2 a^i = a + a^i$, this gives $a^2 a = a + a^i$. Hence $a = EP$.

(4) \implies (1) Using the equality $a^2 a^i = a + a^i$, we have $2a^i = 2a^i a^i = 2a^i a^i = 2a^i + a^2 a^i$, it follows that $aa = a a^i$. Hence $a = EP$.

(5) \implies (1) The equality $a^i = a^i + a^i = 2a^i a^i$ gives $aa = a + a^i = 2a^i a^i$, again we have $a a^i = a a^i$. Hence $a = EP$.

(6) \implies (1) If $a^i + a^i = 2a^i a^i$, then $a^i + a^i = 2a^i a^i + a^i = 2a^i a^i = 2a^i a^i$, one obtains that $a^i = a^i a^i$. Hence $a = a a^i$ and so $a = EP$.

**Remark:** The condition (4) of Theorem 2.6 exists in [12, Theorem 2.1(xv)] for $n = 1$.

**Theorem 2.7.** Let $R$ be a $\ast$--ring. Then $E(R) = PE(R)$ if and only if every element of $E(R)$ is EP.

Proof. Let $e \in E(R)$. If $E(R) = PE(R)$, then $e = e^\ast$. It is not difficult to verify that $e$ is EP with $e^\ast = e^\ast = e$. Conversely, we assume that $e$ is EP. Then $e^\ast = e^\ast$, it follows that $e = e e^\ast e = e e^\ast$ and so $e^\ast = e^\ast = (e e^\ast e)^\ast = e e^\ast e = e$. Hence $e \in PE(R)$. □
Recall that a ring $R$ is directly finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. Clearly, a ring $R$ is directly finite if and only if right invertible element of $R$ is invertible.

**Theorem 2.8.** Let $R$ be a $*$-ring. Then the following conditions are equivalent:

1. $R$ is a directly finite ring;
2. Every right invertible element of $R$ is group invertible;
3. Every right invertible element of $R$ is EP.

**Proof.** (1) $\implies$ (3) It is trivial because every invertible element is EP.

(3) $\implies$ (2) It is evident.

(2) $\implies$ (1) Suppose that $a, b \in R$ with $ab = 1$. By hypothesis, $a \in R^\#$, so $1 = ab = (aa^\#)(ab) = aa^\# = a^\#a$, one obtains that $a$ is invertible. Hence $R$ is directly finite. \hfill \Box

Recall that a ring $R$ is reduced if $N(R) = \{0\}$. Using the EP elements, we can characterize reduced rings as follows.

**Theorem 2.9.** Let $R$ be a $*$-ring. Then the following conditions are equivalent:

1. $R$ is a reduced ring;
2. Every element of $N(R)$ is group invertible;
3. Every element of $N(R)$ is EP.

**Proof.** (1) $\implies$ (3) $\implies$ (2) They are trivial.

(2) $\implies$ (1) Suppose that the condition (2) holds. If $R$ is not reduced, then there exists $b \in R \setminus \{0\}$, let $n$ be the positive integer such that $b^n = 0$ and $b^{n-1} \neq 0$. Choose $a = b^{n-1}$. Then $a \in R \setminus \{0\}$ with $a^2 = 0$. Since $a \in R^\#$, $a = a^\#a^\# = 0$, which is a contradiction. Hence $R$ is reduced. \hfill \Box

**Theorem 2.10.** Let $R$ be a $*$-ring and $a \in R$. Then $a$ is EP if and only if there exists (unique) $p \in PE(R)$ such that $pa = ap = 0$ and $a + p \in U(R)$.

**Proof.** It is similar to the proof of [2, Theorem 2.1]. \hfill \Box

Also, similar to the proof of [2, Theorem 2.1], we have the following corollary.

**Corollary 2.11.** Let $R$ be a $*$-ring and $a \in R$. Then $a$ is EP if and only if there exists unique $p \in PE(R)$ such that $pa = ap = 0$ and $a + p \in U(R)$.

**Corollary 2.12.** Let $R$ be a $*$-ring and $a \in R$. Then $a$ is EP if and only if there exists $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p \in U(R)$.

