Warped Product Skew CR-Submanifolds of Kenmotsu Manifolds and their Applications

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Abstract. In this paper, we introduce the notion of warped product skew CR-submanifolds in Kenmotsu manifolds. We obtain several results on such submanifolds. A characterization for skew CR-submanifolds is obtained. Furthermore, we establish an inequality for the squared norm of the second fundamental form of a warped product skew CR-submanifold $M_1 \times f M_\perp$ of order 1 in a Kenmotsu manifold $\tilde{M}$ in terms of the warping function such that $M_1 = M_T \times M_\theta$, where $M_T$, $M_\perp$, and $M_\theta$ are invariant, anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. Finally, some applications of our results are given.

1. Introduction

The notion of CR-submanifolds was introduced by Bejancu [6] as a generalization of the complex and totally real submanifolds of almost Hermitian manifolds. A more general family of submanifolds are slant submanifolds introduced and defined by B.-Y. Chen [13, 14] in 1990. A generalization of slant submanifolds was given by Papaghiuc [34] by defining semi-slant submanifolds of almost Hermitian manifolds, for which the slant and CR-submanifolds are particular cases. Later on, J.L. Cabrerizo et al. [10, 11] studied slant and semi-slant submanifolds of an almost contact metric manifold.

On the other hand, A. Carriazo defined hemi-slant submanifolds under the name of anti-slant submanifolds [12] and showed that CR-submanifolds and slant submanifolds are hemi-slant submanifolds. In [37], B. Sahin studied these submanifolds under the name of semi-slant submanifolds for their warped products.

In [35], Ronsse introduced skew CR-submanifolds of Kaehler manifolds as a generalization of slant submanifolds and CR-submanifolds. It is important to observe that semi-slant submanifolds [34] and hemi-slant submanifolds [37] are particular cases of skew CR-submanifolds.

In the beginning of this century, B.-Y. Chen introduced the notion of warped product CR-submanifolds [15, 16]. On the basis of Chen’s idea on warped product submanifolds many articles have been appeared (for instance see [4, 5, 9, 17, 29, 32, 31, 36]) and references therein. For a detailed survey on warped product manifolds and warped product submanifolds we refere to Chen’s books [18, 20] and his survey article [19].
Recently, Sahin [38] introduced the notion of skew CR-warped products of Kaehler manifolds which are the generalizations of CR-warped products which are introduced by B.-Y. Chen [15] and warped product hemi-slant submanifolds studied in [37].

As Kenmotsu manifolds are themselves warped product manifolds, it is interesting to study warped product submanifolds of Kenmotsu manifolds. There are many papers on warped product submanifolds of Kenmotsu manifolds. Some basic lemmas are given which are useful in the next sections. In Section 3, we study warped product skew CR-submanifolds of Kenmotsu manifolds. We start with a non-trivial example of warped product skew CR-submanifolds and then we derive some useful lemmas. In Section 5, necessary and sufficient conditions for a skew CR-submanifold to be locally a warped product submanifold are obtained. In Section 6, we establish a sharp relationship for the squared norm of the second fundamental form \( h \) in terms of the warping function \( f \) of a warped product skew CR-submanifold \( M \) of order 1 in Kenmotsu manifolds. The equality case is also considered. In Section 7, some applications of our results are given.

2. Preliminaries

A \((2n + 1)\)-dimensional Riemannian manifold \( \tilde{M} \) is said to be an almost contact metric manifold [8] if it admits a \((1,1)\) tensor field \( \varphi \), a vector field \( \xi \), an 1-form \( \eta \) and a Riemannian metric \( g \), which satisfy the following relations

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad (1)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad (2)
\]

for any vector fields \( X, Y \) on \( \tilde{M} \). In addition, if

\[
(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \tilde{\nabla}_X \xi = X - \eta(X)\xi \quad (3)
\]

where \( \tilde{\nabla} \) is the Reimannian connection with respect to \( g \), then \( (\tilde{M}, \varphi, \xi, \eta, g) \) is called a Kenmotsu manifold [28]. The covariant derivative of \( \varphi \) is defined as

\[
(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y \quad (4)
\]

for any vector fields \( X, Y \) on \( \tilde{M} \).

Let \( M \) be a submanifold of an almost contact metric manifold \( \tilde{M} \) with induced metric \( g \) and if \( \nabla \) and \( \nabla^\perp \) are the induced connections on the tangent and normal bundles \( TM \) and \( T^\perp M \) of \( M \), respectively, then the Gauss and Weingarten formulas are respectively given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X V = -A_N X + \nabla^\perp_X V, \quad (5)
\]

for any vector fields \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), where \( h \) is the second fundamental form of \( M \) and \( A_N \) is the Weingarten endomorphism associated with \( N \). The second fundamental form \( h \) and the shape operator \( A \) are related by

\[
g(h(X, Y), N) = g(A_N X, Y). \quad (6)
\]
For any $X \in \Gamma(TM)$, we write

$$\varphi X = TX + FX,$$

where $TX$ is the tangential component of $\varphi X$ and $FX$ is the normal component of $\varphi X$. Similarly, for any vector field $N$ normal to $M$, we put

$$\varphi N = BN + CN,$$

where $BN$ and $CN$ are the tangential and normal components of $\varphi N$, respectively.

The invariant and anti-invariant submanifolds are defined depending on the behaviour the tangent spaces under the action of the almost contact structure $\varphi$. A submanifold $M$ tangent to the structure vector field $\xi$ is said to be invariant (resp. anti-invariant) if $\varphi(T_p M) \subseteq T_p M$, $\forall p \in M$ (resp. $\varphi(T_p M) \subseteq T_p ^{-}M$, $\forall p \in M$).

We denote by $H$, the mean curvature vector defined as $H(p) = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i)$, where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the tangent space $T_p M$, for any $p \in M$.

Also, we set

$$||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)) \text{ and } h'_{ij} = g(h(e_i, e_j), e_r),$$

for $i, j = 1, \cdots, m$ and $r = m+1, \cdots, 2n+1$, where $\{e_{m+1}, \cdots, e_{2n+1}\}$ is an orthonormal basis of the normal space $T_p ^{-}M$.

For a differentiable function $f$ on an $m$-dimensional manifold $M$, the gradient $\nabla f$ of $f$ is defined as

$$g(\nabla f, X) = X(f)$$

for any $X$ tangent to $M$. As a consequence, we have

$$||\nabla f||^2 = \sum_{i=1}^{m} (e_i(f))^2$$

for an orthonormal frame $\{e_1, \cdots, e_m\}$ on $M$.

A submanifold $M$ of a Riemannian manifold $\tilde{M}$ is said to be totally umbilical if $h(X, Y) = g(X, Y)H$ and totally geodesic if $h(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$. Also, $M$ is minimal in $\tilde{M}$, if $H = 0$.

There are some other classes of submanifolds of almost contact Riemannian manifolds which are defined as follows:

A submanifold $M$ tangent to the structure vector field $\xi$ is said to be a contact CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D} : p \rightarrow D_p$ and $\mathcal{D}^\perp : p \rightarrow D_p ^\perp$, $\forall p \in M$, such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the 1-dimensional distribution spanned by $\xi$.

(ii) $\mathcal{D}$ is invariant by $\varphi$, i.e., $\varphi \mathcal{D} = \mathcal{D}$.

(iii) $\mathcal{D}^\perp$ is anti-invariant by $\varphi$, i.e., $\varphi \mathcal{D}^\perp \subseteq TM ^\perp$.

Invariant and anti-invariant submanifolds are special cases of a contact CR-submanifolds. If we denote the dimensions of the distribution $\mathcal{D}$ and $\mathcal{D}^\perp$ by $d_1$ and $d_2$, respectively, then $M$ is invariant (resp. anti-invariant) if $d_2 = 0$ (resp. $d_1 = 0$).

A submanifold $M$ is called slant [11] if for each $X \in T_p M$ linearly independent on $\xi_p$, the angle $\theta(X)$ between $\varphi X$ and $T_p M$ is a constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_p M - \langle \xi_p \rangle$.

On a slant submanifold, if $\theta = 0$, then $M$ is invariant and if $\theta = \frac{\pi}{2}$ then $M$ is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.
A submanifold $M$ is called semi-slant [10] if it is endowed with two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^0$ such that $\mathcal{D}$ is invariant with respect to $\varphi$ and $\mathcal{D}^0$ is a proper slant distribution.

A submanifold $M$ is called pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^1$ and $\mathcal{D}^0$ such that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \langle \xi \rangle$$

where $\mathcal{D}^1$ is an anti-invariant distribution and its orthogonal complementary distribution $\mathcal{D}^0$ is proper slant.

From the definition of a pseudo-slant submanifold, if we consider the dimensions $\dim \mathcal{D}^1 = d_1$, and $\dim \mathcal{D}^0 = d_2$, then it is clear that contact CR-submanifolds and slant submanifolds are particular classes of pseudo-slant submanifolds with $\theta = 0$ and $d_1 = 0$, respectively. Also, an invariant (resp. anti-invariant) submanifold is a pseudo-slant submanifold with $\theta = 0$ and $\mathcal{D}^1 = 0$ (resp. $\mathcal{D}^2 = 0$).

The normal bundle $T^\perp M$ of a pseudo-slant submanifold $M$ is decomposed as

$$T^\perp M = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \nu$$

where $\nu$ is a $\varphi$-invariant normal subbundle in the normal bundle $T^\perp M$.

A useful characterization of slant submanifolds was given in [11] as follows:

**Theorem 2.1.** [11] Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda(-I + \eta \otimes \xi)$$

(11)

Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequence of the above theorem

$$g(TX, TY) = \cos^2 \theta \left(g(X, Y) - \eta(X)\eta(Y)\right),$$

(12)

$$g(FX, FY) = \sin^2 \theta \left(g(X, Y) - \eta(X)\eta(Y)\right),$$

(13)

for any vector fields $X, Y$ tangent to $M$.

Also, for a slant submanifold of an almost contact metric manifold, we have the following useful result.

**Theorem 2.2.** [41] Let $M$ be a proper slant submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then

(a) $BFX = \sin^2 \theta(-X + \eta(X)\xi)$,  
(b) $CFX = -FTX$

(14)

for any $X \in \Gamma(TM)$.

3. Skew CR-submanifolds of Kenmotsu manifolds

Let $M$ be a submanifold of a Kenmotsu manifold $\tilde{M}$. We recall the definition of skew CR-submanifolds from [35]. Throughout the paper we consider the the structure vector field $\xi$ is tangent to the submanifold otherwise the submanifold is C-totally real [29].

