On the Graph of Modules Over Commutative Rings II

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Abstract. Let $M$ be a module over a commutative ring $R$. In this paper, we continue our study about the quasi-Zariski topology-graph $G(\tau^*_T)$ which was introduced in [On the graph of modules over commutative rings, Rocky Mountain J. Math. 46(3) (2016), 1–19]. For a non-empty subset $T$ of $\text{Spec}(M)$, we obtain useful characterizations for those modules $M$ for which $G(\tau^*_T)$ is a bipartite graph. Also, we prove that if $G(\tau^*_T)$ is a tree, then $G(\tau^*_T)$ is a star graph. Moreover, we study coloring of quasi-Zariski topology-graphs and investigate the interplay between $\chi(G(\tau^*_T))$ and $\omega(G(\tau^*_T))$.

1. Introduction

Throughout this paper $R$ is a commutative ring with a non-zero identity and $M$ is a unital $R$-module.

By $N \leq M$ (resp. $N < M$) we mean that $N$ is a submodule (resp. proper submodule) of $M$.

Define $(N:R)M$ or simply $(N:M) = \{r \in R \mid rM \subseteq N\}$ for any $N \leq M$. We denote $((0):M)$ by $AnnR(M)$ or simply $Ann(M)$. $M$ is said to be faithful if $Ann(M) = (0)$.

Let $N, K \leq M$. Then the product of $N$ and $K$, denoted by $NK$, is defined by $(N:M)(K:M)M$ (see [3]).

A prime submodule of $M$ is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in P (P : M)$ or $e \in P ^1$[13].

The prime spectrum of $M$ is the set of all prime submodules of $M$ and denoted by $\text{Spec}(M)$.

There are many papers on assigning graphs to rings or modules (see, for example, [1, 4–7, 9, 16]). In [5], the present authors introduced and studied the graph $G(\tau^*_T)$ (resp. $AG(M)$, called the quasi-Zariski topology-graph (resp. the annihilating-submodule graph), where $T$ is a non-empty subset of $\text{Spec}(M)$.

$AG(M)$ is an undirected graph with vertices $V(AG(M)) = \{N \leq M\}$ there exists $(0) \neq K < M$ with $NK = (0)$.

In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M\}$ there exists a submodule $K < M$ with $(K : M) \neq Ann(M)$ and $NK = (0)$]. By [4, Theorem 3.4], one conclude that $AG(M)^*$ is a connected subgraph.

$G(\tau^*_T)$ is an undirected graph with vertices $V(G(\tau^*_T)) = \{N < M\}$ there exists $K < M$ such that $V^*(N) \cup V^*(K) = T$ and $V^*(N), V^*(K) \neq T$ and distinct vertices $N$ and $L$ are adjacent if and only if $V^*(N) \cup V^*(L) = T$ (see [5, Definition 2.1]).

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For any submodule $N$ of $M$, $V^*(N)$ is the set of all prime submodules of $M$ containing $N$. Of course, $V^*(M)$ is the empty set and $V^*(0)$ is the set of all prime submodules of $M$. Note that for any family of submodules $N_i$ ($i \in I$) of $M$, $\cap V^*(N_i) = V^*(\langle \cup_i N_i \rangle)$. Thus if $Z^*(M)$ denotes the collection of all subsets $V^*(N)$ of Spec$(M)$, then $Z^*(M)$ contains the empty set and Spec$(M)$, and $Z^*(M)$ is closed under arbitrary intersections. If $Z^*(M)$ is closed under finite unions, i.e. for any submodules $N$ and $K$ of $M$, there exists a submodule $L$ of $M$ such that $V^*(N) \cup V^*(K) = V^*(L)$, for in this case $Z^*(M)$ satisfies the axioms for the closed subsets of a topological space and $M$ is called a top module for short. The quasi-Zariski topology on $X = \text{Spec}(M)$ is the topology $\tau^*_M$ described by taking the set $Z^*(M) = \{V^*(N)| N$ is a submodule of $M\}$ as the set of closed sets of Spec$(M)$, where $V^*(N) = \{P \in X| P \supseteq N\}$ [15].

If Spec$(M) \neq \emptyset$, the mapping $\psi : \text{Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ such that $\psi(P) = (P : M)/\text{Ann}(M)$ for every $P \in \text{Spec}(M)$, is called the natural map of Spec$(M)$ [14].

A topological space $X$ is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets $X_i$ of $X$ with $i = 1, 2$, we have $X = X_1$ or $X = X_2$.

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of $M$ containing $N$, and in case $N$ is not contained in any prime submodule, $\sqrt{N}$ is defined to be $M$ [13].

We recall that $N < M$ is said to be a semiprime submodule of $M$ if for every ideal $I$ of $R$ and every submodule $K$ of $M$ with $IK \subseteq N$ implies that $IK \subseteq N$. Further $M$ is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [17]).

The notations Nil$(R)$, Min$(M)$, and Min$(T)$ will denote the set of all nilpotent elements of $R$ and the set of all minimal prime submodules of $M$ and the set of minimal members of $T$, respectively.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in $G$ is called the clique number of $G$. In this article, we continue our studying about $\text{AG}(\tau^*_G)$ and $\text{AG}(M)$ and we try to relate the combinatorial properties of the above mentioned graphs to the algebraic properties of $M$.