**Proof.** The sufficiency follows from Theorem 2.10.

The necessity: Noting that $p = 1 - a^\#a$ in Theorem 2.10. Then, for any $x \in comm(a)$, we have $(1 - p)xp = a^\#axp = a^\#xp = 0$ and $px(1 - p) = pxa^\# = paxa^\# = 0$, this implies that $px = pxp = xp$. Hence $p \in comm^2(a)$, we are done. \hfill \Box

Similarly, we have the following corollary.

**Corollary 2.13.** Let $R$ be a $*$-ring and $a \in R$. Then $a$ is EP if and only if there exists unique $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p \in U(R)$.

**Theorem 2.14.** Let $R$ be a $*$-ring and $a \in R$. Then $a$ is EP if and only if there exists $b \in comm^2(a)$, $ab = ba \in PE(R)$, $a = a^\#b$ and $b = ab^\#$.

**Proof.** Suppose that $a$ is EP. Then by Corollary 2.12, there exists $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p \in U(R)$. Choose $b = (a + p)^{-1}(1 - p)$. Then clearly, $b \in comm^2(a)$ and $ab = ba = 1 - p \in PE(R)$. By a simple computation, we have $a = a^\#b$ and $b = ab^\#$.

Conversely, assume that there exists $b \in comm^2(a)$, $ab = ba \in PE(R)$, $a = a^\#b$ and $b = ab^\#$. Choose $p = 1 - ab$. Then $p \in PE(R)$, $ap = a - a^\#b = 0 = pa$ and $pb = b - ab^\# = 0 = bp$. Note that $(a + p)(b + p) = ab + p = 1$. Then $a + p \in U(R)$, by Theorem 2.10, $a$ is EP. \hfill \Box
3. $\leftrightarrow$-Strongly Regular Rings

Recall that an element $a$ of a ring $R$ is strongly regular if $a \in a^2R \cap Ra^2$. It is well known that $a \in R$ is strongly regular if and only if there exist $e \in E(R)$ and $u \in U(R)$ such that $a = eu = ue$.

Let $R$ be a $\leftrightarrow$-ring. An element $a \in R$ is called $\leftrightarrow$-strongly regular if there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. A ring $R$ is called $\leftrightarrow$-regular if every element of $R$ is $\leftrightarrow$-strongly regular.

Clearly, $\leftrightarrow$-strongly regular elements are strongly regular, and so $\leftrightarrow$-strongly regular rings are strongly regular. However, the converse is not true by the following example.

**Example 3.1.** Let $D$ be a division ring and $R = D \oplus D$. Set $*$ be an involution of $R$ defined by $*((a, b)) = (b, a)$. Evidently, $R$ is a strongly regular ring, but $R$ is not $\leftrightarrow$-strongly regular. In fact $(1, 0)$ is not a $\leftrightarrow$-strongly regular element.

**Theorem 3.2.** Let $R$ be a $\leftrightarrow$-ring. Then $R$ is a $\leftrightarrow$-strongly regular ring if and only if $R$ is a strongly regular ring with $E(R) = PE(R)$.

**Proof.** Suppose that $R$ is a $\leftrightarrow$-strongly regular ring and $e \in E(R)$. Then there exist $p \in PE(R)$ and $u \in U(R)$ such that $e = pu = up$, this gives $e = pe = ep$. Note that $p = eu^{-1}$. Then $p = ep = e$, so $E(R) \subseteq PE(R)$, this shows that $E(R) = PE(R)$.

The converse is trivial. $\square$

**Theorem 3.3.** Let $R$ be a $\leftrightarrow$-ring and $a \in R$. Then $a$ is $EP$ if and only if $a$ is $\leftrightarrow$-strongly regular.

**Proof.** Suppose that $a$ is $EP$. Then, by Theorem 2.10, there exists $p \in PE(R)$ such that $ap = pa = 0$. Write $a + p = u \in U(R)$. Then $a = a(1 - p) = u(1 - p) = (1 - p)u$. Since $1 - p \in PE(R)$, $a$ is $\leftrightarrow$-strongly regular.