For any $X$ and $Y$ in $T_pM$, we have $g(TX, Y) = -g(X, TY)$. Hence, it follows that $T^2$ is a symmetric operator on the tangent space $T_pM$, for all $p \in M$. Therefore, its eigenvalues are real and it is diagonalizable. Moreover, its eigenvalues are bounded by $-1$ and $0$. For each $p \in M$, we may set

$$\mathcal{D}^1 = \ker[T^2 + \lambda^2(p)]_p,$$

where $I$ is the identity transformation and $\lambda(p) \in [0, 1]$ such that $-\lambda^2(p)$ is an eigenvalue of $T^2(p)$. We note that $\mathcal{D}^1 = \ker F$ and $\mathcal{D}^0 = \ker T$. $\mathcal{D}^1$ is the maximal $\varphi$-invariant subspace of $T_pM$ and $\mathcal{D}^0$ is the maximal
\( \varphi \)-anti-invariant subspace of \( T_pM \). From now on, we denote the distributions \( D^i \) and \( D^0 \) by \( D \oplus \langle \xi \rangle \) and \( D^\perp \), respectively. Since \( T^2 \) is symmetric and diagonalizable, if \(-\lambda_i^2(\varphi), \cdots, -\lambda_1^2(\varphi)\) are the eigenvalues of \( T^2 \) at \( p \in M \), then \( T_pM \) can be decomposed as direct sum of mutually orthogonal eigenspaces, i.e.

\[
T_pM = D^1_p \oplus D^2_p \cdots \oplus D^k_p.
\]

Each \( D^i_p, 1 \leq i \leq k \), is a \( T \)-invariant subspace of \( T_pM \). Moreover if \( \lambda_i \neq 0 \), then \( D^i_p \) is even dimensional. We say that a submanifold \( M \) of a Kenmotsu manifold \( \tilde{M} \) is a generic submanifold if there exists an integer \( k \) and functions \( \lambda_i, 1 \leq i \leq k \) defined on \( M \) with values in \((0,1)\) such that

1. Each \(-\lambda_i^2(\varphi), 1 \leq i \leq k \) is a distinct eigenvalue of \( T^2 \) with

\[
T_pM = D^1_p \oplus D^2_p \oplus D^1_p \oplus \cdots \oplus D^1_p \oplus \langle \xi \rangle_p
\]

for any \( p \in M \).

2. The dimensions of \( D_p, D^p \) and \( D^i_p, 1 \leq i \leq k \) are independent on \( p \in M \).

Moreover, if each \( \lambda_i \) is constant on \( M \), then \( M \) is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with \( k = 0 \), \( D \neq \{0\} \) and \( D^\perp \neq \{0\} \). And slant submanifolds are also a particular class of skew CR-submanifolds with \( k = 1 \), \( D = \{0\} \), \( D^\perp = \{0\} \) and \( \lambda_1 \) is constant. Moreover, if \( D^2 = \{0\}, D \neq 0 \) and \( k = 1 \), then \( M \) is a semi-slant submanifold. Furthermore, if \( D = \{0\}, D^\perp \neq \{0\} \) and \( k = 1 \), then \( M \) is a pseudo-slant (or hemi-slant) submanifold.

A submanifold \( M \) of a Kenmotsu manifold \( \tilde{M} \) is said to be a proper skew CR-submanifold of order 1 if \( M \) is a skew CR-submanifold with \( k = 1 \) and \( \lambda_1 \) is constant. In that case, the tangent bundle of \( M \) is decomposed as

\[
TM = D \oplus D^\perp \oplus D^0 \oplus \langle \xi \rangle
\]

The normal bundle \( T^\perp M \) of a skew CR-submanifold \( M \) is decomposed as

\[
T^\perp M = \varphi D^\perp \oplus PD^0 \oplus \nu,
\]

where \( \nu \) is a \( \varphi \)-invariant normal subbundle of \( T^2 M \).

Now, we give the following results which are useful for the further study.

**Lemma 3.1.** Let \( M \) be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \( \tilde{M} \) such that \( \xi \) is tangent to \( M \). Then

\[
A_{WZ} = A_{WZ}
\]

for any \( Z, W \in \Gamma(D^\perp) \)

**Proof.** The proof of this lemma is similar to Lemma 3.2 [2]. \( \square \)

**Lemma 3.2.** Let \( M \) be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \( \tilde{M} \). Then the anti-invariant distribution \( D^\perp \) is always integrable.

**Proof.** For any \( X_1 \in \Gamma(D), Z, W \in \Gamma(D^\perp) \), we have

\[
g([Z, W], X_1) = g(\nabla_Z W, X_1) - g(\nabla_W Z, X_1)
\]

\[
= g(\varphi \nabla_Z W, \varphi X_1) + \eta(\nabla_Z W)\eta(X_1) - g(\varphi \nabla_W Z, \varphi X_1) - \eta(\nabla_W Z)\eta(X_1).
\]

Using (4), we derive

\[
g([Z, W], X_1) = g(\nabla_Z \varphi W, \varphi X_1) - g((\nabla_Z \varphi) W, \varphi X_1) - g(\nabla_W \varphi Z, \varphi X_1) + g((\nabla_W \varphi) Z, \varphi X_1).
\]
Then from (3) and (5), we have
\[ g([Z, W], X_1) = -g(A_{\psi W} Z, \psi X_1) + g(A_{\psi Z} W, \psi X_1). \]

From (15), we find
\[ g([Z, W], X_1) = 0. \tag{16} \]

Similarly, for any \( X_2 \in \Gamma(D^b) \) and \( Z, W \in \Gamma(D^+), \) we have
\[ g([Z, W], X_2) = g(\tilde{\nabla}_Z W, X_2) - g(\tilde{\nabla}_W Z, X_2) \]
\[ = g(\psi \tilde{\nabla}_Z W, \psi X_2) + \eta(\tilde{\nabla}_Z W)\eta(X_2) - g(\psi \tilde{\nabla}_W Z, \psi X_2) - \eta(\tilde{\nabla}_W Z)\eta(X_2). \]

From (4), we obtain
\[ g([Z, W], X_2) = g(\tilde{\nabla}_Z \psi W, \psi X_2) - g((\tilde{\nabla}_Z \psi) W, \psi X_2) - g(\tilde{\nabla}_W \psi Z, \psi X_2) + g((\tilde{\nabla}_W \psi) Z, \psi X_2). \]

Then from (3) and (7), we derive
\[ g([Z, W], X_2) = g(\tilde{\nabla}_Z \psi W, TX_2) + g(\tilde{\nabla}_Z \psi W, FX_2) - g(\tilde{\nabla}_W \psi Z, TX_2) - g(\tilde{\nabla}_W \psi Z, FX_2). \]

Using (5), we get
\[ g([Z, W], X_2) = g(A_{\psi Z} W, TX_2) + g(W, \psi \tilde{\nabla}_Z FX_2) - g(A_{\psi Z} W, TX_2) - g(Z, \psi \tilde{\nabla}_W FX_2). \]

Again, using (4) and (15), we obtain
\[ g([Z, W], X_2) = g(\tilde{\nabla}_Z \psi F, W) - g((\tilde{\nabla}_Z \psi F) W, Z) - g(\tilde{\nabla}_W \psi F, Z) + g((\tilde{\nabla}_W \psi) F, Z). \]

Then from (4) and (8), we find that
\[ g([Z, W], X_2) = g(\tilde{\nabla}_Z B F, W) + g(\tilde{\nabla}_Z C F, W) - g(\tilde{\nabla}_W B F, Z) - g(\tilde{\nabla}_W C F, Z). \]

Thus by Theorem 2.2, we get
\[ g([Z, W], X_2) = -\sin^2 \theta g(\tilde{\nabla}_Z X_2, W) - g(\tilde{\nabla}_Z F T, W) + \sin^2 \theta g(\tilde{\nabla}_W X_2, Z) + g(\tilde{\nabla}_W F T, Z) \]
\[ = \sin^2 \theta g(\tilde{\nabla}_Z W, X_2) + g(A_{F T} X_2, W) - \sin^2 \theta g(\tilde{\nabla}_W X_2, Z) - g(A_{F T} W, Z). \]

By the symmetric property of the shape operator, we find
\[ \cos^2 \theta g([Z, W], X_2) = 0. \]

Since \( M \) is a proper skew CR-submanifold, thus \( \cos^2 \theta \neq 0. \) Then, we have
\[ g([Z, W], X_2) = 0. \tag{17} \]

Also, for any \( Z, W \in \Gamma(D^+), \) we have
\[ g([Z, W], \xi) = g(\tilde{\nabla}_Z W, \xi) - g(\tilde{\nabla}_W Z, \xi) = -g(\tilde{\nabla}_Z \xi, W) + g(\tilde{\nabla}_W \xi, Z). \]

By using (3), the right hand side of the above relation vanishes identically, hence we find that
\[ g([Z, W], \xi) = 0. \tag{18} \]

By combining (16), (17) and (18), the result follows immediately. \( \Box \)
Let $M$ be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \Gamma(\mathcal{D} \oplus \mathcal{D}^0)$. Then, we have

$$g(\nabla_{X_1} Y_1, Z) = g(A_{\varphi Z} X_1, \varphi Y_1),$$

(19)

$$g(\nabla_{X_1} Y_2, Z) = \sec^2 \theta \left( g(A_{\varphi Z} X_1, TY_2) - g(A_{FTY_2} Z, X_1) \right),$$

(20)

$$g(\nabla_{Y_1} X_1, Z) = g(A_{\varphi Z} \varphi X_1, Y_2)$$

(21)

for any $X_1, Y_1 \in \Gamma(\mathcal{D})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^+)$.

