On a Conjecture of the Harmonic Index and the Minimum Degree of Graphs

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Abstract. The harmonic index of a graph \(G\) is defined as the sum of the weights \(\frac{2}{d(u)+d(v)}\) of all edges \(uv\) of \(G\), where \(d(v)\) denotes the degree of the vertex \(v\) in \(G\). Cheng and Wang [4] proposed a conjecture: For all connected graphs \(G\) with \(n \geq 4\) vertices and minimum degree \(\delta(G) \geq k\), where \(1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1\), then \(H(G) \geq H(K_{k,n-k})\) with equality if and only if \(G = K_{k,n-k}\). \(K_{k,n-k}\) is a complete split graph which has only two degrees, i.e. degree \(k\) and degree \(n-1\), and the number of vertices of degree \(k\) is \(n-k\), while the number of vertices of degree \(n-1\) is \(k\). In this work, we prove that this conjecture is true when \(k \leq \lfloor \frac{n}{2} \rfloor\), and give a counterexample to show that the conjecture is not correct when \(k = \lfloor \frac{n}{2} \rfloor + 1\), \(n\) is even, that is \(k = \frac{n+1}{2} + 1\).

1. Introduction

Throughout this paper we consider only simple connected graphs. Such a graph will be denoted by \(G = (V(G), E(G))\), where \(V(G)\) and \(E(G)\) are the vertex set and edge set of \(G\), respectively. For a vertex \(v\) of a graph \(G\), we denote the degree of \(v\) by \(d_G(v)\) (\(d(v)\) for short). The minimum degree of \(G\) is denoted by \(\delta(G)\). We use \(G - uv\) to denote the graph that arises from \(G\) by deleting the edge \(uv \in E(G)\). Let \(G(n, k)\) be the set of connected simple graphs of order \(n\) with minimum degree \(k\). We use \(K_{k,n-k}\) to denote the graph that arises from the complete bipartite graph \(K_{k,n-k}\) by joining every pair of vertices in the partite set with \(k\) vertices by a new edge. Our other notations are standard and taken mainly from [2].

The Randić index \(R(G)\), proposed by Randić [11] in 1975, is defined as the sum of the weights \(\frac{1}{\sqrt{d(u)d(v)}}\) over all edges \(uv\) of \(G\), that is,

\[
R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.
\]

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationship studies. Mathematical properties of this descriptor have been studied extensively (see [7-9] and the references cited therein).

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Another variant of the Randić index is called the harmonic index. For a graph $G$, the harmonic index $H(G)$ is defined in [6] as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

Zhong [13] determined the minimum and maximum values of the harmonic index for simple connected graphs and trees. Wu et al. [12] determined the graph with minimum harmonic index among all the graphs (or all triangle-free graphs) with minimum degree at least two. Chang and Zhu [3] gave the minimum value of the harmonic index for the graphs with minimum degree at least two and obtained a lower bound on the harmonic index of a triangle-free graph with arbitrary minimum degree. Cheng and Wang [4] determined the graph which attains the minimum harmonic index of connected graphs with minimum degree at least three, and posed the following Conjecture 1.1.

**Conjecture 1.1.** [4] Let $G$ be a graph with $n \geq 4$ vertices and $\delta(G) \geq k$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G \cong K^*_{k,n-k}$.

Ali [1] gave some counterexamples to show that the Conjecture 1.1 is not always true. In this work, using the methods in [5,10], we prove that Conjecture 1.1 is true when $k \leq \frac{n}{2}$, and give a counterexample to show that the conjecture is not correct when $k = \frac{n}{2} + 1$ (n is even).

For an edge $e = uv$ of a graph $G$ with harmonic index $H(G)$, its weight is defined to be $\frac{2}{d(u) + d(v)}$. The harmonic index of $G$ is the sum of weights over all its edges.

**Lemma 1.2.** [12] If $e$ is an edge with maximal weight in $G$, then $H(G - e) < H(G)$.

Let $G$ be a graph with $n \geq 4$ vertices and $\delta(G) \geq k$. If $\delta(G) > k$, then by Lemma 1.2, the deletion of an edge with maximal weight yields a graph $G'$ of minimal degree at least $k$ such that $H(G') < H(G)$. So we only need prove the Conjecture 1.1 is true for $G$ with $\delta(G) = k$. We can modify the Conjecture 1.1 to obtain the following Conjecture 1.3.