Conversely, assume that $a$ is $\leftrightarrow$-strongly regular. Then there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. Since $(a + 1 - p)(u^{-1}p + 1 - p) = (u^{-1}p + 1 - p)a + 1 - p(1 - p) = 1, a + 1 - p \in U(R)$. Noting that $a(1 - p) = (1 - p)a = 0$ and $1 - p \in PE(R)$. Hence $a$ is $EP$ by Theorem 2.10. $\square$

**Theorem 3.4.** Let $R$ be a $\leftrightarrow$-ring. Then $R$ is $\leftrightarrow$-strongly regular if and only if $R$ is Abel and for each $a \in R$, $Ra = Ra^\prime$.

**Proof.** Suppose that $R$ is $\leftrightarrow$-strongly regular. Note that $\leftrightarrow$-strongly regular rings are strongly regular. Then $R$ is also Abel. Now let $a \in R$. Then $a$ is $\leftrightarrow$-strongly regular, so there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. Hence $a^\prime a = a^\prime up$, one obtains that $Ra^\prime a = Rp = Ra$.

Conversely, assume that $R$ is Abel and for each $a \in R$, $Ra = Ra^\prime$. Write that $a = da^\prime a$ for some $d \in R$. Then $(ad^\prime)^2 = da^\prime da^\prime = (da^\prime)da^\prime = d(\alpha d^\prime a)d^\prime = da^\prime ad^\prime = ad^\prime$. Noting that $R$ is Abel, $ad^\prime$ is a central idempotent of $R$, so $d^\prime a$ is a central idempotent of $R$, this gives that $a = (da^\prime)a = a(da^\prime)$. Hence $Ra \subseteq Ra^\prime$. By [4, Proposition 2.7], $R$ is a $\leftrightarrow$-regular ring, so $a \in R^\prime$. Thus by [13, Theorem 3.1], one knows that $a$ is $EP$, by Theorem 3.3, $a$ is $\leftrightarrow$-strongly regular. Hence $R$ is $\leftrightarrow$-strongly regular. $\square$

**Corollary 3.5.** A $\leftrightarrow$-ring $R$ is a $\leftrightarrow$-strongly regular ring if and only if $R$ is an Abel ring and $\leftrightarrow$-regular ring.

Let $R$ be a ring and write $ZE(R) = \{x \in R| \exists e \in E(R), x = xe\}$ for each $e \in E(R)$). It is easy to show that $ZE(R)$ is a subring of $R$ and $Z(R)$, the center, of $R$ is contained in $ZE(R)$.

Let $R$ be a $\leftrightarrow$-ring. Choose $a \in ZE(R)$ and $e \in E(R)$. Since $e \in E(R)$, $ae = ea$ follows. Hence $ae \in ZE(R)$, so $ZE(R)$ becomes a $\leftrightarrow$-ring.

**Theorem 3.6.** Let $R$ be a $\leftrightarrow$-regular ring. Then $ZE(R)$ is a $\leftrightarrow$-strongly regular ring.

**Proof.** Let $a \in ZE(R)$. Since $R$ is a $\leftrightarrow$-regular ring, by [6, Lemma 2.1], there exists $p \in PE(R)$ such that $aR = pR$. Write $p = ab$ for some $b \in R$. Then $a = pa = aba$. Choose $e \in E(R)$. Then $ae = ea$, it follows that $(1 - p)ea = (1 - p)ea = (1 - p)ae = 0$, this gives $(1 - p)ep = 0$, that is, $ep = pep$. Since $e^\prime \in E(R)$, $e^\prime p = pe^\prime p$, one obtains $pe = pep$. Hence $ep = pe$, this implies $p \in ZE(R)$. Note that $ba \in E(R)$. Then
Lemma 4.4. Let $R$ be a ring. If $R$ is a strongly regular ring, then $R$ is $\ast$-regular. Then there exist $u$ and $v$ such that $pu = up = v$ and $v = e$ for some $e \in R$.

Proof. The necessity follows from Corollary 3.5. Conversely, assume that $R$ is a $\ast$-regular ring. Then $R$ is a semiprime ring and $pR(1 - p)R = 0$ for each $p \in PE(R)$. Hence $R$ is $\ast$-Abel, by Corollary 3.5, $R$ is $\ast$-regular.

