Abstract. The main object of this article is to introduce the concepts of $f$–lacunary statistical convergence of order $\alpha$ and strong $f$–lacunary summability of order $\alpha$ of sequences of real numbers and give some inclusion relations between these spaces.

1. Introduction

In 1951, Steinhaus [33] and Fast [18] introduced the concept of statistical convergence and later in 1959, Schoenberg [32] reintroduced independently. Bhardwaj and Dhawan [3], Caserta et al. [4], Connor [5], Çakallı [10], Çınar et al. [11], Çolak [12], Et et al. ([14], [16]), Fridy [20], Işık [24], Salat [31], Di Maio and Kočinac [13] and many authors investigated some arguments related to this notion.

A modulus $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

iii) $f$ is increasing,

iv) $f$ is continuous from the right at 0.

It follows that $f$ must be continuous in everywhere on $[0, \infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f$–density of a subset $E \subset \mathbb{N}$ for any unbounded modulus $f$ by

$$d^f(E) = \lim_{n \to \infty} \frac{f(|\{k \leq n : k \in E\}|)}{f(n)}, \text{if the limit exists}$$

and defined $f$–statistical convergence for any unbounded modulus $f$ by

$$d^f(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0$$

i.e.

$$\lim_{n \to \infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - \ell| \geq \varepsilon\}|) = 0,$$
and we write it as $S^f - \lim x_k = \ell$ or $x_k \rightarrow e(S^f)$. Every $f$–statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f$–statistically convergent for every unbounded modulus $f$.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \rightarrow \infty$ as $r \rightarrow \infty$.

In [21], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence $(x_k)$ of real numbers is called lacunary statistically convergent to a real number $\ell$, if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \|k \in I_r : |x_k - \ell| \geq \varepsilon\| = 0$$

for every positive real number $\varepsilon$.

Throughout this paper the intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_r$. Lacunary sequence spaces were studied in [6], [7], [8], [9], [17], [19], [21], [23], [25], [29], [35], [36].

First of all, the notion of a modulus was given by Nakano [27]. Maddox [26] used a modulus function to construct some sequence spaces. Afterwards different sequence spaces defined by modulus have been studied by Altun and Et [2], Et et al. [15], Işık [24], Gaur and Mursaleen [22], Nuray and Savaş [28], Pehlivan and Fisher [30], Şengül [34] and everybody else.

### 2. Main Results

In this section we will introduce the concepts of $f$–lacunary statistically convergent sequences of order $\alpha$ and strongly $f$–lacunary summable sequences of order $\alpha$ of real numbers, where $f$ is an unbounded modulus and give some inclusion relations between these concepts.

**Definition 2.1.** Let $f$ be an unbounded modulus, $\theta = (k_r)$ be a lacunary sequence and $\alpha$ be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is $f$–lacunary statistically convergent of order $\alpha$, if there is a real number $\ell$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{f(h_r)^{\alpha}} \|k \in I_r : |x_k - \ell| \geq \varepsilon\| = 0,$$

where $I_r = (k_{r-1}, k_r]$ and $f(h_r)^{\alpha}$ denotes the $\alpha$th power of $f(h_r)$, that is $(f(h_r)^{\alpha}) = (f(h_1)^{\alpha}, f(h_2)^{\alpha}, ..., f(h_r)^{\alpha}, ...)$.

This space will be denoted by $S_{\theta, f}^{\alpha}$. In this case, we write $S_{\theta, f}^{\alpha} - \lim x_k = \ell$ or $x_k \rightarrow \ell(S_{\theta, f}^{\alpha})$.

**Definition 2.2.** Let $f$ be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers and $\alpha$ be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is strongly $w^{\alpha}([\theta, f, p])$–summable to $\ell$ (a real number) such that

$$w^{\alpha}([\theta, f, p]) = \left\{x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{p(h_r)^{\alpha}} \sum_{k \in I_r} [f(|x_k - \ell|)]^{\alpha} = 0, \text{ for some } \ell \right\}.$$

In the present case, we denote $w^{\alpha}([\theta, f, p]) - \lim x_k = \ell$.