In section 2 of this paper, we state some properties related to the quasi-Zariski topology-graph that are basic or needed in the later sections. In section 3, we study the bipartite quasi-Zariski topology-graphs of modules over commutative rings (see Proposition 3.1). Also, we prove that if $G(\tau^*_G)$ is a tree, then $G(\tau^*_G)$ is a star graph (see Theorem 3.5). In section 4, we study coloring of the quasi-Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau^*_G))$ and $\omega(G(\tau^*_G))$. We show that under condition over minimal submodules of $M/\bigcup_{\mathfrak{p} \in P} \mathfrak{p}$, we have $\omega(G(\tau^*_G)) = \chi(G(\tau^*_G))$ (see Theorem 4.1). Moreover, we investigate some relations between the existence of cycles in the quasi-Zariski topology-graph of a cyclic module and the number of its minimal members of $T$ (see Proposition 4.9).

Let us introduce some graphical notions and denotations that are used in what follows: A graph $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function $\psi_G$ that associates an unordered pair of distinct vertices with each edge. The edge $e$ joins $x$ and $y$ if $\psi_G(e) = \{x, y\}$, and we say $x$ and $y$ are adjacent. A path in graph $G$ is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where $x_{i-1}$ and $x_i$ are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between $x_{i-1}$ and $x_i$.

A graph $H$ is a subgraph of $G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$; that is, $U$ and $V$ are each independent sets and complete bipartite graph on $n$ and $m$ vertices, denoted by $K_{n,m}$, where $V$ and $U$ are of size $n$ and $m$, respectively, and $E(G)$ connects every vertex in $V$ with all vertices in $U$. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of $U$. For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by $\text{deg}(v)$. If all the vertices of $G$ have the same degree $k$, then $G$ is called $k$-regular, or simply regular. We denote by $C_n$ a cycle of order $n$. Let $G$ and $G'$ be two graphs. A graph homomorphism from $G$ to $G'$ is a mapping $\phi : V(G) \rightarrow V(G')$ such that for every edge $[u,v]$ of $G$, $[\phi(u), \phi(v)]$ is an edge of $G'$. A retract of $G$ is a subgraph $H$ of $G$ such that there exists a homomorphism $\phi : G \longrightarrow H$ such that $\phi(x) = x$, for every
vertex x of H. The homomorphism φ is called the retract (graph) homomorphism (see [10]).

Throughout the rest of this paper, we denote: T is a non-empty subset of Spec(M), Q := ∩_P∈T P,  ̄M := M/ Q,  ̄N := N/ Q, m := m + Q, and I := I/(Q : M), where N is a submodule of M containing Q, m ∈ M, and I is an ideal of R containing (Q : M).

2. Auxiliary results

In this section, we provide some properties related to the quasi-Zariski topology-graph that are basic or needed in the sequel. Throughout this paper M is a top module and by [15, Theorem 3.5], every multiplication module is a top module.

Remark 2.1. By [15, Lemma 2.1], if M is a top module, then for every pair of submodules N and L of M, we have V*(N) ∪ V*(L) = V*( ̄N) ∪ V*( ̄L) = V*( ̄N ∩ ̄L). By [5, Proposition 2.3], we have T is a closed subset of Spec(M) if and only if T = V*(∩_P∈T P) and G(τ^*_T) ≠ ∅ if and only if T = V*(∩_P∈T P) and T is not irreducible. So if N and K are adjacent in G(τ^*_T), then ̄N ∩ ̄K = ∩_P∈T P. Therefore ∩_P∈T P ⊆ ̄N, ̄K.

Lemma 2.2. (See [2, Proposition 7.6].) Let R_1, R_2, . . . , R_n be non-zero ideals of R. Then the following statements are equivalent:

(a) R = R_1 ⊕ . . . ⊕ R_n;
(b) As an abelian group R is the direct sum of R_1, . . . , R_n;
(c) There exist pairwise orthogonal idempotents e_1, . . . , e_n with 1 = e_1 + . . . + e_n, and R_i = Re_i, i = 1, . . . , n.

Proposition 2.3. Suppose that e is an idempotent element of R. We have the following statements.

(a) R = R_1 ⊕ R_2, where R_1 = eR and R_2 = (1 − e)R.
(b) M = M_1 ⊕ M_2, where M_1 = eM and M_2 = (1 − e)M.
(c) For every submodule N of M, N = N_1 ⊕ N_2 such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and (N ∩_R M) = (N_1 ∩_R M_1) ⊕ (N_2 ∩_R M_2).
(d) For submodules N and K of M, NK = N_1K_1 ⊕ N_2K_2, N ∩ K = N_1 ∩ K_1 ⊕ N_2 ∩ K_2 such that N = N_1 ⊕ N_2 and K = K_1 ⊕ K_2.
(e) Prime submodules of M are P ⊕ M_2 and M_1 ⊕ Q, where P and Q are prime submodules of M_1 and M_2, respectively.
(f) For submodule N of M, we have ̄N = ̄N_1 ⊕ ̄N_2 = ̄N_1 ⊕ ̄N_2, where N = N_1 ⊕ N_2.

Proof. This is clear. □

An ideal I < R is said to be nil if I consist of nilpotent elements.