Corollary 3.7. Let $R$ be a $\ast$-ring. Then $R$ is a $\ast$-regular ring if and only if $R$ is an Abelian $\ast$-ring.

Proof. The necessity follows from Corollary 3.5. The converse follows from Corollary 3.3.

Corollary 3.8. Let $R$ be a $\ast$-ring. Then $R$ is a $\ast$-regular ring if and only if $R$ is a $\ast$-quasi-normal $\ast$-regular ring.

Proof. The necessity follows from Corollary 3.5. Conversely, assume that $R$ is a $\ast$-quasi-normal $\ast$-regular ring. Then $R$ is a semiprime ring and $pR(1 - p)R = 0$ for each $p \in PE(R)$, this implies $pR(1 - p) = 0 = (1 - p)R$. Hence $R$ is $\ast$-Abel, by Corollary 3.5, $R$ is $\ast$-regular.

Corollary 3.9. If $R$ is a $\ast$-regular ring, then so is $pR$ for any $p \in PE(R)$.

Proof. It follows from Corollary 3.5 and [6, Proposition 2.8].

4. $\ast$-Exchange Rings

Definition 4.1. Let $R$ be a $\ast$-ring and $a \in R$. If there exists $p \in PE(R)$ such that $p \in aR$ and $1 - p \in (1 - a)R$, then $a$ is called $\ast$-exchange element of $R$. And a $\ast$-ring $R$ is said to be $\ast$-exchange if every element of $R$ is $\ast$-exchange.

Clearly, any $\ast$-exchange element of a $\ast$-ring $R$ is exchange and the converse is true whenever $PE(R) = E(R)$.

Lemma 4.2. Let $R$ be a $\ast$-ring and $x \in R$. If $x$ is $\ast$-regular, then $x$ is $\ast$-exchange.

Proof. Suppose that $x$ is $\ast$-regular. Then there exist $u \in U(R)$ and $p \in PE(R)$ such that $x = pu = up$, and hence $x(1 - p) = 0$. Note that $p = xu^{-1}$ and $(1 - x)(1 - p) = 1 - p$. Hence $x$ is $\ast$-exchange.

Lemma 4.3. Let $R$ be a $\ast$-ring and $x \in R$. Then the following conditions are equivalent:

1. $x$ is $\ast$-exchange;
2. There exists $p \in PE(R)$ such that $p - x \in (x - x^2)R$.

Proof. (1) $\implies$ (2) Assume that $x$ is $\ast$-exchange. Then there exists $p \in PE(R)$ such that $p \in xR$ and $1 - p \in (1 - x)R$, this gives $p - x = (1 - x)p - x(1 - p) \in (x - x^2)R$.

(2) $\implies$ (1) Let $p \in PE(R)$ satisfy $p - x \in (x - x^2)R$. Write $p - x = (x - x^2)c$ for some $c \in R$. It follows that $p = x(1 + (1 - x)c) \in xR$ and $1 - p = (1 - x)(1 - x)c \in (1 - x)R$. Hence $x$ is $\ast$-exchange.

Let $R$ be a $\ast$-ring and $I$ be an (one-sided) ideal of $R$. $I$ is called $\ast$-(one-sided) ideal of $R$ if $a^* \in I$ for each $a \in I$. Clearly, the Jacobson radical $J(R)$ of a $\ast$-ring $R$ is $\ast$-ideal.

Lemma 4.4. Let $R$ be a $\ast$-exchange ring and $I$ a $\ast$-right ideal of $R$. Then the projection elements can be lifted modulo $I$. 
Proof. Let \( x \in R \) satisfy \( x - x^2 \in I \). Since \( R \) is \( \ast \)-exchange, there exists \( p \in PE(R) \) such that \( p - x \in (x - x^2)R \) by Lemma 4.3. Note that \( I \) is a \( \ast \)-right ideal of \( R \). Hence \( p - x \in I \), we are done.

**Lemma 4.5.** If \( R \) is a \( \ast \)-exchange ring, then \( E(R) = PE(R) \).

Proof. Let \( e \in E(R) \). Then by the hypothesis, there exists \( p \in PE(R) \) such that \( p \in eR \) and \( 1 - p \in (1 - e)R \). It follows that \( p = ep = e \). Hence \( e \in PE(R) \), this gives \( E(R) \subseteq PE(R) \). Therefore \( E(R) = PE(R) \).