**Conjecture 1.3.** Let $G \in G(n,k)$, where $n \geq 4$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G \cong K^*_{k,n-k}$.

### 2. Preliminaries

First, we will give some linear equalities and nonlinear inequalities which must be satisfied in any graph from the class $G(n,k)$. Denote by $n_i$ the number of vertices of degree $i$, and by $x_{i,j}$ the number of edges joining the vertices of degrees $i$ and $j$ in $G(n,k)$. The mathematical description of the problem $P$ to determining minimum of $H(G) = \sum_{k \leq i \leq j \leq n-1} \frac{2x_{i,j}}{i+j}$ is:

$$\min \sum_{k \leq i \leq j \leq n-1} \frac{2x_{i,j}}{i+j}$$

subject to:

$$2x_{k,k} + x_{k,k+1} + x_{k,k+2} + \cdots + x_{k,n-1} = kn_k,$n_k + n_{k+1} + n_{k+2} + \cdots + n_{n-1} = (k + 1)n_{k+1},$$

$$x_{k,n-1} + x_{k+1,n-1} + x_{k+2,n-1} + \cdots + 2x_{n-1,n-1} = (n - 1)n_{n-1},$$

$$x_{i,j} \leq n_i n_j, \text{ for } k \leq i \leq n - 1, \text{ i < j \leq n - 1},$$

$$x_{i,j} \leq \left\lfloor \frac{n_i}{2} \right\rfloor, \text{ for } k \leq i \leq n - 1,$$

$$x_{i,j}, n_i \text{ are nonnegative integers, for } k \leq i \leq j \leq n - 1.$$
Constraints (1) – (5) define a nonlinearly optimization problem.

If the first equality from (1) divide by $k$, second by $k + 1$, third by $k + 2$ and so on, the last by $n - 1$ and sum them all, we get

$$
\sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j} = n_k + n_{k+1} + n_{k+2} + \cdots + n_{n-1} = n
$$

because of (2). Then

$$
H(G) = \frac{1}{2} \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j} = \frac{1}{2} \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j} - \frac{1}{2} \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j} = \frac{n}{2} - \frac{1}{2} \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j}
$$

Define

$$
\gamma = \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j}.
$$

Henceforth we will consider the problem of maximizing $\gamma$ instead of minimizing $H(G)$.

3. Main result

**Theorem 3.1.** Let $G \in G(n,k)$, where $n \geq 4$, $1 \leq k \leq \frac{n}{2}$. Then $H(G) \geq H(K^*_{n,n-k})$ with equality if and only if $G \cong K^*_{n,n-k}$.

**Proof.** Since the minimum degree of $G$ is $k$, it is obvious that $n_{n-1} \leq k$. Let $m$ be the index such that $n_m + n_{m+1} + \cdots + n_{n-2} + n_{n-1} \geq k$ and $n_{m+1} + \cdots + n_{n-2} + n_{n-1} < k$. We distinguish two cases: (1) $n_m + \cdots + n_{n-2} + n_{n-1} = k$, (2) $n_m + \cdots + n_{n-2} + n_{n-1} > k$.

**Case 1.** $n_m + \cdots + n_{n-2} + n_{n-1} = k$.

Note that

$$
\gamma = \sum_{k \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j} = \sum_{j=k+1}^{n-1} \left( \frac{1}{k} + \frac{1}{k+j} - \frac{4}{k+j} \right) x_{k,j} + \sum_{j=k+2}^{n-1} \left( \frac{1}{k+1} + \frac{1}{k+1+j} - \frac{4}{k+1+j} \right) x_{k+1,j} + \cdots
$$

$$
+ \sum_{m=1}^{n-1} \left( \frac{1}{m-1} + \frac{1}{m-1+j} - \frac{4}{m-1+j} \right) x_{m-1,j} + \sum_{m \geq k} \left( \frac{1}{m} + \frac{1}{m+j} - \frac{4}{m+j} \right) x_{i,j}.
$$