**Proof.** For any $X_1, Y_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^+)$, we have

$$g(\nabla_{X_1} Y_1, Z) = g(\tilde{\nabla}_{X_1} Y_1, Z) = g(\varphi \tilde{\nabla}_{X_1} Y_1, \varphi Z) + \eta(\tilde{\nabla}_{X_1} Y_1) \eta(Z).$$

Using (4) and the fact that $\xi$ is orthogonal to $\mathcal{D}^+$, we obtain

$$g(\nabla_{X_1} Y_1, Z) = g(\tilde{\nabla}_{X_1} \varphi Y_1, \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) Y_1, \varphi Z).$$

Then from (3) and (5), we get

$$g(\nabla_{X_1} Y_1, Z) = g(h(X_1, \varphi Y_1), \varphi Z).$$

Thus, (19) follows from the above relation by using (6). Also, for any $X_1 \in \Gamma(\mathcal{D})$, $Y_2 \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^+)$, we have

$$g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} Y_2, Z) = g(\varphi \tilde{\nabla}_{X_1} Y_2, \varphi Z) + \eta(\tilde{\nabla}_{X_1} Y_2) \eta(Z).$$

Again, using (4), we get

$$g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} \varphi Y_2, \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) Y_2, \varphi Z).$$

From (3) and (7), we derive

$$g(\nabla_{X_1} Y_2, Z) = g(h(X_1, TY_2), \varphi Z) + g(\tilde{\nabla}_{X_1} FY_2, \varphi Z)
\quad = g(h(X_1, TY_2), \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) FY_2, Z) + g((\tilde{\nabla}_{X_1} \varphi) FY_2, Z).$$

The last term in the right hand side vanishes identically by using (3). Then from (8), the above equation takes the form

$$g(\nabla_{X_1} Y_2, Z) = g(h(X_1, TY_2), \varphi Z) - g(\tilde{\nabla}_{X_1} BFY_2, Z) - g(\tilde{\nabla}_{X_1} CFY_2, Z).$$

Thus, using Theorem 2.2, we find

$$g(\nabla_{X_1} Y_2, Z) = g(h(X_1, TY_2), \varphi Z) + \sin^2 \theta g(\tilde{\nabla}_{X_1} Y_2, Z) - \sin^2 \theta \eta(Y_2) g(\tilde{\nabla}_{X_1} \xi, Z) + g(\tilde{\nabla}_{X_1} FTY_2, Z).$$

Again, using (3) and (5), we get (20). Similarly, we have

$$g(\nabla_{X_2} X_1, Z) = g(\tilde{\nabla}_{X_2} X_1, Z) = g(\varphi \tilde{\nabla}_{X_2} X_1, \varphi Z) + \eta(\tilde{\nabla}_{X_2} X_1) \eta(Z).$$

Then from (3), we get

$$g(\nabla_{X_2} X_1, Z) = g(\tilde{\nabla}_{X_2} \varphi X_1, \varphi Z) = g(h(Y_2, \varphi X_1), \varphi Z) = g(A_{\varphi Z} \varphi X_1, Y_2),$$

which is (21). Hence, the lemma is proved completely. □

**Lemma 3.4.** Let $M$ be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $\xi$ is orthogonal to $\mathcal{D}^+$. Then, the following hold:
(i) If \( \xi \in \Gamma(\mathcal{D} \oplus \mathcal{D}^0) \), then
\[
g(\nabla_{X_2} Y_2, Z) = \sec^2 \theta \left( g(A_{\psi \varphi} X_2, TY_2) - g(A_{F_{TY_2}} Z, X_2) \right)
\]  
(22)
for any \( X_2, Y_2 \in \Gamma(\mathcal{D}^0) \), \( Z \in \Gamma(\mathcal{D}^\perp) \).

(ii) If \( \xi \in \Gamma(\mathcal{D}) \), then
\[
g(\nabla_Z V, X_2) = \sec^2 \theta \left( g(A_{F_{TX_2}^0} Z, V) - g(A_{\varphi \psi} Z, TX_2) \right),
\]  
(23)
\[
g(\nabla_Z V, X_1) = -g(A_{\varphi \psi} Z, \varphi X_1) - \eta(X_1) g(Z, V),
\]  
(24)
for any \( X_1 \in \Gamma(\mathcal{D} \oplus (\xi)) \), \( X_2 \in \Gamma(\mathcal{D}^0) \) and \( Z, V \in \Gamma(\mathcal{D}^\perp) \).

(iii) If \( \xi \in \Gamma(\mathcal{D}^0) \), then
\[
g(\nabla_Z V, X_2) = \sec^2 \theta \left( g(A_{F_{TX_2}^0} Z, V) - g(A_{\varphi \psi} Z, TX_2) \right) - \eta(X_2) g(Z, V),
\]  
(25)
\[
g(\nabla_Z V, X_1) = -g(A_{\varphi \psi} Z, \varphi X_1)
\]  
(26)
for any \( X_1 \in \Gamma(\mathcal{D}) \), \( X_2 \in \Gamma(\mathcal{D}^0 \oplus (\xi)) \) and \( Z, V \in \Gamma(\mathcal{D}^\perp) \).

Proof. For any \( X_2, Y_2 \in \Gamma(\mathcal{D}^0) \) and \( Z \in \Gamma(\mathcal{D}^\perp) \), we have
\[
g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} Y_2, Z) = g(\varphi \tilde{\nabla}_{X_1} Y_2, \varphi Z) + \eta(\tilde{\nabla}_{X_1} Y_2) \eta(Z).
\]
Using (4), we get
\[
g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} \varphi Y_2, \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) Y_2, \varphi Z).
\]
The second term in the right hand side is identically zero by using (3). Then from (7), we derive
\[
g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} TY_2, \varphi Z) + g(\tilde{\nabla}_{X_1} FY_2, \varphi Z).
\]
Using (4) and (7), we find
\[
g(\nabla_{X_1} Y_2, Z) = g(\tilde{\nabla}_{X_1} (TY_2), \varphi Z) - g((\tilde{\nabla}_{X_1} \varphi) FY_2, Z) + g(\tilde{\nabla}_{X_1} CFY_2, Z) + g(\tilde{\nabla}_{X_1} FY_2, Z) - g(\tilde{\nabla}_{X_1} BFY_2, Z).
\]
Then using Theorem 2.2, we arrive at
\[
g(\nabla_{X_1} Y_2, Z) = g(A_{\psi \varphi} TY_2, X_2) + \sin^2 \theta g(\tilde{\nabla}_{X_1} Y_2, Z) + g(\tilde{\nabla}_{X_1} FTY_2, Z).
\]
Hence, the first part of the Lemma follows from the above relation by using (5) and (6). Now, for any \( X_2 \in \Gamma(\mathcal{D}^0) \) and \( Z, V \in \Gamma(\mathcal{D}^\perp) \), we have
\[
g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z V, X_2) = g(\varphi \tilde{\nabla}_Z V, \varphi X_2) + \eta(X_2) \eta(\nabla_Z V).
\]
Using (4), we obtain
\[
g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, \varphi X_2) - g((\tilde{\nabla}_Z \varphi) V, \varphi X_2).
\]
Then from (3) and (7), we find that
\[
g(\nabla_Z V, X_2) = g(\tilde{\nabla}_Z \varphi V, TX_2) + g(\tilde{\nabla}_Z \varphi V, FX_2).
\]
Again, using (4) and (5), we obtain
\[
g(\nabla_Z V, X_2) = g(\varphi \tilde{V} FX_2, V) - g(A_{\varphi} V, TX_2)
\]
\[
= g(\nabla_Z \varphi FX_2, V) - g(A_{\varphi} V, TX_2) - g(\tilde{V} Z \varphi FX_2, V)
\]
\[
= g(\tilde{V} Z BFX_2, V) - g(A_{\varphi} V, TX_2) + g(\tilde{V} Z CFX_2, V).
\]
Hence by Theorem 2.2, we derive
\[
g(\nabla_Z V, X_2) = -g(A_{\varphi} V, TX_2) - \sin^2 \theta g(\nabla_Z X_2, V) - g(\nabla_Z FTX_2, V)
\]
\[
= -g(A_{\varphi} V, TX_2) + \sin^2 \theta g(\nabla_Z V, X_2) + g(A_{FTX_2} Z, V)
\]
or,
\[
\cos^2 \theta g(\nabla_Z V, X_2) = g(A_{FTX_2} Z, V) - g(A_{\varphi} V, TX_2)
\]
which gives (23). Also, for any \(X_1 \in \Gamma(\mathcal{O} \oplus \langle \xi \rangle)\) and \(Z, V \in \Gamma(\mathcal{O}^+)\), we have
\[
g(\nabla_Z V, X_1) = g(\tilde{V} Z V, X_1) = g(\varphi \tilde{V} Z \varphi X_1) + \eta(X_1) \eta(\tilde{V} Z V).
\]
Using (3)-(5), we derive
\[
g(\nabla_Z V, X_1) = g(\tilde{V} Z \varphi V, \varphi X_1) - g(\tilde{V} Z \varphi V, \varphi X_1) + \eta(X_1) g(\tilde{V} Z V, \xi)
\]
\[
= g(\tilde{V} Z \varphi V, \varphi X_1) - \eta(X_1) g(\tilde{V} Z \xi, V)
\]
\[
= -g(A_{\varphi} V, \varphi X_1) - \eta(X_1) g(Z, V),
\]
which is (24). Now, to prove the last part of the lemma, consider any \(X_2 \in \Gamma(\mathcal{O}^+ \oplus \langle \xi \rangle)\) and \(Z, V \in \Gamma(\mathcal{O}^+)\). Then, we have
\[
g(\nabla_Z V, X_2) = g(\tilde{V} Z V, X_2) = g(\varphi \tilde{V} Z \varphi X_2) + \eta(\tilde{V} Z V) \eta(X_2).
\]
Using (4), we obtain
\[
g(\nabla_Z V, X_2) = g(\tilde{V} Z \varphi V, \varphi X_2) - g(\tilde{V} Z \varphi V, \varphi X_2) + \eta(X_2) g(\tilde{V} Z V, \xi).
\]
Then from (3) and (7), we derive
\[
g(\nabla_Z V, X_2) = g(\tilde{V} Z \varphi V, TX_2) + g(\tilde{V} Z \varphi V, FX_2) - \eta(X_2) g(Z, V).
\]
Again, using (4) and (5), we get
\[
g(\nabla_Z V, X_2) = -g(A_{\varphi} V, TX_2) - g(\tilde{V} Z FX_2, \varphi V) - \eta(X_2) g(Z, V)
\]
\[
= -g(A_{\varphi} V, TX_2) + g(\varphi \tilde{V} Z FX_2, V) - \eta(X_2) g(Z, V)
\]
\[
= -g(A_{\varphi} V, TX_2) + g(\tilde{V} Z BFX_2, V) + g(\tilde{V} Z CFX_2, V) - \eta(X_2) g(Z, V).
\]
Hence, by Theorem 2.2, we obtain
\[
g(\nabla_Z V, X_2) = -g(A_{\varphi} V, TX_2) - \sin^2 \theta g(\nabla Z X_2, V) + \sin^2 \theta \eta(X_2) g(\nabla Z \xi, V) + g(\tilde{V} Z FTX_2, V) - \eta(X_2) g(Z, V)
\]
\[
= -g(A_{\varphi} V, TX_2) + \sin^2 \theta g(\nabla Z V, X_2) + \sin^2 \theta \eta(X_2) g(Z, V) + g(A_{FTX_2} Z, V) - \eta(X_2) g(Z, V)
\]
or,
\[
\cos^2 \theta g(\nabla_Z V, X_2) = g(A_{FTX_2} Z, V) - g(A_{\varphi} V, TX_2) - \cos^2 \theta \eta(X_2) g(Z, V)
\]
which gives (25). Similarly, for any \( X_1 \in \Gamma(D \oplus \langle \xi \rangle) \) and \( Z, V \in \Gamma(D^\perp) \), we have
\[
g(V_z V, X_1) = g(\hat{V}_z V, X_1) = g(\varphi \hat{V}_z V, \varphi X_1) + \eta(X_1)\eta(\hat{V}_z V).
\]
Using (3) and the fact that \( \xi \in \Gamma(D^\perp) \), we derive
\[
g(V_z V, X_1) = g(\hat{V}_z \varphi V, \varphi X_1) - g((\hat{V}_z \varphi)V, \varphi X_1) = -g(A_\varphi Z, \varphi X_1),
\]
which is (26). Hence, the proof of the lemma is complete. \( \square \)

**Lemma 3.5.** Let \( M \) be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold \( M \) such that \( \xi \) is orthogonal to \( D^\perp \). Then, we have
\[
g(V_z X_1, Y_2) = \csc^2 \theta \left( g(A_{FY_2} Z, \varphi X_1) - g(A_{FTY_2} Z, X_1) \right)
\]
for any \( X_1 \in \Gamma(D) \), \( Y_2 \in \Gamma(D^\perp) \) and \( Z \in \Gamma(D^\perp) \).