Lemma 2.4. (See [12, Theorem 21.28].) Let I be a nil ideal in R and u ∈ R be such that u + I is an idempotent in R/I. Then there exists an idempotent e in uR such that e − u ∈ I.

Lemma 2.5. (See [6, Lemma 2.4].) Let N be a minimal submodule of M and let Ann(M) be a nil ideal. Then we have N^2 = (0) or N = eM for some idempotent e ∈ R.

Lemma 2.6. Assume that T is a closed subset of Spec(M) and ̄M is a multiplication module. Then AG( ̄M) is isomorphic with a (an induced) subgraph of G(τ^*_T).
Proof. Let $\bar{N} \in V(\text{Spec}(\bar{M}))$. Then there exists a nonzero submodule $\bar{K}$ of $\bar{M}$ such that it is adjacent to $\bar{N}$. So we have $NK \subseteq Q$. Hence $V^\ast(NK) = T$. If $V^\ast(N) = T$, then $N = Q$, a contradiction. Hence $\bar{N}$ is a vertex in $G(\tau^\ast_T)$ which is adjacent to $K$. □

Lemma 2.7. If $\bar{M}$ is a faithful multiplication module, then $G(\tau^\ast_{\text{Spec}(\bar{M})})$ and $AG(M)$ are the same.

Proof. $\bar{M}$ is a faithful module so that $T = \text{Spec}(M)$. If $G(\tau^\ast_{\text{Spec}(\bar{M})}) \neq \emptyset$, then there exist non-trivial submodules $N$ and $K$ of $M$ which is adjacent in $G(\tau^\ast_{\text{Spec}(\bar{M})})$. Hence $V^\ast(NK) = \text{Spec}(M)$ which implies that $NK = (0)$ so that $AG(M) \neq \emptyset$. By Lemma 2.6, $AG(M)$ is isomorphic with a subgraph of $G(\tau^\ast_{\text{Spec}(\bar{M})})$. One can see that the vertex map $\phi : V(G(\tau^\ast_{\text{Spec}(\bar{M})})) \longrightarrow V(AG(M))$, defined by $N \longrightarrow N$ is an isomorphism. □

Recall that $\Delta(G(\tau^\ast_T))$ is the maximum degree of $G(\tau^\ast_T)$ and the length of an $R$-module $M$, is denoted by $l_R(M)$.

Lemma 2.8. Let every nontrivial submodule of $M$ be a vertex in $G(\tau^\ast_T)$. If $\Delta(G(\tau^\ast_T)) < \infty$, then $l_R(M) \leq \Delta(G(\tau^\ast_T)) + 1$. Also, every non-trivial submodule of $M$ has finitely many submodules.

Proof. Straightforward. □

Theorem 2.9. Let $\bar{M}$ be a multiplication module and $G(\tau^\ast_T) \neq \emptyset$. Then $M$ has acc (resp. dcc) on vertices of $G(\tau^\ast_T)$ if and only if $\bar{M}$ is a Noetherian (resp. an Artinian) module.

Proof. Suppose that $G(\tau^\ast_T)$ has acc (resp. dcc) on vertices. By [5, Proposition 2.3 (iii)], $\bar{M}$ is not a prime module and hence there exist $r \in R$ and $m \in M$ such that $rm = 0$ but $m \neq 0$ and $r \notin \text{Ann}(M)$. Now $\bar{r}M \cong M/(0 :_M r)$. Further, $\bar{r}M$ and $(0 :_M r)$ are vertices because $(0 :_M r)(\bar{r}M) = ((0 :_M r) : M)(\bar{r}M : M)M \subseteq \bar{r}M((0 :_M r) : M) \subseteq r(0 :_M r) = 0$. Then $\{\bar{N} \bar{N} \subseteq \bar{M}, \bar{N} \subseteq \bar{r}M\} \cup \{\bar{N} \subseteq \bar{M}, \bar{N} \subseteq (0 :_M r)\} \subseteq V(G(\tau^\ast_T))$. It follows that the $R$-modules $\bar{r}M$ and $(0 :_M r)$ have acc (resp. dcc) on submodules. Since $\bar{r}M \cong M/(0 :_M r)$, $\bar{M}$ has acc on submodules and the proof is completed. □

3. Quasi-Zariski topology-graph of modules

First, in this section we give the more notation to be used throughout the remainder of this article. Suppose that $e (e \neq 0,1)$ is an idempotent element of $R$. Let $M_1 := eM, M_2 := (1 - e)M, T_1 := \{p_1 \in \text{Spec}(M_1) | p_1 \not\subseteq \text{Ann}(M_1)\}, T_2 := \{p_2 \in \text{Spec}(M_2) | p_2 \not\subseteq \text{Ann}(M_2)\}, Q_1 := \cap_{p_1 \in T_1}p_1, Q_2 := \cap_{p_2 \in T_2}p_2, \bar{M}_1 = e\bar{M} = eM/\bar{Q}_1$, and $\bar{M}_2 = (e - 1)\bar{M} = (e - 1)M/\bar{Q}_2$. Consequently we have, $Q = Q_1 \oplus Q_2$, where $Q = \cap_{p \in T}p$ and $\bar{M} \cong M_1 \oplus M_2$.

We recall that a submodule $N$ of $M$ is a prime $R$-module if and only if it is a prime $R/\text{Ann}(M)$-module (see [4, Result 1.2]).