Weights of all edges which join vertices of degree $i$, with vertices of degree $j$, $i+1 \leq j \leq n-1$ are represented by $\sum_{j=i+1}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j}$. We give the maximum possible weights to these edges. Since $n_m + \cdots + n_{n-2} + n_{n-1} = k$ and $\sum_{j=i} x_{i,j} \leq n_i$, first we join a vertex of degree $i$ to all $k$ vertices of degrees $n-i, \cdots, m$ (maximum weights) and with $i-k$ vertices of other degrees $j, i+1 \leq j \leq m-1$. Furthermore, $h(x,y) = \frac{1}{x} + \frac{1}{y} - \frac{4}{x+y}$ is increasing for $y$, where $k \leq x < y \leq m-1$ (since $\frac{\partial h(x,y)}{\partial y} = \frac{(y-x)(4y+3)}{y(x+y)^2} > 0$ for $k \leq x < y \leq m-1$). We will
maximize the weights of these last \( i - k \) edges joining a vertex of degree \( i \) to \( i - k \) vertices of degree \( m - 1 \). Thus,

\[
\sum_{j=m+1}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j} \\
= \sum_{j=m}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j} + \sum_{j=m+1}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j} \\
\leq \sum_{j=m}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) n_i n_j + \left( \frac{1}{i} + \frac{1}{m-1} - \frac{4}{i+m-1} \right) (i-k)n_i \\
= n_i \sum_{j=m}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) n_j + \left( \frac{1}{i} + \frac{1}{m-1} - \frac{4}{i+m-1} \right) (i-k).
\]

Then

\[
y \leq n_k \left( \frac{1}{k} + \frac{1}{n-1} - \frac{4}{k+n-1} \right) n_{n-1} + \left( \frac{1}{k} + \frac{1}{n-2} - \frac{4}{k+n-2} \right) n_{n-2} + \cdots + \left( \frac{1}{k} + \frac{1}{m-1} - \frac{4}{k+m-1} \right) n_m \\
+ n_{k+1} \left( \frac{1}{k+1} + \frac{1}{n-1} - \frac{4}{k+n-1} \right) n_{n-1} + \left( \frac{1}{k+1} + \frac{1}{n-2} - \frac{4}{k+n-1} \right) n_{n-2} + \cdots \\
+ \left( \frac{1}{k+1} + \frac{1}{m-1} - \frac{4}{k+m-1} \right) n_m + \left( \frac{1}{k+1} + \frac{1}{m-1} - \frac{4}{k+m-1} \right) n_m (m-1-k) \\
+ \sum_{m \leq i \leq j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) x_{i,j} \\
= y_1 + y_2
\]

where \( f(i) = \sum_{j=i}^{n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) n_j + \left( \frac{1}{i} + \frac{1}{m-1} - \frac{4}{i+m-1} \right) (i-k) \). Let \( g(x) = x \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right) \). Note that \( g'(x) = \frac{(x-y)(x+3y)}{y(x+y)^2} < 0 \) for \( 0 < x < y \), then \( g(x) \) is monotonous decreasing for \( 0 < x < y \). Thus we have for \( k+1 \leq i \leq m-1, m \leq j \leq n-1 \):