**Proof.** For any \( X_1 \in \Gamma(D) \), \( Y_2 \in \Gamma(D^\perp) \) and \( Z \in \Gamma(D^\perp) \), we have
\[
g(V_z X_1, Y_2) = g(\hat{V}_z X_1, Y_2) = g(\varphi \hat{V}_z X_1, \varphi Y_2) + \eta(Y_2)\eta(\hat{V}_z X_1).
\]
Using (4), we find that
\[
g(V_z X_1, Y_2) = g(\varphi \hat{V}_z X_1, \varphi Y_2) - g((\varphi \hat{V}_z) X_1, \varphi Y_2) - \eta(Y_2) g(\hat{V}_z \xi, X_1).
\]
Then from (3) and (7), we obtain
\[
g(V_z X_1, Y_2) = g(\hat{V}_z \varphi X_1, TY_2) + g(\hat{V}_z \varphi X_1, FY_2)
= g(X_1, \varphi \hat{V}_z TY_2) + g(h(Z, \varphi X_1), FY_2)
= g(X_1, \varphi \hat{V}_z TY_2) - g(X_1, (\varphi \hat{V}_z) TY_2) + g(h(Z, \varphi X_1), FY_2).
\]
By using (3), (7) and (12), we derive
\[
g(V_z X_1, Y_2) = -\cos^2 \theta g(\hat{V}_z Y_2, X_1) + \cos^2 \theta \eta(Y_2) g(X_1, \hat{V}_z \xi) - g(A_{FTY_2} Z, X_1) + g(A_{FY_2} Z, \varphi X_1)
= \cos^2 \theta g(V_z X_1, Y_2) + g(A_{FY_2} Z, \varphi X_1) - g(A_{FTY_2} Z, X_1).
\]
which gives (3.13), hence the lemma is proved. \( \square \)

4. Warped product skew CR-submanifolds of Kenmotsu manifolds

In [7], R.L. Bishop and B. O'Neill introduced the notion of warped product manifolds to study the manifolds of negative curvatures. These manifolds are natural generalizations of Riemannian product manifolds. The definition of a warped product is formulated as: Let \( (M_1, g_1) \) and \( (M_2, g_2) \) be two Riemannian manifolds and \( f \) a positive differentiable function on \( M_1 \). Consider the product manifold \( M_1 \times M_2 \) with its canonical projections \( \pi_1 : M_1 \times M_2 \rightarrow M_1 \) and \( \pi_2 : M_1 \times M_2 \rightarrow M_2 \). The warped product \( M = M_1 \times_f M_2 \) is the product manifold \( M_1 \times M_2 \) equipped with the Riemannian metric \( g \) given by
\[
g(X, Y) = g_1(\pi_1(X), \pi_1(Y)) + (f \circ \pi_1)^2 g_2(\pi_2(X), \pi_2(Y))
\]
for any tangent vector \( X, Y \in TM \), where \( * \) is the symbol for the tangent maps. If \( X \) is tangent to \( M_1 \) and \( V \) is tangent to \( M_2 \), then from lemma 7.3 of [7] we have
\[
\nabla_X V = \nabla_V X = X(\ln f)V.
\]
Recall that if \( M = M_1 \times_f M_2 \) is a warped product manifold, then \( M_1 \) is totally geodesic in \( M \) and \( M_2 \) is totally umbilical in \( M \).
In this section, we consider a warped product $M = M_1 \times M_2$ in a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_0$, where $M_T$, $M_0$ and $M_2$ are invariant, proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. Throughout this section we consider the structure vector field $\xi$ is tangent to the submanifold $M$. Therefore, two possible cases arise:

**Case 1.** When $\xi$ is tangent to $M_2$, then it is easy to see that the warped product is simply a Riemannian product. Thus, we will not discuss this case anymore for the non-existence of such properly warped products.

**Case 2.** When $\xi$ is tangent to $M_1 = M_T \times M_0$. In this case either $\xi$ is tangent to $M_T$ or $M_0$ and in both subcases the warped product exists and we will discuss these kinds of warped products in our further study.

Let $M = M_1 \times M_2$ be a warped product skew CR-submanifold of order 1 of Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_0$ and the structure vector field $\xi$ is tangent to $M_1$. Then, we call such submanifolds skew CR-warped products analogous to the CR-warped products introduced by Chen in [15, 16]. If we consider the dimensions of these submanifolds as $\dim M_T = d_1$, $\dim M_0 = d_2$ and $\dim M_2 = d_3$, then it is obvious that $M$ is a CR-warped product if $d_2 = 0$ and $M$ is a warped product pseudo-slant (or hemi-slant) submanifold if $d_1 = 0$.

Now, we provide the following non-trivial example of warped product skew CR-submanifolds of order 1 of an almost contact metric manifold.

**Example 4.1.** Consider a submanifold of $\mathbb{R}^{11}$ with the cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, t)$ and the almost contact structure

$$\varphi \left( \frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_j}, \quad \varphi \left( \frac{\partial}{\partial y_i} \right) = \frac{\partial}{\partial x_j}, \quad \varphi \left( \frac{\partial}{\partial t} \right) = 0, \quad 1 \leq i, j \leq 5.$$  

It is easy to show $\mathbb{R}^{11}$ is an almost contact metric manifold with respect to the Euclidean metric tensor of $\mathbb{R}^{11}$. Let us consider a submanifold $M$ of $\mathbb{R}^{11}$ defined by the immersion $\chi$ as follows

$$\chi(u, v, w, s, r, t) = (u \cos w, u \sin w, u + v, s, 0, v \cos w, v \sin w, u - v, r, 0, t).$$

Then the tangent space of $M$ is spanned by the following vectors

$$Z_1 = \cos w \frac{\partial}{\partial x_1} + \sin w \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad Z_2 = \cos w \frac{\partial}{\partial y_1} + \sin w \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3},$$

$$Z_3 = -u \sin w \frac{\partial}{\partial x_1} + u \cos w \frac{\partial}{\partial x_2} - v \sin w \frac{\partial}{\partial y_1} + v \cos w \frac{\partial}{\partial y_2}, \quad Z_4 = \frac{\partial}{\partial x_4}, \quad Z_5 = \frac{\partial}{\partial y_4}, \quad Z_6 = \frac{\partial}{\partial t}.$$  

Then, we find

$$\varphi Z_1 = -\cos w \frac{\partial}{\partial y_1} - \sin w \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}, \quad \varphi Z_2 = \cos w \frac{\partial}{\partial x_1} + \sin w \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}, \quad \varphi Z_3 = u \sin w \frac{\partial}{\partial y_1} - u \cos w \frac{\partial}{\partial y_2} + v \sin w \frac{\partial}{\partial x_1} + v \cos w \frac{\partial}{\partial x_2}; \quad \varphi Z_4 = -\frac{\partial}{\partial t}, \quad \varphi Z_5 = \frac{\partial}{\partial x_4}, \quad \varphi Z_6 = 0.$$  

It is easy to see that $D = \text{Span}[Z_4, Z_5]$ is an invariant distribution, $D^\perp = \text{Span}[Z_3]$ is an anti-invariant distribution and $D^0 = \text{Span}[Z_1, Z_2]$ is a slant distribution with slant angle $\theta = \arccos(\frac{1}{3}) = 70^\circ 52'$ such that $\xi = \frac{\partial}{\partial t}$ is tangent to $D \oplus D^0$. Hence, we conclude that $M$ is a proper skew CR-submanifold of order 1 of $\mathbb{R}^{11}$.

It is easy to observe that $D \oplus D^0$ and $D^\perp$ are integrable. Denoting the integral manifolds of $D$, $D^0$ and $D^\perp$ by $M_T$, $M_0$ and $M_2$, respectively. Then the induced metric tensor $g$ of $M$ is given by

$$ds^2 = 3(du^2 + dv^2) + dw^2 + dr^2 + dt^2 + (u^2 + v^2)du^2 = g_{M_0} + (u^2 + v^2)g_{M_2}.$$  

Thus $M$ is a warped product skew CR submanifold of $\mathbb{R}^{11}$ with the warping function $f = \sqrt{u^2 + v^2}$ such that $M_1 = M_T \times M_0$.  

Now, we prove the following useful lemmas for a warped product skew CR-submanifold of a Kenmotsu manifold.

**Lemma 4.2.** Let $M = M_1 \times M_2$ be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $\xi$ is tangent to $M_1$ and $M_1 = M_T \times M_0$, where $M_T$ and $M_0$ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then, the following hold:

(i) $\xi(\ln f) = 1$,
(ii) $g(h(X_1, Y_1), \varphi Z) = 0$,
(iii) $g(h(X_1, Z), F_Y) = h(X_1, Y_2), \varphi Z) = 0$,
(iv) $g(h(X_2, Z), F_Y) = g(h(X_2, Y_2), \varphi Z)$

for any $X_1, Y_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_0)$ and $Z \in \Gamma(TM_\perp)$.