Proposition 3.1. Suppose that $\bar{M}$ is a multiplication module. Then the following statements hold.

(a) If there exists a vertex of $G(\tau^\ast_T)$ which is adjacent to every other vertex, then $\bar{M} \cong M_1 \oplus M_2$, where $M_1$ is a simple module and $M_2$ is a prime module for some idempotent element $e \in R$.

(b) If $\bar{M}_1$ and $\bar{M}_2$ are prime modules for some idempotent element $e \in R$, then $G(\tau^\ast_T)$ is a complete bipartite graph.

Proof. (a) Suppose that $N$ is adjacent to every other vertex of $G(\tau^\ast_T)$. Since $V^\ast(N) = V^\ast(\sqrt{N})$, we have $N = \sqrt{N}$. It is clear that $\bar{N}$ is a minimal submodule of $\bar{M}$. We have $(N)^2 \neq (0)$ because $V^\ast(N) \neq T$. Then Lemma 2.5, implies that $\bar{M} \cong \bar{e}M \oplus (e - 1)\bar{M}$ for some idempotent element $e$ of $R$. Without loss of generality we may assume that $M_1 \oplus Q_2$ is adjacent to every other vertex. We claim that $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module. Let $Q_1 \subseteq K < \bar{M}_2$. We have $V^\ast(K \oplus Q_2) \neq T$ because $Q_1 \oplus Q_2 \subseteq K \oplus Q_2$. Since $V^\ast(K \oplus Q_2) \cup V^\ast(Q_1 \oplus M_2) = T$, we have $K \oplus Q_2$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. Therefore $V^\ast(K \oplus Q_2) \cup V^\ast(M_1 \oplus Q_2) = V^\ast(K \oplus Q_2) = T$, a contradiction. It implies that $\bar{M}_1$ is a simple module. Now, we
show that \( \bar{M}_2 \) is a prime module. It is enough to show that is a prime \( R/(Q_2 : M_2) \)-module. Otherwise, \( \bar{I}K = (0) \), where \( (Q_2 : M_2) \subseteq I < R \) and \( Q_2 \subseteq K < M \). It follows that \( V'(M_1 \oplus K) \cap V'(Q_1 \oplus IM_2) = V'(Q_1 \oplus K(IM_2)) = T \) because \( K(IM_2) \subseteq IK \subseteq Q_2 \) and \( (Q_2 : M_2)^2M_2 \subseteq K(IM_2) \). Therefore \( V'(M_1 \oplus K) \cup V'(M_1 \oplus Q_2) = T = V'(M_1 \oplus Q_2) \), a contradiction.

(b) Assume that \( N_1 \oplus N_2 \) is adjacent to \( K_1 \oplus K_2 \). One can see that \( \sqrt{N_1K_1} \oplus \sqrt{N_2K_2} = \sqrt{Q_1} \oplus \sqrt{Q_2} \). It implies that \( (\sqrt{K_1 : M_1}M_1 : M_1) \cap (\sqrt{N_1 : M_1}M_1) = (Q_1 : M_1) \) or \( (\sqrt{N_1 : M_1}M_1) = Q_1 \) and \( \sqrt{(K_2 : M_2)M_2 : M_2} = (Q_2 : M_2) \) or \( \sqrt{(N_2 : M_2)M_2} = Q_2 \). Therefore \( G(\tau^*_1) \) is a complete bipartite graph with two parts \( U \) and \( V \) such that \( N \in U \) if and only if \( V'(N) = V'(M_1 \oplus Q_2) \) and \( K \in V \) if and only if \( V'(K) = V'(Q_1 \oplus M_2) \).

**Corollary 3.2.** Let \( M \) be a faithful multiplication module. Then the following statements are equivalent.

(a) There is a vertex of \( G(\tau^*_1) \) which is adjacent to every other vertex of \( G(\tau^*_1) \).

(b) \( G(\tau^*_1) \) is a star graph.

(c) \( M = F \oplus D \), where \( F \) is a simple module and \( D \) is a prime module.

**Proof.** (a) \( \Rightarrow \) (b) Let \( \bar{M} \) be a faithful module. Then \( Q = (0) \) and we have \( T = \text{Spec}(M) \). By Proposition 3.1, \( M = M_1 \oplus M_2 \), where \( M_1 \) is a simple module and \( M_2 \) is a prime module. Then every non-zero submodule of \( M \) is of the form \( M_1 \oplus N_2 \) or \( (0) \oplus N_2 \), where \( N_2 \) is a non-zero submodule of \( M_2 \). By our hypothesis, we can not have any vertex of the form \( M_1 \oplus N_2 \), where \( N_2 \) is a non-zero proper submodule of \( M_2 \). Also \( M_1 \oplus (0) \) is adjacent to every other vertex, and non of the submodules of the form \( (0) \oplus N_2 \) can be adjacent to each other. So \( G(\tau^*_1) \) is a star graph.

(b) \( \Rightarrow \) (c) This follows by Proposition 3.1 (a).