\[
k \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k+j} \right) \geq \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j} \right).
\]
Therefore
\[
f(i) \leq k \left( \sum_{j=m}^{n-1} \left( \frac{1}{k} \cdot \frac{1}{j} - \frac{4}{k + j} \right) n_j \right) + \left( \frac{1}{k} + \frac{1}{m - 1} - \frac{4}{k + m - 1} \right) (i - k)
\]
\[
= \left( \frac{1}{i} - \frac{i - k}{i} \right) \left( \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j \right) + \frac{k(i - k)}{i} \left( \frac{1}{k} + \frac{1}{m - 1} - \frac{4}{k + m - 1} \right)
\]
\[
= \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j + \frac{i - k}{i} \left( \frac{1}{k} + \frac{1}{m - 1} - \frac{4}{k + m - 1} \right) \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j
\]
\[
\leq \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j,
\]
because \( \sum_{j=m}^{n-1} n_j = k \) and \( m \leq j \leq n - 1 \). Since \( n_k + \cdots + n_{m-1} = n - k \), we have
\[
y_1 = \sum_{k \leq i \leq n-1} f(i)n_i \leq \left( \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j \right) \sum_{i=k}^{m-1} n_i
\]
\[
= (n - k) \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) n_j = \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{n - 1} - \frac{4}{k + n - 1} \right) (n - k)n_j
\]
\[
- \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) (n - k)n_j + \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{1}{j} - \frac{4}{k + j} \right) (n - k)n_j
\]
\[
= \left( \frac{1}{k} + \frac{1}{n - 1} - \frac{4}{k + n - 1} \right) (n - k) - \sum_{j=m}^{n-1} \left( \frac{1}{k} + \frac{4}{k + j} - \frac{4}{k + n - 1} - \frac{1}{j} \right) (n - k)n_j.
\]
Since \( x_{i,j} \leq n_i n_j, m \leq j \leq n - 1, \) and \( n_{m-1} = k - \sum_{j=m}^{n-2} n_j \), we have
\[
y_2 = \sum_{m \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i + j} \right) x_{i,j} \leq \sum_{m \leq i < j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i + j} \right) n_i n_j
\]
\[
= \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i + \left( n - 1 \right) \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i n_j
\]
\[
= k \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i - \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i^2
\]
\[
+ \sum_{m \leq i < j \leq n-2} \left( \frac{1}{i} + \frac{1}{j} - \frac{4}{i + j} \right) - \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) + \left( \frac{1}{j} + \frac{1}{n - 1} - \frac{4}{j + n - 1} \right)
\]
\[
= k \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i - \sum_{i=m}^{n-2} \left( \frac{1}{i} + \frac{1}{n - 1} - \frac{4}{i + n - 1} \right) n_i^2
\]
\[
- 2 \sum_{m \leq i < j \leq n-2} \left( \frac{2}{i + j} + \frac{1}{n - 1} - \frac{2}{i + n - 1} - \frac{2}{j + n - 1} \right) n_i n_j.
\]
Thus,

\[ y \leq y_1 + y_2 \leq \left(1 + \frac{1}{k} \right) + \left(1 - \frac{4}{k + n - 1}\right)(n - k)k - \sum_{i=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + k + n - 1} - \frac{1}{i}\right)(n - k)\gamma_i \]

\begin{align*}
+ k & \sum_{j=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + n - 1}\right)\gamma_i - \sum_{j=1}^{n-2} \left(\frac{1}{j - 1} + \frac{4}{j + n - 1}\right)\gamma_j^2 \\
- 2 & \sum_{m \leq k \leq j < n-2} \left(\frac{2}{i + j} + \frac{1}{i + n - 1} - \frac{2}{j + n - 1}\right)\gamma_i \gamma_j \\
+ \sum_{j=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + n - 1}\right)k - \left(\frac{1}{i - 1} + \frac{4}{k + n - 1} - \frac{1}{i}\right)(n - k)\gamma_i \\
\leq & \left(1 + \frac{1}{k} \right) + \left(1 - \frac{4}{k + n - 1}\right)(n - k)k - \sum_{i=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + n - 1}\right)\gamma_i^2 \\
- 2 & \sum_{m \leq k \leq j < n-2} \left(\frac{2}{i + j} + \frac{1}{i + n - 1} - \frac{2}{j + n - 1}\right)\gamma_i \gamma_j \\
+ (n - k) & \sum_{i=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + n - 1}\right)\gamma_i - \left(\frac{1}{i - 1} + \frac{4}{k + n - 1} - \frac{1}{i}\right)(n - k)\gamma_i \\
= & \left(1 + \frac{1}{k} \right) + \left(1 - \frac{4}{k + n - 1}\right)(n - k)k - \left(\sum_{i=1}^{n-2} \left(\frac{1}{i - 1} + \frac{4}{i + n - 1}\right)\gamma_i^2 \\
+ 2 & \sum_{m \leq k \leq j < n-2} \frac{(n - j - 1)(2(n - 1)(n + j - 1) - (i + j)(n + i - 1))}{(i + j)(n - 1)(n + i - 1)(n + j - 1)}\gamma_i \gamma_j \\
- 2(n - k) & \sum_{i=1}^{n-2} \frac{(i - k)(i + n - 1)(k + n - 1) - (i + j)(k + i)}{i(k + i)(i + n - 1)(k + n - 1)}\gamma_i \\
\leq & \left(1 + \frac{1}{k} \right) + \left(1 - \frac{4}{k + n - 1}\right)(n - k)k.
\end{align*}