Let \( R \) be a \( \ast \)-ring and \( I \) a \( \ast \)-ideal of \( R \). For each \( a = a + I \) in \( R = R/I \), we define \( a^* = a^* + I \). Then \( R/I \) becomes a \( \ast \)-ring.

**Theorem 4.6.** Let \( R \) be a \( \ast \)-ring. Then \( R \) is a \( \ast \)-exchange ring if and only if
1. \( R/J(R) \) is \( \ast \)-exchange ring;
2. Projection elements can be lifted modulo \( J(R) \);
3. \( E(R) = PE(R) \).

Proof. Suppose that \( R \) is \( \ast \)-exchange. Then the projection elements can be lifted modulo \( J(R) \) by Lemma 4.4 and \( E(R) = PE(R) \) by Lemma 4.5. Note that \( R \) is exchange. Then \( R/J(R) \) is exchange, it follows that \( R/J(R) \) is \( \ast \)-exchange because \( E(R) = PE(R) \).

Conversely, let \( a \in R \). Since \( R = R/J(R) \) is \( \ast \)-exchange, there exists \( p \in R \) such that \( \bar{p} \in PE(\bar{R}) \cap \bar{a} \bar{R} \) and \( \bar{1} - \bar{p} \in (1 - \bar{a})\bar{R} \). Note that the projection elements can be lifted modulo \( J(R) \). Then we can assume that \( p \in PE(R) \). Let \( b, c \in R \) satisfy \( p - ab \in J(R) \) and \( 1 - p - (1 - a)c \in J(R) \). Write \( u = 1 - p + ab \). Then \( u \in U(\bar{R}) \). Let \( e = upu^{-1} \). Then we have \( e^2 = e = abpu^{-1} \in aR \). Note that \( E(R) = PE(R) \). Then \( e \in PE(R) \). Since \( p - ab \in J(R) \), \( \bar{a} \bar{b} = \bar{p} \), it follows that \( \bar{a} \bar{b} = \bar{p} \bar{a} \bar{b} = \bar{1} \), so \( \bar{e} = \bar{a} \bar{b} \bar{p} \bar{u}^{-1} = \bar{p} \), \( e - p \in J(R) \), it follows that \( 1 - e - (1 - a)c = 1 - p - (1 - a)c + p - e \in J(R) \). Write \( 1 - e - (1 - a)c = d \in J(R) \). Then \( 1 = e(1 - d)^{-1} + (1 - a)c(1 - d)^{-1} \). Choose \( f = e + c(1 - d)^{-1}(1 - e) \). Then \( f \in PE(R) \cap aR \) and \( 1 - f = (1 - e)(1 - d)^{-1}(1 - e) = (1 - a)c(1 - d)^{-1}(1 - e) \). Therefore \( a \) is \( \ast \)-exchange and so \( R \) is \( \ast \)-exchange.

Theorem 4.6 implies the following corollary.

**Corollary 4.7.** A \( \ast \)-ring \( R \) is \( \ast \)-exchange if and only if \( R \) is exchange and \( PE(R) = E(R) \).

**Lemma 4.8.** Let \( R \) be a \( \ast \)-ring. Then \( E(R) = PE(R) \) if and only if for each \( e, g \in E(R) \), \( e^*e = ee^* \) and \( g^*g = 0 \) implies \( g = 0 \).

Proof. Suppose that \( E(R) = PE(R) \) and \( e \in E(R) \). We claim that \( eR(1 - e) = 0 \). If not, then there exists \( a \in R \) such that \( ae(1 - e) \neq 0 \). Note that \( g = e + e(1 - e) \in E(R) \). Then \( e + e(1 - e) = g = g^* = e^* + (1 - e^*)a^*e^* = e + (1 - e^*)e^* = e + e(1 - e) = e - e(1 - e) \), it follows that \( e(1 - e) = (1 - e^*)a^*e^* = e - e^*e^* = 0 \), which is a contradiction. Hence \( eR(1 - e) = 0 \).