(c) \( \Rightarrow \) (a) Assume that \( M = F \oplus D \), where \( F \) is a simple module and \( D \) is a prime module. It is easy to see that for some minimal submodule \( N \) of \( M \), we have \( N^2 \neq (0) \). Since \( M \) is a faithful module, Lemma 2.5 implies that \( F \cong eM \), where \( e \) is an idempotent element of \( R \). Finally Proposition 3.1 (a) completes the proof. \( \square \)

**Lemma 3.3.** Let \( e \in R \) be an idempotent element of \( R \) and let \( \bar{M} \) be a multiplication module. If \( G(\tau^*_1) \) is a triangle-free graph, then both \( M_1 \) and \( M_2 \) are prime \( R \)-modules. Moreover, if \( G(\tau^*_1) \) has no cycle, then \( M_1 \) is a simple module and \( M_2 \) is a prime module.

**Proof.** Without loss of generality, we can assume that \( \bar{M}_1 \) is a prime module. Then \( \bar{I}K = (0) \), where \( (Q_2 : M_2) \subseteq I < R \) and \( Q_2 \subseteq K < M \). It follows that \( V'(M_1 \oplus K) \cap V'(Q_1 \oplus IM_2) = V'(Q_1 \oplus K(IM_2)) = T \) (if \( IM_2 = K \), then \( V'(Q_1 \oplus K) = V'(Q_1 \oplus K^2) = V'(Q_1 \oplus K(IM_2)) = T \), a contradiction). So both \( M_1 \) and \( M_2 \) are prime \( R \)-modules. Now suppose that \( G(\tau^*_1) \) has no cycle. If none of \( M_1 \) and \( M_2 \) is a simple module, then we choose non-trivial submodules \( N_i \), in \( M_i \) for some \( i = 1, 2 \). So \( N_1 \oplus Q_2, Q_1 \oplus N_2, M_1 \oplus Q_2, \) and \( Q_1 \oplus M_2 \) form a cycle, a contradiction. \( \square \)

**Corollary 3.4.** Assume that \( \bar{M} \) is a multiplication module. Then \( G(\tau^*_1) \) is a star graph if and only if \( M_1 \) is a simple module and \( \bar{M}_2 \) is a prime module for some idempotent \( e \in R \).

**Proof.** The necessity is clear by Proposition 3.1 (a). For the converse, assume that \( M = M_1 \oplus M_2 \), where \( M_1 \) is a simple module and \( M_2 \) is a prime for some idempotent \( e \in R \). Using the Proposition 3.1 (b), \( G(\tau^*_1) \) is a complete bipartite graph with two parts \( U \) and \( V \) such that \( N \in U \) if and only if \( V'(N) = V'(M_1 \oplus Q_2) \) and \( K \in V \) if and only if \( V'(K) = V'(Q_1 \oplus M_2) \). We claim that \( |U| = 1 \). Otherwise, \( V'(M_1 \oplus Q_2) = V'(N_1 \oplus Q_2) \), where \( Q_1 \neq N_1 < M_1 \). It follows that \( (N_1 : M_1)M_1 = M_1 \), a contradiction (note that if \( M \) is a multiplication module, then \( \sqrt{N} \neq M \), where \( N < M \)). So \( G(\tau^*_1) \) is a star graph. \( \square \)

**Theorem 3.5.** If \( G(\tau^*_1) \) is a tree, then \( G(\tau^*_1) \) is a star graph.
Proof. Suppose that $G(\tau^*_T)$ is not a star graph. Then $G(\tau^*_T)$ has at least four vertices. Obviously, there are two adjacent vertices $L$ and $K$ of $G(\tau^*_T)$ such that $|N(L) \setminus |K|| \geq 1$ and $|N(K) \setminus |L|| \geq 1$. Let $N(L) \setminus |K| = \{L_i\}_{i \in \Lambda}$ and $N(K) \setminus |L| = \{K_j\}_{j \in \Gamma}$. Since $G(\tau^*_T)$ is a tree, we have $N(L) \cap N(K) = \emptyset$. By [5, Theorem 2.6], $\text{diam}(G(\tau^*_T)) \leq 3$. So every edge of $G(\tau^*_T)$ is of the form $[L, K], [L, L_i]$ or $[K, K_j]$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, Pick $p \in \Lambda$ and $q \in \Gamma$. Since $G(\tau^*_T)$ is a tree, $\sqrt{L_p} \cap \sqrt{K_q}$ is a vertex of $G(\tau^*_T)$. If $\sqrt{L_p} \cap \sqrt{K_q} = L_u$ for some $u \in \Lambda$, then $V^*(K) \cup V^*(L_u) = T$, a contradiction. If $\sqrt{L_p} \cap \sqrt{K_q} = K_v$, for some $v \in \Gamma$, then $V^*(L) \cup V^*(K_v) = T$, a contradiction. If $\sqrt{L_p} \cap \sqrt{K_q} = L$ or $\sqrt{L_p} \cap \sqrt{K_q} = K$, then $V^*(L) = T$ or $V^*(K) = T$, respectively, a contradiction. So the claim is proved. \qed

Proposition 3.6. Let $\tilde{M}$ be a multiplication module. Then in each case of the following statements, $|T| = 2$ and $G(\tau^*_T) \cong K_2$.

(a) $R$ be an Artinian ring and $G(\tau^*_T)$ is a bipartite graph.

(b) $\text{Ann}(\tilde{M})$ is a nil ideal of $R$ and $G(\tau^*_T)$ is a finite bipartite graph.

(c) $\text{Ann}(\tilde{M})$ is a nil ideal of $R$ and $G(\tau^*_T)$ is a regular graph of finite degree.