**Definition 2.3.** Let $f$ be an unbounded modulus, $p = (p_k)$ be a sequence of strictly positive real numbers and $\alpha$ be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is strongly $w^{f,\alpha}_{\theta} (p)$–summable to $\ell$ (a real number) such that

$$w^{f,\alpha}_{\theta} (p) = \left\{x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f(|x_k - \ell|)]^{\alpha} = 0, \text{ for some } \ell \right\}.$$

In the present case, we write $w^{f,\alpha}_{\theta} (p) - \lim x_k = \ell$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $w^{f,\alpha}_{\theta} [p]$ instead of $w^{f,\alpha}_{\theta} (p)$.
Theorem 2.5. Let $f$ be an unbounded modulus. The classes of sequences $w_{0,f}^{f,\alpha}$ and $\delta_{0}^{f,\alpha}$ are linear spaces.

Theorem 2.6. The space $w_{0,f}^{f,\alpha}$ is paranormed by

$$g(x) = \sup_{r} \left\{ \frac{1}{f(h_{r})^{\alpha}} \sum_{k \in I} [f(|x_{k}|r)^{\alpha} \right\}^{\frac{1}{\alpha}}$$

where $0 < \alpha \leq 1$ and $M = \max(1, H)$.

Proposition 2.7. (30) Let $f$ be a modulus and $0 < \delta < 1$. Then for each $||u|| \geq \delta$, we have $f(||u||) \leq 2f(1)\delta^{-1}||u||$.

Theorem 2.8. Let $f$ be an unbounded modulus, $\alpha$ be a real number such that $0 < \alpha \leq 1$ and $p > 1$. If $\lim_{u \to 0} \inf \frac{f(u)}{u} > 0$, then $w_{0,f}^{f,\alpha} = w_{0,f}^{\alpha}$.

Proof. Let $p > 1$ be a positive real number and $x \in w_{0,f}^{f,\alpha}$. If $\lim_{u \to 0} \inf \frac{f(u)}{u} > 0$ then there exists a number $c > 0$ such that $f(u) > cu$ for $u > 0$. Clearly

$$\frac{1}{f(h_{r})^{\alpha}} \sum_{k \in I} [f(|x_{k} - \ell|r)]^{\alpha} \geq \frac{1}{f(h_{r})^{\alpha}} \sum_{k \in I} [c|x_{k} - \ell|]^{\alpha} = \frac{c^{\alpha}}{f(h_{r})^{\alpha}} \sum_{k \in I} |x_{k} - \ell|^{\alpha},$$

and therefore $w_{0,f}^{f,\alpha} \subset w_{0,f}^{\alpha}$.

Now let $x \in w_{0,f}^{\alpha}$. Then we have

$$\frac{1}{f(h_{r})^{\alpha}} \sum_{k \in I} |x_{k} - \ell|^{\alpha} \to 0 \text{ as } r \to \infty.$$

Let $0 < \delta < 1$. We can write

$$\frac{1}{f(h_{r})^{\alpha}} \sum_{k \in I} |x_{k} - \ell|^{\alpha} \geq \frac{1}{f(h_{r})^{\alpha}} \sum_{|x_{k} - \ell| \geq \delta} |x_{k} - \ell|^{\alpha} \geq \frac{1}{f(h_{r})^{\alpha}} \sum_{|x_{k} - \ell| \geq \delta} \left[ f(|x_{k} - \ell|) \right]^{\alpha} \geq \frac{1}{f(h_{r})^{\alpha}} \frac{\delta^{\alpha}}{2f(1)\delta^{-1}} \sum_{k \in I} [f(|x_{k} - \ell|)]^{\alpha}.$$
by Proposition 2.7. Therefore \( x \in \omega_{\theta}^{f,\alpha} [p] \).

If \( \lim_{u \to \infty} \inf \frac{f(u)}{u} = 0 \), the equality \( \omega_{\theta}^{f,\alpha} [p] = \omega_{\theta,f}^{\alpha} [p] \) can not be hold as shown the following example:

Let \( f(x) = 2 \sqrt{x} \) and define a sequence \( x = (x_k) \) by

\[
x_k = \begin{cases} \sqrt{h_r}, & \text{if } k = k_r, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( \ell = 0 \), \( \alpha = \frac{4}{5} \) and \( p = \frac{6}{5} \), we have

\[
\frac{1}{f(h_r)^2} \sum_{k \in \mathbb{I}} (f([|x_k|]))^p = \left( \frac{2h_r^2}{(2 \sqrt{h_r})^2} \right) \to 0 \text{ as } r \to \infty
\]

hence \( x \in \omega_{\theta}^{f,\alpha} [p] \), but

\[
\frac{1}{f(h_r)^2} \sum_{k \in \mathbb{I}} |x_k|^p = \left( \frac{\sqrt{h_r}}{2 \sqrt{h_r}} \right)^2 \to \infty \text{ as } r \to \infty
\]

and so \( x \notin \omega_{\theta,f}^{\alpha} [p] \). \( \square \)

Maddox [26] showed that the existence of an unbounded modulus \( f \) for which there is a positive constant \( c \) such that \( f(xy) \geq cf(x)f(y) \), for all \( x \geq 0, y \geq 0 \).