**Proof.** For any $Z \in \Gamma(TM_\perp)$, we have $\nabla_Z \xi = Z$. Then from (5), we get

$$\nabla_Z \xi + h(Z, \xi) = Z.$$

Equating the tangential components and then using (28), we obtain $\xi(\ln f)Z = Z$. Taking the inner product with $Z$, we get (i). Now, for the other parts of the lemma we consider any $X_1, Y_1 \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$. Then, we have

$$g(h(X_1, Y_1), \varphi Z) = g(\nabla_{X_1}Y_1, \varphi Z) = -g(\varphi \nabla_{X_1}Y_1, Z).$$

Then from (4), we arrive at

$$g(h(X_1, Y_1), \varphi Z) = g((\nabla_{X_1}Y_1, Z) - g(\nabla_{X_1}Y_1, Z) = g(\nabla_{X_1}Z, \varphi Y_1).$$

Thus, on using (28), we get $g(h(X_1, Y_1), \varphi Z) = X_1(\ln f) g(\varphi Y_1, Z) = 0$, which is (ii). To prove the third part of the lemma, consider any $X_1 \in \Gamma(TM_T)$, $Y_2 \in \Gamma(TM_0)$, and $Z \in \Gamma(TM_\perp)$. Then, we have

$$g(h(X_1, Y_2), \varphi Z) = g(\nabla_{X_1}Y_2, \varphi Z) = -g(\varphi \nabla_{X_1}Y_2, Z).$$

Using (4), we obtain

$$g(h(X_1, Y_2), \varphi Z) = g((\nabla_{X_1}Y_2, Z) = g(\nabla_{X_1}Y_2, Z).$$

First term in the right hand side vanishes identically by using (3). Then from (7), we get

$$g(h(X_1, Y_2), \varphi Z) = -g(\nabla_{X_1}TY_2, Z) - g(\nabla_{X_1}FY_2, Z).$$

Using (5) and (28), we find that

$$g(h(X_1, Y_2), \varphi Z) = X_1(\ln f) g(TY_2, Z) + g(A_{FY_2}Z, X_1).$$

Hence, first equality of (iii) follows from the above relation by using (6) and the orthogonality of vector fields. For the second equality of (iii), we have

$$g(h(X_1, Y_2), \varphi Z) = g(\nabla_{Y_2}X_1, \varphi Z) = -g(\varphi \nabla_{Y_2}X_1, Z) = g((\nabla_{Y_2}\varphi)X_1, Z) - g(\varphi \nabla_{Y_2}X_1, Z).$$

From (3), (5) and (28), we derive

$$g(h(X_1, Y_2), \varphi Z) = Y_2(\ln f) g(\varphi X_1, Z) = 0,$$

which is the second equality of (iii). Similarly, for any $X_2, Y_2 \in \Gamma(TM_0)$, and $Z \in \Gamma(TM_\perp)$, we have

$$g(h(X_2, Y_2), \varphi Z) = g(\nabla_{X_2}Y_2, \varphi Z) = -g(\varphi \nabla_{X_2}Y_2, Z).$$
From (4), we find
\[ g(h(X_2,Y_2),\varphi Z) = g((\tilde{\nabla}_{X_2}\varphi)Y_2,Z) - g(\tilde{\nabla}_{X_2}\varphi Y_2,Z) = g(\tilde{\nabla}_{X_2}Z,TY_2) + g(A_{FY_2}X_2,Z). \]

Thus, the fourth part of the lemma follows from the above relation by using the orthogonality of vector fields. Hence, the lemma is proved completely.

\[ \square \]

**Lemma 4.3.** Let \( M = M_1 \times fM_\perp \) be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \( \tilde{M} \) such that \( \xi \) is tangent to \( M_1 \), where \( M_1 = M_T \times M_0 \). Then, we have
\[ g(h(X_1,Z),\varphi V) = -\varphi X_1(\ln f) g(Z,V) \] (29)
for any \( X_1 \in \Gamma(TM_T) \) and \( Z, V \in \Gamma(TM_\perp) \).

**Proof.** For any \( X_1 \in \Gamma(TM_T) \) and \( Z, V \in \Gamma(TM_\perp) \), we have
\[ g(h(X_1,Z),\varphi V) = g(\tilde{\nabla}_Z X_1,\varphi V) = -g(\varphi \tilde{\nabla}_V X_1, V). \]

Then from (4), we obtain
\[ g(h(X_1,Z),\varphi V) = g((\tilde{\nabla}_Z \varphi)X_1,V) - g(\tilde{\nabla}_V \varphi X_1,V). \]

First term in the right hand side is identically zero by using (3). Then from (5) and (28), we get
\[ g(h(X_1,Z),\varphi V) = -\varphi X_1(\ln f) g(Z,V), \]
which is (29). Thus, the proof is complete. \( \square \)

If we interchange \( X_1 \) by \( \varphi X_1 \) in (29) for any \( X_1 \in \Gamma(TM_T) \), then two cases arise:

(i) When \( \xi \in \Gamma(TM_T) \), then
\[ g(h(\varphi X_1,Z),\varphi V) = (X_1(\ln f) - \eta(X_1)) g(Z,V), \]
for any \( X_1 \in \Gamma(TM_T) \) and \( Z, V \in \Gamma(TM_\perp) \).

(ii) When \( \xi \in \Gamma(TM_0) \), then
\[ g(h(\varphi X_1,Z),\varphi V) = X_1(\ln f) g(Z,V), \]
for any \( X_1 \in \Gamma(TM_T) \) and \( Z, V \in \Gamma(TM_\perp) \).

Let \( M = M_1 \times_f M_\perp \) be a warped product skew CR-submanifold of a Kenmotsu manifold \( \tilde{M} \) such that \( M_1 = M_T \times M_0 \). We denote the tangent spaces of \( M_T, M_0 \) and \( M_\perp \) by \( D, D^0 \) and \( D^\perp \), respectively. Then \( M \) is called \( D - D^\perp \) mixed totally geodesic if \( h(X_1,Z) = 0 \), for any \( X_1 \in \Gamma(D) \) and \( Z \in \Gamma(D^\perp) \), respectively. Similarly, \( M \) is a \( D^0 - D^\perp \) mixed totally geodesic if \( h(X_2,Z) = 0 \), for any \( X_2 \in \Gamma(D^0) \) and \( Z \in \Gamma(D^\perp) \), respectively.

The following theorem is a consequence of Lemma 4.3.

**Theorem 4.4.** Let \( M = M_1 \times_f M_\perp \) be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \( \tilde{M} \) such that \( M_1 = M_T \times M_0 \), where \( M_T \) and \( M_0 \) are invariant and proper slant submanifolds of \( \tilde{M} \), respectively. If \( M \) is \( D - D^\perp \) mixed totally geodesic warped product, then \( f \) is constant on \( M \).

**Proof.** The proof follows from Lemma 4.3. \( \square \)
Lemma 4.5. Let \( M = M_1 \times \tilde{M}_1 \) be a warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \( \tilde{M} \) such that \( \xi \) is tangent to \( M_1 \), where \( M_1 = M_\perp \times M_{\theta} \). Then, we have

\[
g(h(Z, V), FX_2) - g(h(Z, X_2), \varphi V) = TX_2(\ln f) g(Z, V)
\]

for any \( X_2 \in \Gamma(TM_\theta) \) and \( Z, V \in \Gamma(TM_\perp) \).

Proof. For any \( X_2 \in \Gamma(TM_\theta) \) and \( Z, V \in \Gamma(TM_\perp) \), we have

\[
g(h(X_2, Z), \varphi V) = g(\tilde{\nabla}_Z X_2, \varphi V) = -g(\varphi \tilde{\nabla}_Z X_2, V).
\]

Then (4), we derive

\[
g(h(X_2, Z), \varphi V) = g((\tilde{\nabla}_Z \varphi) X_2, V) - g(\tilde{\nabla}_Z \varphi X_2, V).
\]

First term in the right hand side identically vanishes by using (3). Then from (7), we get

\[
g(h(X_2, Z), \varphi V) = -g(\tilde{\nabla}_Z FX_2, V) - g(\tilde{\nabla}_Z FX_2, V).
\]

Using (5) and (28), we obtain

\[
g(h(X_2, Z), \varphi V) = -TX_2(\ln f) g(Z, V) + g(A_{FX_2} Z, V),
\]

which gives (32). Hence the proof is complete. \( \square \)

If we interchange \( X_2 \) by \( TX_2 \) in (32) for any \( X_2 \in \Gamma(TM_\theta) \), then two cases arise:

(i) When \( \xi \in \Gamma(TM_\perp) \), then

\[
g(h(Z, V), FTX_2) - g(h(TX_2, Z), \varphi V) = -\cos^2 \theta X_2(\ln f) g(Z, V),
\]

for any \( X_2 \in \Gamma(TM_\theta) \) and \( Z, V \in \Gamma(TM_\perp) \).

(ii) When \( \xi \in \Gamma(TM_\theta) \), then

\[
g(h(Z, V), FX_2) - g(h(TX_2, Z), \varphi V) = \cos^2 \theta (\eta(X_2) - X_2(\ln f)) g(Z, V),
\]

for any \( X_2 \in \Gamma(TM_\theta) \) and \( Z, V \in \Gamma(TM_\perp) \).

5. A characterization of skew CR-warped products

As we have seen that there is no proper warped product skew CR-submanifold \( M \) of order 1 of a Kenmotsu manifold \( \tilde{M} \), if \( M \) is \( D - D^\perp \) mixed totally geodesic (Theorem 4.4). Thus, for further study, we consider the warped product skew CR-submanifold of order 1 of a Kenmotsu manifold, when it is a \( D^\theta - D^\perp \) mixed totally geodesic. Before proving a characterization, we need the following definitions.

Definition 5.1. A foliation on a manifold \( M \) is an integrable subbundle \( \mathcal{F} \) of the tangent bundle of \( M \), i.e., for any sections \( X \) and \( Y \) of \( \mathcal{F} \), then the Lie bracket \([X, Y]\) is a section of \( \mathcal{F} \) as well.

Definition 5.2. A foliation \( L \) on a Riemannian manifold \( M \) is called totally umbilical if every leaf of \( L \) is a totally umbilical Riemannian submanifold of \( M \). If, in addition, the mean curvature vector of every leaf is parallel in the normal bundle, then \( L \) is called a spherical foliation, because in this case each leaf of \( L \) is an extrinsic sphere in \( M \). If every leaf of \( L \) is a totally geodesic submanifold of \( M \), then \( L \) is called a totally geodesic foliation.
Now, we recall the following well-known result of S. Hiepko [26].

**Hiepko’s Theorem.** Let $D_1$ and $D_2$ be two orthogonal distribution on a Riemannian manifold $M$. Suppose that both $D_1$ and $D_2$ are involutive such that $D_1$ is a totally geodesic foliation and $D_2$ is a spherical foliation. Then $M$ is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where $M_1$ and $M_2$ are integral manifolds of $D_1$ and $D_2$, respectively.

Now, we prove the following characterization by using Hiepko’s Theorem and useful lemmas of Sections 3 and Sections 4.

**Theorem 5.3.** Let $M$ be a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$. Then $M$ is locally a $D^0 – D^+$ mixed totally geodesic warped product skew CR-submanifold if and only if

(i) $A_{\varphi Z} X$ has no component in $\Gamma(D^0)$ and $\Gamma(D)$, i.e., $A_{\varphi Z} X \in \Gamma(D^+)$, for any $X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle)$ and $Z \in \Gamma(D^+)$. (ii) For any $X_1 \in \Gamma(D)$, $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$, we have

$$A_{\varphi Z} X_1 = -\varphi X_1(\mu)Z, \quad A_{\varphi Z} X_2 = 0, \quad A_{\varphi Z} Z = TX_2(\mu)Z, \quad (\xi \mu) = 1$$

for some smooth function $\mu$ on $M$ satisfying $V(\mu) = 0$, for any $V \in \Gamma(D^+)$. 