Proof. (a) First we may assume that $G(\tau^*_T)$ is not empty. Then $R$ can not be a local ring. Otherwise, $T = V^*(mM)$, where $m$ is the unique maximal ideal of $R$. Therefore [5, Proposition 2.3] implies that $mM = M$ and hence $T$ is empty, a contradiction. Hence by [8, Theorem 8.9], $R = R_1 \oplus \ldots \oplus R_n$, where $R_i$ is an Artinian local ring for $i = 1, \ldots, n$ and $n \geq 2$. By Lemma 2.2 and Proposition 2.3, since $G(\tau^*_T)$ is a bipartite graph, we have $n = 2$ and hence $\tilde{M} \cong M_1 \oplus M_2$ for some idempotent $e \in R$. If $M_1$ is a prime module, then it is easy to see that $M_1$ is a vector space over $R/\text{Ann}(M_1)$ and so is a semisimple $R$-module. A similar argument as we did in proof of Corollary 3.4 implies that $|T| = 2$ and $G(\tau^*_T) \cong K_2$.

(b) By Theorem 2.9, $\tilde{M}$ is an Artinian and Noetherian module so that $R/\text{Ann}(\tilde{M})$ is an Artinian ring. A similar arguments in part (a) says that, $R/\text{Ann}(\tilde{M})$ is a non-local ring. So by [8, Theorem 8.9] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo $\text{Ann}(\tilde{M})$. By lemma 2.4, $\tilde{M} \cong M_1 \oplus M_2$, for some idempotent $e$ of $R$. Now, the proof that $G(\tau^*_T) \cong K_2$ is similar to the proof of Corollary 3.4.

(c) We may assume that $G(\tau^*_T)$ is not empty. So $\tilde{M}$ is not a prime module by [5, Proposition 2.3] and a similar manner in proof of Theorem 2.9, shows that $\tilde{M}$ has a finite length so that $R/\text{Ann}(\tilde{M})$ is an Artinian ring. As in the proof of part (b), $\tilde{M} \cong M_1 \oplus M_2$ for some idempotent $e \in R$. If $M_1$ has one non-trivial submodule $N$, then $\text{deg}(Q_1 \oplus M_2) > \text{deg}(N \oplus M_2)$ (we note that by [7, Proposition 2.5], $NK = (0)$ for some $(0) \neq K < M_1$) and this contradicts the regularity of $G(\tau^*_T)$. Hence $M_1$ is a simple module. Finally a similar argument as we have seen in Corollary 3.4 gives $G(\tau^*_T) \cong K_2$. \qed

Theorem 3.7. Assume that $\tilde{M}$ is a multiplication module and $|\text{Min}(\tilde{M})| \geq 3$. Then $G(\tau^*_T)$ contains a cycle.

Proof. If $G(\tau^*_T)$ is a tree, then by Theorem 3.5, $G(\tau^*_T)$ is a star graph. Suppose that $G(\tau^*_T)$ is a star graph. Then by Corollary 3.4, $\tilde{M} \cong M_1 \oplus M_2$, where $M_1$ is a simple module and $\tilde{M}$ is a prime module and hence by Proposition 2.3 (e), $\text{Min}(\tilde{M}) = \{0 \oplus M_2, M_1 \oplus 0\}$, that is $|\text{Min}(\tilde{M})| = 2$, a contradiction. Therefore $G(\tau^*_T)$ contains a cycle. \qed

4. Coloring of the quasi-Zariski topology-graph of modules

The purpose of this section is to study coloring of the quasi-Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau^*_T))$ and $\omega(G(\tau^*_T))$. We note that since $E(G(\tau^*_T)) \geq 1$ when $G(\tau^*_T) \neq \emptyset$, then $\chi(G(\tau^*_T)) \geq 2$.

Theorem 4.1. Let $\tilde{M}$ be an Artinian module such that for every minimal submodule $N$ of $\tilde{M}$, $N$ is a vertex in $G(\tau^*_T)$. Then $\omega(G(\tau^*_T)) = \chi(G(\tau^*_T))$. 

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Proof.  $\bar{M}$ is Artinian, so it contains a minimal submodule. Clearly, for every minimal submodule $\bar{N}$ of $\bar{M}$, $V^*(N) \neq \emptyset$. Also, $N \cap L = Q$, where $N$ and $L$ are minimal submodules of $\bar{M}$. It follows that $N$ and $L$ are adjacent in $G(\tau_+)$, where $\bar{N}$ and $\bar{L}$ are minimal submodules of $\bar{M}$. First, suppose that $M$ has infinitely many minimal submodules. Then $\omega(G(\tau_+)) = \infty$ and there is nothing to prove. Next, assume that $\bar{M}$ has $k$ minimal submodules, where $k$ is finite. We conclude that $\chi(G(\tau_+)) = k = \omega(G(\tau_+))$. Obviously, $\omega(G(\tau_+)) \geq k$. If possible, assume that $\omega(G(\tau_+)) > k$. Let $\Sigma = \{N_i \mid i \in \omega\}$, where $|\Sigma| = \omega(G(\tau_+))$ be a maximum clique in $G(\tau_+)$. As every $N_i \in \omega$, $\sqrt{N_i}$ contains a minimal submodule, there exists a minimal submodule $\bar{K}$ and submodules $N_i$ and $N_j$ in $\omega$, such that $\bar{K} \subset \sqrt{N_i} \cap \sqrt{N_j}$, and hence $V^*(K) = T$, a contradiction. Hence $\omega(G(\tau_+)) = k$. Next, we claim that $G(\tau_+)$ is $k$-colorable. In order to prove, put $A = \{K_1, \ldots, K_k\}$ be the set of all minimal submodules of $\bar{M}$. Now, we define a coloring $f$ on $G(\tau_+)$ by setting $f(N) = \min\{|i| K_i \subseteq \sqrt{N}\}$ for every vertex $N$ of $G(\tau_+)$.