**Theorem 2.9.** Let \( f \) be an unbounded modulus, \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \). If \( \lim_{u \to \infty} \frac{f(u)}{u^\alpha} > 0 \), then \( w^* [\theta, f, p] \subset S_{\theta}^{f,\alpha} \).

**Proof.** Let \( x \in w^* [\theta, f, p] \) and \( \lim_{u \to \infty} \frac{f(u)^\alpha}{u^\alpha} > 0 \). For \( \epsilon > 0 \), we have

\[
\frac{1}{h_r^2} \sum_{k \in \mathbb{I}} f(|x_k - \ell|) \geq \frac{1}{h_r^2} f \left( \sum_{k \in \mathbb{I}} |x_k - \ell| \right) \geq \frac{1}{h_r^2} f \left( \sum_{k \in \mathbb{I}, |x_k - \ell| \geq \epsilon} |x_k - \ell| \right) \geq \frac{1}{h_r^2} f \left( \| \left\{ k \in \mathbb{I}_r : |x_k - \ell| \geq \epsilon \right\} \right) \geq \frac{c}{h_r^2} f \left( \| \left\{ k \in \mathbb{I}_r : |x_k - \ell| \geq \epsilon \right\} \right) f(\epsilon) = \frac{c}{h_r^2} \frac{f(\left\{ k \in \mathbb{I}_r : |x_k - \ell| \geq \epsilon \right\})}{f(h_r)^\alpha} f(h_r)^\alpha f(\epsilon).
\]

Therefore, \( w^* [\theta, f, p] \) implies \( S_{\theta}^{f,\alpha} \) implies \( w^* \) implies \( S_{\theta}^{f,\alpha} \).

**Theorem 2.10.** Let \( \alpha_1, \alpha_2 \) be two real numbers such that \( 0 < \alpha_1 \leq \alpha_2 \leq 1 \), \( f \) be an unbounded modulus function and let \( \theta = (k_r) \) be a lacunary sequence, then we have \( w_{\theta}^{f,\alpha_1}(p) \subset S_{\theta}^{f,\alpha_2} \).
Proof. Let \( x \in w^{f,\alpha}_0(p) \) and \( \varepsilon > 0 \) be given and \( \sum_1, \sum_2 \) denote the sums over \( k \in I_r, |x_k - \ell| \geq \varepsilon \) and \( k \in I_r, |x_k - \ell| < \varepsilon \) respectively. Since \( f(h_r) < f(h_r)^2 \) for each \( r \), we may write

\[
\frac{1}{f(h_r)^a} \sum_{k \in I_r} f(|x_k - \ell|) \leq \frac{1}{f(h_r)^a} \sum_{k \in I_r} f(|x_k - \ell|) + \sum_{k \in I_r} f(|x_k - \ell|) \leq \frac{1}{f(h_r)^a} \sum_{k \in I_r} f(|x_k - \ell|) + \sum_{k \in I_r} f(|x_k - \ell|)
\]

Hence \( x \in S^{f,\alpha}_0 \). \( \square \)

**Theorem 2.11.** Let \( \theta = (k_r) \) be a lacunary sequence and \( \alpha \) be a fixed real number such that \( 0 < \alpha \leq 1 \). If \( \liminf q_r > 1 \) and \( \lim_{m \to \infty} \frac{f(q_m)}{f(q_{m+1})} > 0 \), then \( S^{f,\alpha}_0 \subset S^{f,\alpha}_0 \).