**Proof.** Let $M = M_1 \times_f M_2$ be a $D^0 – D^+$ mixed totally geodesic proper warped product skew CR-submanifold of order 1 of a Kenmotsu manifold $\tilde{M}$ such that $M_1 = M_T \times M_0$. In this theorem the tangent spaces of $M_T$, $M_0$ and $M_1$ are also denoted by $D$, $D^0$ and $D^+$, respectively. Then, from Lemma 4.2 (ii), we have

$$A_{\varphi Z} X_1 \perp D, \quad \forall X_1 \in \Gamma(D), \quad Z \in \Gamma(D^+). \quad (36)$$

Similarly, from the second equality of lemma 4.2 (iii), we have

$$A_{\varphi Z} X_1 \perp D^0, \quad \forall X_1 \in \Gamma(D), \quad Z \in \Gamma(D^+). \quad (37)$$

Also, for any $X_1 \in \Gamma(D)$ and $Z \in \Gamma(D^+)$, we have

$$g(A_{\varphi Z} X_1, \xi) = g(h(X_1, \xi), \varphi Z) = 0, \quad (38)$$

since for a submanifold of a Kenmotsu manifold $h(U, \xi) = 0$, $\forall U \in \Gamma(TM)$. Thus, from (36)-(38), we conclude that

$$A_{\varphi Z} X_1 \in \Gamma(D^+), \quad \forall X_1 \in \Gamma(D), \quad Z \in \Gamma(D^+). \quad (39)$$

Similarly, from the second equality of Lemma 4.2 (iii), we have

$$A_{\varphi Z} X_2 \perp D, \quad \forall X_2 \in \Gamma(D^0), \quad Z \in \Gamma(D^+). \quad (40)$$

Also, for a $D^0 – D^+$ mixed totally geodesic warped product skew CR-submanifold, from Lemma 4.2 (iv), we have

$$A_{\varphi Z} X_2 \perp D^0, \quad \forall X_2 \in \Gamma(D^0), \quad Z \in \Gamma(D^+). \quad (41)$$

On the other hand, for any $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$, we have

$$g(A_{\varphi Z} X_2, \xi) = g(h(X_2, \xi), \varphi Z) = 0. \quad (42)$$

Then, from (40)-(42), we conclude that

$$A_{\varphi Z} X_2 \in \Gamma(D^+), \quad \forall X_2 \in \Gamma(D^0), \quad Z \in \Gamma(D^+). \quad (43)$$
Thus, from (39), (43) and (44), we get $A_{q,y}X \in \Gamma(D^+)$, for any $X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle)$ and $Z \in \Gamma(D^+)$, which is (i).

For (ii), we proceed as follows: From Lemma 4.2 (ii), we have $g(A_{q,y}X_1, Y_1) = 0$, for any $X_1, Y_1 \in \Gamma(D)$, and $Z \in \Gamma(D^+)$. And, from the second equality of lemma 4.2 (iii), we have $g(A_{q,y}X_1, Y_2) = 0$, for any $X_1 \in \Gamma(D)$, $Y_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$.

And, for any $X_1 \in \Gamma(D)$ and $Z \in \Gamma(D^+)$, we have $g(A_{q,y}X_1, \xi) = g(h(X_1, \xi), \varphi Z) = 0$. Thus, we conclude that $g(A_{q,y}X_1, X) = 0$, for any $X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle)$, which means that either $A_{q,y}X_1 \in \Gamma(D^2)$ or $A_{q,y}X_1 = 0$. If $A_{q,y}X_1 \in \Gamma(D^2)$, then taking the inner product with $V \in \Gamma(D^2)$ and using Lemma 4.3, we get the first relation of (ii).

Now, for the second relation of (ii), form Lemma 4.2 (iii), we have $g(A_{q,y}X_2, X_1) = 0$, for any $X_1 \in \Gamma(D)$, $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$.

And, for a $D^0 - D^1$ mixed totally geodesic warped product submanifold, we have $g(A_{q,y}X_2, Y_1) = 0$, for any $X_2, Y_1 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$.

On the other hand, for any $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$, we have $g(A_{q,y}X_2, \xi) = g(h(X_2, \xi), \varphi Z) = 0$. Hence, we conclude that $g(A_{q,y}X_2, X) = 0$, for any $X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle)$, which means that either $A_{q,y}X_2 \in \Gamma(D^2)$ or $A_{q,y}X_2 = 0$. If $A_{q,y}X_2 \in \Gamma(D^2)$, then taking the inner product with $V \in \Gamma(D^2)$, we get $g(A_{q,y}X_2, V) = g(h(X_2, V), \varphi Z) = 0$, by using the $D^0 - D^2$ mixed totally geodesic condition. Hence, in both cases $A_{q,y}X_2 = 0$, which is the second relation of (ii).

Similarly, from Lemma 4.2 (iii), we have $g(A_{q,y}X_2, X_1) = 0$, for any $X_1 \in \Gamma(D)$, $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$. And, for a $D^0 - D^1$ mixed totally geodesic warped product submanifold, we have $g(A_{q,y}X_2, Y_1) = 0$, for any $X_2, Y_1 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$. And, for any $X_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$, we have $g(A_{q,y}X_2, \xi) = g(h(Z, \xi), FX_2) = 0$. Thus, we conclude that $g(A_{q,y}X_2, X) = 0$, for any $X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle)$, which means that either $A_{q,y}X_2 \in \Gamma(D^2)$ or $A_{q,y}X_2 = 0$. If $A_{q,y}X_2 \in \Gamma(D^2)$, then from Lemma 4.5, for a $D^0 - D^1$ mixed totally geodesic warped product submanifold, we find the third relation of (ii). The last relation of (ii) follows from Lemma 4.3 (i).

Conversely, suppose that $M$ is a proper skew CR-submanifold of order 1 of a Kenmotsu manifold $	ilde{M}$ such that (i) and (ii) hold. Then, from Lemma 3.3 and the given conditions of (ii), we have

$$g(V_1, Y_1, Z) = 0, \quad g(V_1, Y_2, Z) = 0, \quad g(V_2, X_1, Z) = 0 \quad (45)$$

for any $X_1 \in \Gamma(D)$, $Y_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$.

Similarly, from Lemma 3.4 (i) and the given conditions of (ii), we find that

$$g(V_1, Y_2, Z) = 0, \quad (46)$$

for any $X_2, Y_2 \in \Gamma(D^0)$ and $Z \in \Gamma(D^+)$. Thus, the relations (45) and (46) imply that the leaves of $D \oplus D^0 \oplus \langle \xi \rangle$ are totally geodesic in $M$. Consider $M_1$ be a leaf of $D \oplus D^0 \oplus \langle \xi \rangle$, thus $M_1$ is totally geodesic in $M$. On the other hand, from Lemma 3.2, $D^1$ is always integrable. If we consider the integral manifold $M_1$ of $D^1$ and $h^1$ be the second fundamental form of $M_1$ in $M$, then for any $X_1 \in \Gamma(D)$ and $Z, V \in \Gamma(D^+)$, we have

$$g(h^1(Z, V), X_1) = g(V_2V, X_1) = g(\nabla_2V, X_1) = -g(V_2X_1, V).$$

Using (2), (4) and the fact that $\xi$ is orthogonal to $D^1$, we obtain

$$g(h^1(Z, V), X_1) = g((\nabla_2\varphi)X_1, \varphi V) - g(V_2\varphi X_1, \varphi V).$$

Then from (3) and (5), we arrive at

$$g(h^1(Z, V), X_1) = -\eta(X_1)g(Z, V) - g(h(\varphi X_1, Z), \varphi V) = -\eta(X_1)g(Z, V) - g(A_{q,y}\varphi X_1, Z).$$

Using the given hypothesis of the theorem i.e., the first relation of (ii) by interchanging $X_1$ by $\varphi X_1$, we derive

$$g(h^1(Z, V), X_1) = -X_1(\mu) g(Z, V).$$
Thus, from the gradient definition, we find
\[ g(h^+(Z, V), X_1) = -g(\tilde{\nabla}_\mu, X_1) g(Z, V). \]  
(47)

Similarly, for any \( X_2 \in \Gamma(D^0) \) and \( Z, V \in \Gamma(D^+) \), we have
\[ g(h^+(Z, V), TX_2) = g(\tilde{\nabla}_Z V, TX_2) = g(\tilde{\nabla}_Z V, \varphi X_2) - g(\tilde{\nabla}_Z V, FX_2). \]

Using the covariant derivative property of the connection and (2), we obtain
\[ g(h^+(Z, V), TX_2) = g(\tilde{\nabla}_Z FX_2, V) - g(\varphi \tilde{\nabla}_Z V, X_2) = -g(A_{FX_2} Z, V) + g((\tilde{\nabla}_Z \varphi) V, X_2) - g(\tilde{\nabla}_Z \varphi V, X_2) \]

Then from (3), (5) and the hypothesis of the theorem, i.e., the third relation of (ii), we derive
\[ g(h^+(Z, V), TX_2) = -TX_2(\mu) g(Z, V) + g(A_{\varphi V} Z, X_2). \]

From the gradient definition and the symmetric property of shape operator, we find that
\[ g(h^+(Z, V), TX_2) = -g(\tilde{\nabla}_\mu, TX_2) g(Z, V) + g(A_{\varphi V} X_2, Z). \]

Second term in the right hand side of the above equation vanishes identically by using the second relation of (ii), thus, we obtain
\[ g(h^+(Z, V), TX_2) = -g(\tilde{\nabla}_\mu, TX_2) g(Z, V) + g(A_{\varphi V} X_2, Z). \]  
(48)

Also, for any \( Z, V \in \Gamma(D^+) \), we have
\[ g(h^+(Z, V), \xi) = g(\tilde{\nabla}_Z V, \xi) = -g(\tilde{\nabla}_Z \xi, V) = -g(Z, V). \]

Then, from the hypothesis of the theorem, i.e., the last relation of (ii), we find that
\[ g(h^+(Z, V), \xi) = -(\xi \mu) g(Z, V) = -g(\tilde{\nabla}_\mu, \xi) g(Z, V). \]  
(49)

Thus, from (47)-(49), we conclude that
\[ g(h^+(Z, V), X) = -g(\tilde{\nabla}_\mu, X) g(Z, V), \]  
(50)

for any \( X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle) \), which means that
\[ h^+(Z, V) = -\tilde{\nabla}_\mu g(Z, V). \]  
(51)