The second inequality holds because \( k \leq n - k \). So \( H(G) = \frac{n}{2} - \frac{y}{2} \geq \frac{n}{2} - \frac{1}{2} k - \frac{4}{k + n - 1} \gamma_k k = \frac{k(k - 1)}{2(k - 2)} + \frac{2k(k - 1)}{k + n - 1} = H(K_{k+2,n-2,k}). \) Equality holds when \( \gamma_i = 0 \) for \( k + 1 \leq i \leq n - 2 \), \( n_k = n - k \), \( n_{k-1} = k \), \( x_{k+2,n} = (n - k)k \), \( x_{k+1,n-1} = k \), and all other \( x_{i,j} \) are equal to zero. Thus, graphs for which the harmonic index attains its minimum value are \( K_{k+2,n-2,k}. \)

**Case 2.** \( n_m + \cdots + n_{n-2} + n_{n-1} > k. \)

We put \( n_m = n_{m+1} + \cdots + n_{n-1} = k \). Then \( n_k + \cdots + n_{m+1} + n_{m} = n - k \). We will color the vertices of degree \( m \) with red and white, such that the number of red vertices is \( n_{m+1} \). Denote by \( x_{i,m} \) (resp. \( x_{i,m} \)) for \( i \neq m \), the number of edges between vertices of degree \( i \) and the white (resp. red) vertices of degree \( m \), by \( x_{m,m} \) (resp. \( x_{m,m} \)) the number of edges between white (resp. red) vertices of degree \( m \), and by \( x_{m,m} \) the number of edges between white and red vertices of degree \( m \). Then \( x_{i,m} = x_{i,m} + x_{i,m} \) for
When \( i \neq m \), and \( x_{m,m} = x_{m',m'} + x_{m',m''} + x_{m'',m''} \). We will replace system (1) by:

\[
\begin{align*}
  x_{i,i} + \cdots + x_{i,m-1} + x_{i,m} + x_{i,m'} + x_{i,m+1} + \cdots + x_{i,n-1} &= m_{i,i}, \quad k \leq i \leq n-1, i \neq m, \\
  x_{k,k'} + \cdots + x_{k-1,m'} + 2x_{k,m'} + x_{k,m'',m'} + x_{k',m+1} + \cdots + x_{k',n-1} &= mn_{k,k'}, \\
  x_{k,m'} + \cdots + x_{m-1,m' + m''} + x_{m',m''} + 2x_{m'',m''} + x_{m''',m''} + x_{m''',m''} + \cdots + x_{m''',n-1} &= mm_{m''}, \\
  x_{m',m'} + \cdots + x_{m-1,m''} + x_{m',m'} + 2x_{m',m' + m''} + x_{m''',m''} + x_{m''',m''} + \cdots + x_{m''',n-1} &= mm_{m''}.
\end{align*}
\]

We will proceed similarly as in the Case 1. The rest of the proof is omitted, because it is similar to the one of Case 1. □

**Remark** When \( k = \frac{n}{2} + 1 \), \( n \equiv 0 \pmod{4} \) and \( n \geq 8 \), we give a counterexample to show that the Conjecture 1.1 is not correct. Let \( G' \) be an \( n \)-vertex graph obtained from \( K_n \) by deleting the edges of an \( (\frac{n}{2} - 2) \)-regular graph on \( \frac{n}{2} \) vertices. Then \( G' \in G(\frac{n}{2} + 1) \). By some elementary calculations, we have \( H(K_{\frac{n}{2} + 1, \frac{n}{2} - 1}) = \frac{n^2 + 2n}{8(n-1)} + \frac{n^2 - 4}{8n} \), \( H(G') = \frac{n^2 - 2n}{8(n-1)} + \frac{n}{2(n+2)} + \frac{n}{4} \). Thus, \( H(K_{\frac{n}{2} + 1, \frac{n}{2} - 1}) - H(G') = \frac{(n-4)^2}{6n(n-1)(n+2)} > 0 \) for \( n \geq 8 \). This implies \( H(K_{\frac{n}{2} + 1, \frac{n}{2} - 1}) \neq H(G') \).

When \( k = \frac{n+1}{2} + 1 \), \( n \) is odd, that is \( k = \frac{n+1}{2} \), the graph \( G_1 \) in [1] is a counterexample which can show that the Conjecture 1.1 is also not correct.

**References**