Theorem 4.2. Assume that $\bar{M}$ is a faithful multiplication module. Then the following statements are equivalent.

(a) $\chi(G(\tau_{Spec(M)}^*)) = 2$.

(b) $G(\tau_{Spec(M)}^*)$ is a bipartite graph.

(c) $G(\tau_{Spec(M)}^*)$ is a complete bipartite graph.

(d) Either $R$ is a reduced ring with exactly two minimal prime ideals or $G(\tau_{Spec(M)}^*)$ is a star graph with more than one vertex.

Proof. By using Lemma 2.7, $G(\tau_{Spec(M)}^*)$ and $AG(M)$ are the same and so [6, Theorem 3.2] completes the proof. ☐

Lemma 4.3. Assume that $T$ is a finite closed subset of $Spec(M)$. Then $\chi(G(\tau_+))$ is finite. In particular, $\omega(G(\tau_+))$ is finite.

Proof. Suppose that $T = \{P_1, P_2, \ldots, P_n\}$ is a finite set of distinct prime submodules of $M$. Define a coloring $f(N) = \min\{|n \in N| P_n \notin V^*(N)\}$, where $N$ is a vertex of $G(\tau_+)$. We can see that $\chi(G(\tau_+)) \leq k$. ☐

Corollary 4.4. Assume that $e \in R$ is an idempotent element and $M$ is a multiplication module. Then $G(\tau_{eM})$ is a complete bipartite graph if and only if $M_1$ and $M_2$ are prime modules.

Proof. Assume that $G(\tau_{eM})$ is a complete bipartite graph. Therefore $G(\tau_{eM})$ is a triangle-free graph. So Lemma 3.3 follows that $M_1$ and $M_2$ are prime modules. The conversely holds by Proposition 3.1 (b). ☐

Remark 4.5. Assume that $S$ is a multiplicatively closed subset of $R$ such that $S \cap (\cup_{P \in T}(P : M)) = \emptyset$. Let $T_S = \{S^{-1}P : P \in T\}$. One can see that $V^*(N) = T$ if and only if $V^*(S^{-1}N) = T_S$, where $M$ is a finitely generated module.

Theorem 4.6. Let $S$ be a multiplicatively closed subset of $R$ defined in Remark 4.5 and $M$ is a finitely generated module. Then $G(\tau_{eM})$ is a retract of $G(\tau_S)$ and $\omega(G(\tau_{eM})) = \omega(G(\tau_S))$.

Proof. Consider a vertex map $\phi : V(G(\tau_{S})) \rightarrow V(G(\tau_{eM})); N \rightarrow N_S$. Clearly, $N_S \neq K_S$ implies that $N \neq K$ and $V^*(N) \cup V^*(K) = T$ if and only if $V^*(N_S) \cup V^*(K_S) = T_S$. Thus $\phi$ is surjective and hence $\omega(G(\tau_{eM})) \leq \omega(G(\tau_S))$. If $N \neq K$ and $V^*(N) \cup V^*(K) = T$, then we show that $N_S \neq K_S$. On the contrary suppose that $N_S = K_S$. Then $V^*(N_S) = V^*(\sqrt{N_S}) = V^*(\sqrt{N_S} \cap \sqrt{K_S}) = V^*(N_S) \cup V^*(K_S) = T_S$ and so $V^*(N) = T$, a contradiction. This shows that the map $\phi$ is a graph homomorphism. Now, for any vertex $N_S$ of $G(\tau_{eM})$, we can choose a fixed vertex $N$ of $G(\tau_S)$. Then $\phi$ is a retract (graph) homomorphism which clearly implies that $\omega(G(\tau_{eM})) = \omega(G(\tau_S))$ under the assumption. ☐
Corollary 4.7. Let $S$ be a multiplicatively closed subset of $R$ defined in Remark 4.5 and let $M$ be a finitely generated module. Then $\chi(AG(M_S)) = \chi(AG(M))$.

Corollary 4.8. Assume that $M$ is a semiprime module and $AG(M)\tau$ does not have an infinite clique. Then $M$ is a faithful module and $0 = (P_1 \cap \ldots \cap P_k : M)$, where $P_i$ is a prime submodule of $M$ for $i = 1, \ldots, k$.

Proof. By [6, Theorem 3.7 (b)], $M$ is a faithful module and the last assertion follows directly from the proof of [6, Theorem 3.7 (b)]. □

Proposition 4.9. Let $M$ be a cyclic module and let $T$ be a closed subset of $\text{Spec}(M)$. We have the following statements.

(a) If $\{P_1, \ldots, P_k\} \subseteq \text{Min}(T)$, then there exists a clique of size $n$ in $G(\tau^*_T)$.