**Proof.** Suppose first that \( \liminf q_r > 1 \); then there exists a \( \lambda > 0 \) such that \( q_r > 1 + \lambda \) for sufficiently large \( r \), which implies that

\[
\frac{h_r}{k_r} \geq \frac{\lambda}{1 + \lambda} \implies \left( \frac{h_r}{k_r} \right)^\alpha \geq \left( \frac{\lambda}{1 + \lambda} \right)^\alpha.
\]

If \( S^{f,\alpha}_0 - \lim x_k = \ell \), then for every \( \varepsilon > 0 \) and for sufficiently large \( r \), we have

\[
\frac{1}{f(h_r)^a} f \left( \left| k \leq k_r : |x_k - \ell| \geq \varepsilon \right| \right) \geq \frac{1}{f(h_r)^a} f \left( \left| k \in I_r : |x_k - \ell| \geq \varepsilon \right| \right)
\]

This proves the sufficiency. \( \square \)

**Theorem 2.12.** Let \( f \) be an unbounded modulus and \( 0 < \alpha \leq 1 \). If \( (x_k) \in S^{f,\alpha}_0 \), then \( S^{f,\alpha}_0 - \lim x_k = S^{f,\alpha}_0 - \lim x_k \).

**Proof.** Suppose \( S^{f,\alpha}_0 - \lim x_k = \ell_1, S^{f,\alpha}_0 - \lim x_k = \ell_2 \) and \( \ell_1 \neq \ell_2 \). Let \( 0 < \varepsilon < \frac{|\ell_1 - \ell_2|}{2} \). Then for \( \varepsilon > 0 \) we have

\[
\lim_{n \to \infty} \frac{f \left( \left| k \leq n : |x_k - \ell| \geq \varepsilon \right| \right)}{f(n)} = 0,
\]
and
\[ \lim_{r \to \infty} \frac{f([k \in I_r : |x_k - \ell_2| \geq \varepsilon])}{f(h_r)^a} = 0. \]

On the other hand we can write
\[ f \left( \frac{[k \leq n : |\ell_1 - \ell_2| \geq 2\varepsilon]}{f(n)} \right) \leq f \left( \frac{[k \leq n : |x_k - \ell_1| \geq \varepsilon]}{f(n)} \right) + f \left( \frac{[k \leq n : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right). \]

Taking limit as \( n \to \infty \), we get
\[ 1 \leq 0 + \lim_{n \to \infty} \frac{f([k \leq n : |x_k - \ell_2| \geq \varepsilon])}{f(n)} \leq 1, \]
and so
\[ \lim_{n \to \infty} \frac{f([k \leq n : |x_k - \ell_2| \geq \varepsilon])}{f(n)} = 1. \]

We consider the subsequence
\[ \frac{1}{f(h_m)} f \left( \frac{[k \leq k_m : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \]

Then
\[ \frac{1}{f(h_m)} f \left( \frac{[k \leq k_m : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) = \frac{1}{f(h_m)} f \left( \frac{[k \in \bigcup_{r=1}^m I_r : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \]
\[ = \frac{1}{f(h_m)} f \left( \sum_{r=1}^m [k \in I_r : |x_k - \ell_2| \geq \varepsilon] \right) \]
\[ \leq \frac{1}{f(h_m)} \sum_{r=1}^m f \left( \frac{[k \in I_r : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \]
\[ = \frac{1}{f(h_m)} \sum_{r=1}^m f(h_r)^a \frac{1}{f(h_m)} f \left( \frac{[k \in I_r : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \]

and
\[ \sum_{r=1}^m f(h_r)^a = f(h_1)^a + f(h_2)^a + \ldots + f(h_m)^a = f(k_1-k_0)^a + f(k_2-k_1)^a + \ldots + f(k_m-k_{m-1})^a = f(k_1-k_0)^a + f(k_2-k_1)^a + \ldots + f(k_m-k_{m-1})^a \]
\[ \leq f(k_1) - f(k_0) + f(k_2) - f(k_1) + \ldots + f(k_m) - f(k_{m-1}) = f(k_m). \]

Using (2) in (1), we have
\[ \frac{1}{f(h_m)} f \left( \frac{[k \leq k_m : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \leq \frac{\sum_{r=1}^m f(h_r)^a}{\sum_{r=1}^m f(h_r)^a} \frac{1}{f(h_m)} f \left( \frac{[k \in I_r : |x_k - \ell_2| \geq \varepsilon]}{f(n)} \right) \]
Proof. Let \( \lim f \) and \( \sup f \). Theorem 2.14. Let \( f \) be an unbounded modulus. If \( \|k \leq k_m : |x_k - \ell_2| \geq \varepsilon \| \) then \( \lim \theta = (k, r) \) and \( \theta' = (s, r) \) be two lacunary sequences and \( 0 < \alpha \leq 1 \). If \( (x_k) \in S' \cap \left( S^f_{\theta} \cap S^f_{\theta'} \right) \), then \( S_{\theta}^f - \lim x_k = S_{\theta'}^f - \lim x_k. \)

Corollary 2.13. Let \( \lim p_k > 0 \), then \( w_{\theta}^{f, \alpha} (p) - \lim x_k = \ell \) uniquely.