The relation (51) implies that \( M_\perp \) is totally umbilical in \( M \) with mean curvature vector \( H^+ = -\tilde{\nabla}_\mu \). Now, we have to show that \( H^+ \) is parallel with respect to the normal connection \( D^0 \) of \( M_\perp \) in \( M \). For this, consider any \( X \in \Gamma(D \oplus D^0 \oplus \langle \xi \rangle) \) and \( Z \in \Gamma(D^+) \), thus we have
\[ g(D_Z^N \tilde{\nabla}_\mu, X) = g(\nabla_Z \tilde{\nabla}_\mu, X_1) + g(\nabla_Z \tilde{\nabla}^0_\mu, X_2) + g(\nabla_Z \tilde{\nabla}^\xi_\mu, \xi), \]

where \( \tilde{\nabla}^T_\mu, \tilde{\nabla}^0_\mu \) and \( \tilde{\nabla}^\xi_\mu \) are the gradient components of \( \mu \) on \( M \) along \( D, D^0 \) and \( \langle \xi \rangle \), respectively. Using the Riemannian metric property, we derive
\[
g(D_Z^N \tilde{\nabla}_\mu, X) = \begin{align*}
Zg(\tilde{\nabla}^T_\mu, X_1) - g(\tilde{\nabla}^T_\mu, \nabla_Z X_1) + Zg(\tilde{\nabla}^0_\mu, X_2) - g(\tilde{\nabla}^0_\mu, \nabla_Z X_2) + Zg(\tilde{\nabla}^\xi_\mu, \xi) - Zg(\tilde{\nabla}^\xi_\mu, \nabla_Z \xi) \\
= Z(X_1 \mu) - g(\tilde{\nabla}^T_\mu, [Z, X_1]) - g(\tilde{\nabla}^T_\mu, \nabla_X Z) + Z(X_2 \mu) - g(\tilde{\nabla}^0_\mu, [Z, X_2]) - g(\tilde{\nabla}^0_\mu, \nabla_X Z) \\
+ Z(\xi \mu) - g(\tilde{\nabla}^\xi_\mu, [Z, \xi]) - g(\tilde{\nabla}^\xi_\mu, \nabla_Z Z).
\end{align*}
\]
Now, using the definition of Lie bracket and a property of Riemannian connection, the above relation will be

\[ g(D_2^X \bar{V}_\mu, X) = X_1(Z_\mu) + g(V_{X_1} \bar{V}_T \mu, Z) + X_2(Z_\mu) + g(V_{X_2} \bar{V}_0 \mu, Z) + \xi(Z_\mu) + g(V_{\xi} \bar{V}_2 \mu, Z) = 0, \]

since \((Z_\mu) = 0\), for any \(Z \in \Gamma(D^+)\) and \(V_{X_1} \bar{V}_T \mu + V_{X_2} \bar{V}_0 \mu + V_{\xi} \bar{V}_2 \mu = V_X \bar{V} \mu\) is orthogonal to \(D^+\), for any \(X \in \Gamma(D \oplus D^+ \oplus \langle \xi \rangle)\) as we know that \(\bar{V}_\mu\) is the gradient along \(M_1\) and \(M_1\) is totally geodesic in \(M\). This means that the mean curvature vector \(H^+\) of \(M_1\) is parallel. Thus, the leaves of \(D^+\) are totally umbilical with non vanishing parallel mean curvature vector \(-\bar{V}_\mu\), where \(\bar{V}_\mu\) is the gradient of the function \(\mu\), i.e., \(M_1\) is an extrinsic sphere in \(M\). Hence, by Hiepko’s Theorem, \(M\) is a warped product submanifold, which completes the proof. \(\square\)

6. Inequalities for skew CR-warped products

In this section, we establish two estimates for the squared norm of the second fundamental form of a warped product skew CR-submanifold in Kenmotsu manifolds. Let us consider the dimensions \(\dim M_1 = m\), such that \(M_1 = M_T \times M_0\), where \(M_T\) and \(M_0\) are invariant and proper slant submanifolds of \(M\), respectively. First, we construct the following frame fields for a warped product skew CR-submanifold.

Let \(M = M_1 \times M_2\) be an \(m\)-dimensional warped product skew CR-submanifold of order 1 of a \((2n + 1)\)-dimensional Kenmotsu manifold \(M\) such that the structure vector field \(\xi\) tangent to \(M_T\), where \(M_1 = M_T \times M_0\). Let us consider the dimensions \(\dim M_T = 2p + 1\), \(\dim M_0 = 2q\) and \(\dim M_1 = s\) and their corresponding tangent spaces are denoted by \(D \oplus \langle \xi \rangle, D^0\) and \(D^+\), respectively. We set the orthonormal frame fields of \(D \oplus \langle \xi \rangle\) as follows

\[
\{e_1, e_2, \ldots, e_p, e_{p+1} = \varphi e_1, \ldots, e_{2p} = \varphi e_p, e_{2p+1} = e_1\}
\]

and the orthonormal frame fields of \(D^0\) and \(D^+\), respectively are

\[
\{e_{2p+2} = e_1, \ldots, e_{2p+q+1} = e_q, e_{2p+q+2} = e_{q+1} = \sec \theta T e_1, \ldots, e_{2p+2q+2} = e_{2q} = \sec \theta T e_q\}
\]

and

\[
\{e_{2p+1+2q+1} = \hat{e}_1, \ldots, e_m = e_{2p+1+2q+s} = \hat{e}_s\}.
\]

Then the orthonormal frames of the normal subbundles \(FD^0, \varphi D^+\) and \(\nu\), respectively are

\[
\{e_{m+1} = \hat{e}_1 = \csc \theta F e_1, \ldots, e_{m+q} = \hat{e}_q = \csc \theta F e_q, e_{m+q+1} = \hat{e}_{q+1} = \csc \theta \sec \theta F e_1, \ldots, e_{m+2q} = \hat{e}_q = \csc \theta \sec \theta F e_q\},
\]

\[
\{e_{m+2q+1} = \hat{e}_{2q+1} = \varphi \hat{e}_1, \ldots, e_{m+2q+s} = \hat{e}_{2q+s} = \varphi \hat{e}_s\}
\]

and

\[
\{e_{m+2q+s+1}, \ldots, e_{2n+1}\}.
\]

It is clear that \(\dim \nu = (2n + 1 - m - 2q - s)\).

Now, we establish the following relationship for the squared norm of the second fundamental form of the warped product skew CR-submanifold in Kenmotsu manifolds.

**Theorem 6.1.** Let \(M = M_1 \times M_2\) be a \(D^0 - D^+\) mixed totally geodesic warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \(M\) such that \(\xi\) is tangent to \(M_T\), where \(M_1 = M_T \times M_0\). Then
(i) The squared norm of the second fundamental form satisfies
\[
\|h\|^2 \geq s \left( \cot^2 \theta \|\nabla f\|^2 \right) + 2s \left( \|\nabla f\|^2 - 1 \right)
\]
where \(\nabla f\) and \(\nabla f\) are the gradient components of the function \(\ln f\) along \(M_T\) and \(M_0\), respectively and \(s = \dim M_\perp\).

(ii) If equality sign in (i) holds, then \(M_1\) is a totally geodesic submanifold and \(M_\perp\) is a totally umbilical submanifold of \(M\).

Proof. From the definition of \(h\), we have
\[
\|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j)) = \sum_{m+1}^{2m+1} g(h(e_i,e_j),e_r)^2.
\]
Using the constructed frame fields, we find
\[
\|h\|^2 = \sum_{r=m+1}^{2m+1} \sum_{i,j=1}^{2m+1} g(h(e_i,e_j),e_r)^2 + 2 \sum_{r=m+1}^{2m+1} \sum_{i=1}^{2m+1} \sum_{j=1}^{2m+1} g(h(e_i,e_j),e_r)^2 + 2 \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2 + 2 \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2 + 2 \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2
\]

Fourth term in the right hand side vanishes identically by using the \(D^0 - D^\perp\) mixed totally geodesic condition, thus we derive
\[
\|h\|^2 = \sum_{m+1}^{2m+1} \sum_{i,j=1}^{2m+1} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2 + \sum_{r=m+1}^{2m} \sum_{i=1}^{2m} \sum_{j=1}^{2m} g(h(e_i,e_j),e_r)^2
\]

Since we could not find the relations for a warped product in the form \(g(h(U,W),v)\), for any \(U,W\) either in \(D \oplus \langle \xi \rangle\) or \(D^0\) or \(D^\perp\), therefore we will leave the positive third, sixth, ninth, twelfth and fifteenth terms in...
the right hand side of (55). Then, we find

\[ ||h||^2 \geq \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \overline{e_i})^2 + \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \eta(e_i))^2 + 2 \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \overline{e_i} \eta(e_i))^2 \]

The second and fourth terms vanish identically by using Lemma 4.2 (ii) and Lemma 4.2 (iii), respectively and for a \( D^0 - D^1 \) mixed totally geodesic warped product, the sixth term vanishes identically by using Lemma 4.2 (iv). Also, we could not find the relations for a warped product in the forms \( g(h(X_1, Y_1), F D^0) \), \( g(h(X_2, Y_2), F D^0) \), \( g(h(X_1, X_2), F D^0) \) and \( g(h(Z, V), q D^0) \), for any \( X_1, Y_1 \in \Gamma(D^0) \), \( X_2, Y_2 \in \Gamma(D^0) \) and \( Z, V \in \Gamma(D^1) \). Hence, by leaving these positive terms in the right hand side of (56) and using the constructed frame fields, we obtain

\[ ||h||^2 \geq \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \csc \theta e_i)^2 + \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \csc \theta \sec \theta F e_i)^2 + 2 \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \overline{e_i} \csc \theta e_i)^2 \]

Since \( e_{2p+1} = \xi \) and for a submanifold of a Kenmotsu manifold, we have \( h(\xi, U) = 0 \), for any \( U \in \Gamma(TM) \), thus the last term in the right hand side of (57) vanishes identically. Then, we derive

\[ ||h||^2 \geq \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{s} g(h(e_i, e_j), F e_i)^2 + \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{s} g(h(e_i, e_j), F e_i)^2 + 2 \sum_{r=1}^{q} \sum_{j=1}^{p} g(h(e_i, e_j), \overline{e_i} e_j)^2 + 2 \sum_{j=1}^{p} \sum_{r=1}^{q} g(h(\overline{e_i} e_j), \overline{e_i} e_j)^2 \]

Then, from (29), (30), (32) and (33), we arrive at

\[ ||h||^2 \geq \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{s} (Te_i(\ln f) g(\hat{e}_i, \hat{e}_j))^2 + \cot^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{s} (\hat{e}_i(\ln f) g(\hat{e}_i, \hat{e}_j))^2 + 2 \sum_{j=1}^{p} \sum_{r=1}^{q} (\overline{e}_i(\ln f) - \eta(e_i))^2 (\overline{e}_i, \overline{e}_j)^2 \]