(b) We have $\omega(G(\tau^*_T)) \geq |\text{Min}(T)|$ and if $|\text{Min}(T)| \geq 3$, then $gr(G(\tau^*_T)) = 3$.

(c) If $\sqrt{(0)} = (0)$, then $\chi(G(\tau^*_\text{Spec}(M))) = \omega(G(\tau^*_\text{Spec}(M))) = |\text{Min}(T)|$.

Proof. (a) The proof is straightforward by the facts that $AG(M) = AG(M)\tau$ has a clique of size $n$ by [7, Theorem 2.18] and $AG(M)$ is isomorphic with a subgraph of $G(\tau^*_T)$ by Lemma 2.6.

(b) This is clear by item (a).

(c) If $|\text{Min}(T)| = \infty$, then by Proposition 4.9 (b), there is nothing to prove. Otherwise, [7, Theorem 2.20] implies that $AG(M)$ does not have an infinite clique. So $M$ is a faithful module by Corollary 4.8. Next, Lemma 2.7 says that $G(\tau^*_\text{Spec}(M))$ and $AG(M)$ are the same. Now the result follows by [7, Theorem 2.20]. □

Lemma 4.10. Assume that $M$ is a semiprime multiplication module. Then the following statements are equivalent.

(a) $\chi(G(\tau^*_\text{Spec}(M)))$ is finite.

(b) $\omega(G(\tau^*_\text{Spec}(M)))$ is finite.

(c) $G(\tau^*_\text{Spec}(M))$ does not have an infinite clique.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (d) Suppose that $G(\tau^*_\text{Spec}(M))$ does not have an infinite clique. By Lemma 2.6, $AG(M)$ does not have an infinite clique and so by Corollary 4.8, there exists a finite number of prime submodules $P_1, \ldots, P_k$ of $M$ such that $\cap_{i \in T} P = P_1 \cap \ldots \cap P_k$. Define a coloring $f(N) = \min \{n \in \mathbb{N} | P_n \notin V^*(N)\}$, where $N$ is a vertex of $G(\tau^*_T)$. Then we have $\chi(G(\tau^*_\text{Spec}(M))) \leq k$. □

Corollary 4.11. Assume that $M$ is a multiplication module and $AG(M)$ does not have an infinite clique. Then $G(\tau^*_\text{Spec}(M))$ and $AG(M)\tau$ are the same. Also, $\chi(G(\tau^*_\text{Spec}(M)))$ is finite.

Proof. Since $M$ is a semiprime module, by Corollary 4.8, $M$ is a faithful module and there exists a finite number of prime submodules $P_1, \ldots, P_k$ of $M$ such that $\cap_{i \in T} P = P_1 \cap \ldots \cap P_k$. So the result follows by Lemma 2.7 and from the proof of (c) $\Rightarrow$ (d) of Lemma 4.10. □

Proposition 4.12. Suppose that $\sqrt{(0)} = (0)$ and $M$ is a multiplication module. Then the following statements are equivalent.

(a) $\chi(G(\tau^*_\text{Spec}(M)))$ is finite.

(b) $\omega(G(\tau^*_\text{Spec}(M)))$ is finite.

(c) $G(\tau^*_\text{Spec}(M))$ does not have an infinite clique.

(d) $\text{Min}(T)$ is a finite set.
Proof. (a) \(\implies\) (b) \(\implies\) (c) is clear.

(c) \(\implies\) (d) Suppose \(G(\tau_{\text{Spec}(M)})\) does not have an infinite clique. By Lemma 2.6, \(AG(M)\) does not have an infinite clique and hence by Corollary 4.8, there exists a finite number of prime submodules \(P_1, ..., P_k\) of \(M\) such that \(\bigcap_{P \in T} P = P_1 \cap P_2 \cap ... \cap P_k\). By assumptions, one can see that \(\text{Min}(T)\) is a finite set.

(d) \(\implies\) (a) Assume that \(\text{Min}(T)\) is a finite set (equivalently, \(M\) has a finite number of minimal prime submodules) so that \(\bigcap_{P \in T} P = P_1 \cap P_2 \cap ... \cap P_k\), where \(\text{Min}(T) = \{P_1, ..., P_k\}\). Define a coloring \(f(N) = \min\{n \in N | P_n \notin V^*(N)\}\), where \(N\) is a vertex of \(G(\tau_{\text{Spec}(M)})\). Then we have \(\chi(G(\tau_{\text{Spec}(M)})) \leq k\).

Proposition 4.13. Assume that \(\sqrt{(0)} = (0)\) and \(M\) is a faithful multiplication module. Then the following statements are equivalent.

(a) \(\chi(G(\tau_{\text{Spec}(M)}))\) is finite.

(b) \(\omega(G(\tau_{\text{Spec}(M)}))\) is finite.

(c) \(G(\tau_{\text{Spec}(M)})\) does not have an infinite clique.

(d) \(R\) has a finite number of minimal prime ideals.

(e) \(\chi(G(\tau_{\text{Spec}(M)})) = \omega(G(\tau_{\text{Spec}(M)})) = |\text{Min}(R)| = k\), where \(k\) is finite.

Proof. This is clear by Lemma 2.7, [6, Proposition 3.11], and [6, Corollary 3.12].

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