Proof. Let \( \lim p_k = s > 0 \). Assume that \( w_{\theta}^{f, \alpha} (p) - \lim x_k = \ell_1 \) and \( w_{\theta'}^{f, \alpha} (p) - \lim x_k = \ell_2 \). Then

\[
\lim r \frac{1}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|x_k - \ell_1|)]^{p_k} = 0,
\]

and

\[
\lim r \frac{1}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|x_k - \ell_2|)]^{p_k} = 0.
\]

By definition of \( f \), we have

\[
\frac{1}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|\ell_1 - \ell_2|)]^{p_k} \leq \frac{D}{f(h_r)^{\alpha}} \left( \sum_{k \in I_r} [f (|x_k - \ell_1|)]^{p_k} + \sum_{k \in I_r} [f (|x_k - \ell_2|)]^{p_k} \right) \]

\[
= \frac{D}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|x_k - \ell_1|)]^{p_k} + \frac{D}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|x_k - \ell_2|)]^{p_k}
\]

where \( \sup_k p_k = H \) and \( D = \max \{1, 2^{k-1} \} \). Hence

\[
\lim r \frac{1}{f(h_r)^{\alpha}} \sum_{k \in I_r} [f (|\ell_1 - \ell_2|)]^{p_k} = 0.
\]

Since \( \lim_{k \to \infty} p_k = s \) we have \( \ell_1 - \ell_2 = 0 \). Thus the limit is unique. \( \square \)

Theorem 2.15. Let \( \theta = (k, r) \) and \( \theta' = (s, r) \) be two lacunary sequences such that \( I_r \subset I_r \) for all \( r \in \mathbb{N} \) and \( \alpha_1, \alpha_2 \) two real numbers such that \( 0 < \alpha_1 \leq \alpha_2 \leq 1 \). If

\[
\lim \inf r \to \infty \frac{f(h_r)^{\alpha_1}}{f(h_r)^{\alpha_2}} > 0
\]

(3)

where \( I_r = (k_{r-1}, k_r) \), \( h_r = k_r - k_{r-1} \) and \( f_r = (s_{r-1}, s_r) \), \( \ell_r = s_r - s_{r-1} \), then \( w_{\theta}^{f, \alpha_1} (p) \subset w_{\theta'}^{f, \alpha_2} (p) \).
Proof. Let $x \in w^{f,\alpha_1}_{\theta} (p)$. We can write
\[
\frac{1}{f(\ell_r)^{\alpha_2}} \sum_{k \in \ell_r} [f(|x_k - \ell|)]^p \leq \frac{1}{f(\ell_r)^{\alpha_2}} \sum_{k \in \ell_r} [f(|x_k - \ell|)]^p + \frac{1}{f(\ell_r)^{\alpha_2}} \sum_{k \in \ell_r} [f(|x_k - \ell|)]^p
\]
\[
\geq \frac{1}{f(\ell_r)^{\alpha_1}} \sum_{k \in \ell_r} [f(|x_k - \ell|)]^p
\]
\[
\geq \frac{f(h_r)^{\alpha_1}}{f(\ell_r)^{\alpha_1}} \frac{1}{f(\ell_r)^{\alpha_2}} \sum_{k \in \ell_r} [f(|x_k - \ell|)]^p.
\]
Thus if $x \in w^{f,\alpha_1}_{\theta} (p)$, then $x \in w^{f,\alpha_1}_{\theta} (p)$.
\[\square\]

From Theorem 2.15 we have the following results.

Corollary 2.16. Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and $\alpha_1, \alpha_2$ two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If (3) holds then
\begin{enumerate}[(i)]  
\item $w^{f,\alpha_1}_{\theta} (p) \subset w^{f,\alpha_1}_{\theta'} (p)$, if $\alpha_1 = \alpha_2 = \alpha$,  
\item $w^{f,\alpha_1}_{\theta} (p) \subset w^{f,\alpha_1}_{\theta'} (p)$, if $\alpha_2 = 1$,  
\item $w^{f,\alpha_1}_{\theta} (p) \subset w^{f,\alpha_1}_{\theta'} (p)$, if $\alpha_1 = \alpha_2 = 1$.
\end{enumerate}

References