Since \( \eta(e_i) = 0 \), \( \forall i = 1, \ldots, 2p \) and \( \eta(e_{2p+1}) = 1 \), thus we obtain

\[ ||h||^2 \geq s \csc^2 \theta \sum_{r=1}^{2q} (Te_i(\ln f))^2 - s \csc^2 \theta \sum_{r=q+1}^{2q} (Te_i(\ln f))^2 + s \cot^2 \theta \sum_{r=1}^{q} (\hat{e}_i(\ln f))^2 + 2 \sum_{r=1}^{2p+1} (\overline{e}_i(\ln f))^2 - 2s(\overline{e}_{2p+1}(\ln f))^2. \]
Using (10) and Lemma 4.2 (i), we find
\[ \|h\|^2 \geq s \csc^2 \theta \|T^\theta \ln f\|^2 - s \csc^2 \theta \sum_{r=1}^{q} g(e^r, T^\theta \ln f)^2 \]
\[ + s \cot^2 \theta \sum_{r=1}^{q} (e^r \ln f)^2 + 2s \|T^\theta \ln f\|^2 - 2s \]
\[ = s \cot^2 \theta \|T^\theta \ln f\|^2 - s \csc^2 \theta \sum_{r=1}^{q} g(\sec \theta e^r, T^\theta \ln f)^2 \]
\[ + s \cot^2 \theta \sum_{r=1}^{q} (e^r \ln f)^2 + 2s \left(\|T^\theta \ln f\|^2 - 1\right). \]

Then, from the gradient definition, we obtain
\[ \|h\|^2 \geq s \cot^2 \theta \|T^\theta \ln f\|^2 - s \cot^2 \theta \sum_{r=1}^{q} (e^r \ln f)^2 \]
\[ + s \cot^2 \theta \sum_{r=1}^{q} (e^r \ln f)^2 + 2s \left(\|T^\theta \ln f\|^2 - 1\right) \]
which is inequality (i). To prove the equality case of (53), we proceed as follows: From the given mixed totally geodesic condition, we have
\[ h(D^\theta, D^\perp) = 0. \quad (58) \]
On the other hand, leaving the third term in (55) and the first term in (56), we respectively have
\[ h(D, D^\perp) \perp \nu \quad \text{and} \quad h(D, D^\perp) \perp F D^\theta, \quad \Rightarrow \quad h(D, D^\perp) \subseteq \varphi D^\perp. \quad (59) \]
Also, from Lemma 4.2 (ii), we have
\[ h(D, D^\perp) \perp \varphi D^\perp. \quad (60) \]
Then, from (59) and (60), we conclude that
\[ h(D, D) = 0. \quad (61) \]
Similarly, from the leaving ninth term in the right hand side of (55) and leaving fifth term in the right hand side of (56), we find
\[ h(D^\theta, D^\perp) \perp \nu \quad \text{and} \quad h(D^\theta, D^\perp) \perp F D^\theta, \quad \Rightarrow \quad h(D^\theta, D^\perp) \subseteq \varphi D^\perp. \quad (62) \]
And for a \( D^\theta - D^\perp \) mixed totally geodesic warped product, from Lemma 4.2 (iv), we have
\[ h(D^\theta, D^\perp) \perp \varphi D^\perp. \quad (63) \]
Thus, from (62) and (63), we arrive at
\[ h(D^\theta, D^\perp) = 0. \quad (64) \]
From the leaving sixth term in the right hand side of (55) and leaving third term in (56), we respectively find that
\[ h(D, D^\theta) \perp \nu \quad \text{and} \quad h(D, D^\theta) \perp F D^\theta, \quad \Rightarrow \quad h(D, D^\theta) \subseteq \varphi D^\perp. \quad (65) \]
Also, from Lemma 4.2 (iii), we obtain
\[ h(D, D^0) \perp \varphi D^\bot. \] (66)

Then, from (65) and (66), we conclude that
\[ h(D, D^0) = 0. \] (67)

Since \( M_1 \) is totally geodesic in \( M \) [7, 15], using this fact with (58), (61), (64) and (67), we get \( M_1 \) is totally geodesic in \( \bar{M} \). On the other hand, leaving the fifteenth term in the right hand side of (55), we find \( h(D, D^+) \perp v \). Also, from Lemma 4.2 (iii), we obtain \( h(D, D^+) \perp \mathcal{F} D^0 \). Thus, we conclude that
\[ h(D, D^+) \subseteq \varphi D^\bot. \] (68)

And, the leaving twelfth term in the right hand side of (55) and the leaving sixth term in the right hand side of (56), we respectively have
\[ h(D^+, D^+) \perp v \quad \text{and} \quad h(D^+, D^+) \perp \varphi D^\bot, \quad \Rightarrow \quad h(D^+, D^+) \subseteq \mathcal{F} D^0. \] (69)

Also, from Lemma 4.3 and Lemma 4.5, we respectively have
\[ g(h(X_1, Z), \varphi V) = -\varphi X_1(\ln f) g(Z, V) \] (70)

and
\[ g(h(Z, V), \mathcal{F} X_2) = TX_2(\ln f) g(Z, V), \] (71)

for any \( X_1 \in \Gamma(D \oplus \{ \xi \}), X_2 \in \Gamma(D^0) \) and \( Z, V \in \Gamma(D^+) \). Since \( M_\perp \) is totally umbilical in \( M \) [7, 15], using this fact with (58) and (68)-(71), we observe that \( M_\perp \) is a totally umbilical submanifold of \( M \). Hence, the theorem is proved completely. \( \square \)

If the structure vector field \( \xi \) is tangent to \( M_0 \), then we have the following result.

**Theorem 6.2.** Let \( M = M_1 \times \ell M_2 \) be a \( D^0 - D^\bot \) mixed totally geodesic warped product skew CR-submanifold of order 1 of a Kenmotsu manifold \( \bar{M} \) such that \( \xi \) is tangent to \( M_0 \), where \( M_1 = M_T \times M_0 \). Then

(i) The squared norm of the second fundamental form satisfies
\[ ||h||^2 \geq s \csc^2 \theta (||\bar{V}^f \ln f||^2 - 1) + 2s||\bar{V}^T \ln f||^2 \] (72)

where \( \bar{V}^f \ln f \) and \( \bar{V}^T \ln f \) are the gradient components of the function \( \ln f \) along \( M_T \) and \( M_0 \), respectively.

(ii) If the equality sign in (i) holds, then \( M_1 \) is a totally geodesic submanifold and \( M_\perp \) is a totally umbilical submanifold of \( \bar{M} \).

We can prove this theorem like Theorem 5.3, just we have to handle the structure vector field \( \xi \). In this case the dimensions of \( M_T \) and \( M_0 \) respectively are \( 2p \) and \( 2q + 1 \) and the orthonormal frames of their tangent spaces \( D \) and \( D^0 \oplus \{ \xi \} \), respectively are \( \{ e_1, e_2, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots, e_{2p} = \varphi e_p \} \) and \( \{ e_{2p+1} = e'_1, \cdots, e_{2p+q} = e'_q, e_{2p+q+1} = e'_{q+1} = \sec \theta Te'_1, \cdots, e_{2p+2q} = e'_{2q} = \sec \theta Te'_q, e_{2p+2q+1} = e'_{2q+1} = \xi \} \).

7. Some Applications

In this section, we give some applications of our derived results.

For the warped product skew CR-submanifolds of the form \( M = M_1 \times_f M_\perp \) of a Kenmotsu manifold \( \bar{M} \) such that \( M_1 = M_T \times M_0 \), if \( \dim M_0 = 0 \), then the warped product skew CR-submanifolds turn into CR-warped products \( M = M_T \times_f M_\perp \) which have been studied in [3, 27]. Hence, Theorem 5.3 generalise a result of [27] as follows:

If we put \( \dim M_0 = 0 \) in Theorem 5.3, then the warped product is of the form \( M = M_T \times_f M_\perp \), a contact CR-warped product in a Kenmotsu manifold \( \bar{M} \). Thus, we have the following special case of Theorem 5.3.
Corollary 7.1. (Theorem 3.4 [27]) A proper contact CR-submanifold of a Kenmotsu manifold $\tilde{M}$ is locally a contact CR-warped product if and only if
\[ A_{\phi Z}X_1 = -(\phi X_1 \mu) Z, \quad \forall \ X_1 \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle), \quad Z \in \Gamma(\mathcal{D}^\perp) \]  
for some function $\mu$ on $M$ satisfying $V \mu = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$.

On the other hand, in a warped product skew CR-submanifold $M = M_1 \times_f M_\perp$ such that $M_1 = M_T \times M_0$, if $\dim M_T = 0$, then the warped product skew CR-submanifold turns into a warped product pseudo-slant submanifold $M = M_0 \times_f M_\perp$ and the case has been considered in [2]. In this case, Theorem 4.1 of [2] is a special case of Theorem 5.3, by interchanging $X_2$ by $TX_2$ in the third relation of Theorem 5.3 as follows:

Corollary 7.2. (Theorem 4.1 [2]) Let $M$ be a proper pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then $M$ is locally a mixed totally geodesic warped product submanifold if and only if
\[ A_{\phi Z}X_2 = 0 \quad \text{and} \quad A_{TX_2}Z = \cos^2 \theta (\eta(X_2) - (X_2 \mu)) Z \]  
for any $Z \in \Gamma(\mathcal{D}^\perp)$ and $X_2 \in \Gamma(\mathcal{D}^0 \oplus \langle \xi \rangle)$ for some smooth function $\mu$ on $M$ such that $V \mu = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$.

Similarly, Theorem 3.1 of [3] is a special case of Theorem 6.1 as follows:

If we consider $\dim M_0 = 0$ in Theorem 6.1, then the inequality (53) is true for contact CR-warped products which have been considered in [3].

Corollary 7.3. (Theorem 3.1 [3]) Let $\tilde{M}$ be a $(2n + 1)$-dimensional Kenmotsu manifold and $M = M_T \times_f M_\perp$ an $m$-dimensional contact CR-warped product submanifold, such that $M_T$ is a $(2p + 1)$-dimensional invariant submanifold tangent to $\xi$ and $M_\perp$ a $s$-dimensional anti-invariant submanifold of $\tilde{M}$. Then

(i) The squared norm of the second fundamental form of $M$ satisfies
\[ \|h\|^2 \geq 2s (\|\tilde{\nabla}^\theta \ln f\|^2 - 1) \]  
where $\tilde{\nabla}^\theta \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (75) holds identically, then $M_T$ is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Moreover, $M$ is a minimal submanifold of $\tilde{M}$.

On the other hand, if we consider $\dim M_T = 0$ in Theorem 6.2, then the warped product skew CR-submanifold $M$ turns to the warped product pseudo-slant submanifold $M = M_0 \times_f M_\perp$ and the inequality (72) generalise Theorem 5.1 of [2] as follows.

Corollary 7.4. (Theorem 5.1 [2]) Let $M = M_0 \times_f M_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $M_0$ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$ with their real dimensions $(2q + 1)$ and $s$, respectively. Then

(i) The squared norm of the second fundamental form $h$ of $M$ satisfies
\[ \|h\|^2 \geq sc \cot^2 \theta (\|\tilde{\nabla}^\theta \ln f\|^2 - 1) \]  
where $\tilde{\nabla}^\theta \ln f$ is gradient of the function $\ln f$ along $M_0$.

(ii) If equality sign of (76) holds identically, then $M_0$ is totally geodesic and $M_\perp$ is totally umbilical in $\tilde{M}$. 