Similarly, we can show that \( (1 - e)R = 0 \). Hence \( e^*e = ee^* = e^*e \).

Now assume that \( g \in E(R) \) and \( g^*g = 0 \). Noting that \( E(R) = PE(R) \). Then \( g^* = g \), so \( g = 0 \).

Conversely, let \( e \in E(R) \). Then by hypothesis, one has \( e^*e = ee^* \). Since \( e - e^*e \in E(R) \) and \( (e - e^*e)^*(e - e^*e) = 0 \), again by hypothesis, one obtains that \( e - e^*e = 0 \), this implies \( e \in PE(R) \). Hence \( E(R) = PE(R) \).

By the proof of Lemma 4.8, we have the following corollary.

**Corollary 4.9.** Let \( R \) be a \( \ast \)-ring and \( E(R) = PE(R) \). Then \( R \) is an Abel ring.

It is known that Abel exchange rings are clean. Hence Theorem 4.4 and Corollary 4.9 imply the following corollary.

**Corollary 4.10.** \( \ast \)-exchange rings are clean.
Corollary 4.11. Let $R$ be a $\ast-$ring. Then the following conditions are equivalent:
(1) $R$ is a $\ast-$exchange ring;
(2) $R$ is an exchange ring and $E(R) = PE(R)$;
(3) $R$ is a clean ring and $E(R) = PE(R)$.

The following corollary follows from [17, Theorem 3.3, Corollary 3.4, Theorem 3.12, Corollary 4.9], Corollary 4.7 and Corollary 4.9.

Corollary 4.12. Let $R$ be a $\ast-$exchange ring and $P$ is an ideal of $R$.
(1) If $P$ is a prime ideal of $R$, then $R/P$ is a local ring;
(2) If $P$ is a left (right) primitive ideal of $R$, then $R/P$ is a division ring;
(3) $R$ is a left and right quasi-duo ring;
(4) $R$ has stable range one.

Theorem 4.13. The following conditions are equivalent for a $\ast-$ring $R$:
(1) $R$ is a $\ast-$strongly regular ring;
(2) $R$ is a semiprime $\ast-$exchange ring and every prime ideal of $R$ is maximal;
(3) $R$ is a semiprime $\ast-$exchange ring and every prime ideal of $R$ is left (right) primitive.

Proof. (1) $\implies$ (2) Suppose that $R$ is $\ast-$strongly regular. Then, by Lemma 4.2, $R$ is $\ast-$exchange, this implies $R$ is left and right quasi-duo by Corollary 4.12. Note that $R$ is strongly regular. Hence, by [19, Theorem 2.6], $R$ is a semiprime and every prime ideal of $R$ is maximal.
(2) $\implies$ (3) It is trivial.
(3) $\implies$ (1) Suppose that $R$ is a semiprime $\ast-$exchange ring and every prime ideal of $R$ is left (right) primitive. Then $R$ is left and right quasi-duo by Corollary 4.12 and $PE(R) = E(R)$ by Theorem 4.6. Note that $R$ is strongly regular by [19, Theorem 2.6]. Hence $R$ is $\ast-$strongly regular by Theorem 3.2.

Corollary 4.14. Let $R$ be a $\ast-$exchange semiprimitive ring such that every left $R$-module has a maximal submodule, then $R$ is $\ast-$strongly regular.

Proof. Note that $R$ is left and right quasi-duo and $PE(R) = E(R)$ by Corollary 4.7 and Corollary 4.12. Then, by [19, Lemma 3.2], $R$ is von neumann regular, it follows that $R$ is $\ast-$strongly regular by Theorem 3.2.

Corollary 4.15. Let $R$ be a $\ast-$exchange ring. If every prime ideal of $R$ is left (right) primitive, then $R/J(R)$ is $\ast-$strongly regular.

Proof. Since $R$ is a $\ast-$exchange ring, by Theorem 4.6, $R/J(R)$ is $\ast-$exchange. Note that $R/J(R)$ is semiprime and every prime ideal of $R/J(R)$ is left (right) primitive. Then, by Theorem 4.13, one obtains that $R/J(R)$ is $\ast-$strongly regular.

References


