Spectral Properties of $n$-Normal Operators

Muneo Chô\textsuperscript{a}, Biljana Načevska\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan.
\textsuperscript{b}Department of Mathematics and Physics, Faculty of Electrical Engineering and Information Technologies, Ss. Cyril and Methodius University in Skopje, Macedonia.

Abstract. For a bounded linear operator $T$ on a complex Hilbert space and $n \in \mathbb{N}$, $T$ is said to be $n$-normal if $T^n = T^n$. In this paper we show that if $T$ is a $2$-normal operator and satisfies $\sigma(T) \cap (-\sigma(T)) \subseteq \{0\}$, then $T$ is isoloid and $\sigma(T) = \sigma_{r}(T)$. Under the same assumption, we show that if $z$ and $w$ are distinct eigenvalues of $T$, then $\ker(T - z) \perp \ker(T - w)$. And if non-zero number $z \in \mathbb{C}$ is an isolated point of $\sigma(T)$, then we show that $\ker(T - z)$ is a reducing subspace for $T$. We show that if $T$ is a $2$-normal operator satisfying $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then Weyl’s theorem holds for $T$. Similarly, we show spectral properties of $n$-normal operators under similar assumption. Finally, we introduce $(n, m)$-normal operators and show some properties of this kind of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, the spectrum, the approximate point spectrum and the point spectrum of $T$ are denoted by $\sigma(T)$, $\sigma_{a}(T)$ and $\sigma_{p}(T)$, respectively. The residual spectrum $\sigma_{r}(T)$ of $T$ is $\sigma_{r}(T) = \{ z \in \mathbb{C} : \exists x \in \mathcal{H}; x \neq 0, (T - z)x = 0 \}$. It is well known that $\sigma(T) = \sigma_{p}(T) \cup \sigma_{a}(T)$. For $T \in B(\mathcal{H})$, $T^*$ denotes the adjoint operator of $T$. $T$ is said to be normal and $n$-normal ($n \in \mathbb{N}$) if $T^* T = TT^*$ and $T^n T^n = T^n T^n$, respectively. Hence $1$-normal is normal. For $T \in B(\mathcal{H})$, we denote $T \geq 0$ if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. An operator $T$ is said to be hyponormal if $T^* T - TT^* \geq 0$, quasi-nilpotent if $\sigma(T) = \{0\}$, and nilpotent if there exists $k \in \mathbb{N}$ such that $T^k = 0$, respectively.

In [1], S.A. Alzraiqi and A.B. Patel introduced $n$-normal operators and showed interesting properties of this class. The class of $n$-normal operators is so wide. For example, $n$-nilpotent operators are clearly $n$-normal. Alzraiqi and Patel studied this condition

\begin{equation}
\sigma(T) \cap (-\sigma(T)) = \emptyset.
\end{equation}

Under the condition (\textasteriskcentered), they proved some interesting results. But if an operator $T \in B(\mathcal{H})$ satisfies (\textasteriskcentered), then the operator $T$ is invertible automatically. We try to set a little bit weaker assumption than this condition (\textasteriskcentered). The following result is well-known:

**Theorem 1.1.** (Stampfli [7], Theorem 2) Let $T \in B(\mathcal{H})$ be hyponormal. If $z$ is an isolated point of $\sigma(T)$, then $z \in \sigma_{r}(T)$.

\textsuperscript{2010 Mathematics Subject Classification.} Primary 47B15; Secondary 47A15

\textsuperscript{Keywords.} Hilbert space, linear operator, normal operator, spectrum, Weyl’s Theorem

\textsuperscript{Received:} 18 March 2018; \textsuperscript{Accepted:} 18 July 2018

\textsuperscript{Communicated by Dragan S. Djordjević}

\textsuperscript{Research is partially supported by Grant-in-Aid Scientific Research No.15K04910}

\textsuperscript{Email addresses:} chiyom01@kanagawa-u.ac.jp (Muneo Chô), bibarnmath@gmail.com (Biljana Načevska)
An operator $T \in B(\mathcal{H})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ belongs to the point spectrum of $T$. Hence, hyponormal operators are isoloid. Of course, there are many other classes of operators, weaker than hyponormal, which are isoloid. For example, let $p$ be $0 < p \leq 1$. An operator $T \in B(\mathcal{H})$ is said to be $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$. It is known that if $T$ is $p$-hyponormal, then $T$ is isoloid (see [4]).

Alzraiqi and Patel showed the following result which is of great significance for our work.

**Theorem 1.2.** (Alzraiqi and Patel [1], Proposition 2.2) Let $T \in B(\mathcal{H})$ and $n \in \mathbb{N}$. Then $T$ is $n$-normal if and only if $T^n$ is normal.

The following are the fundamental properties of $n$-normal operator $T$.

**Theorem 1.3.** (Alzraiqi and Patel [1], Proposition 2.6) Let $T \in B(\mathcal{H})$ be an $n$-normal operator. Then the following statements hold.

1. $T^n$ is $n$-normal.
2. If $T^{-1}$ exists, then $T^{-1}$ is $n$-normal.
3. If $S \in B(\mathcal{H})$ is unitary equivalent to $T$, then $S$ is $n$-normal.
4. If $M$ is a reducing subspace for $T$, then $T_M$ is an $n$-normal operator on $M$.

2. Spectral properties of 2-normal operators

For $T \in B(\mathcal{H})$, we set the following property:

$$(** \quad \sigma(T) \cap (\sigma(T))^c \subset \{0\}).$$

Then we begin with the following lemma.

**Lemma 2.1.** Let $T \in B(\mathcal{H})$ satisfy (**). If $z$ is an isolated point of $\sigma(T)$, then $z^2$ is an isolated point of $\sigma(T^2)$.

*Proof.* Assume that $z$ is an isolated point of $\sigma(T)$ and $z^2$ is not isolated point of $\sigma(T^2)$. Then there exists a sequence $\{z_n\} \subset \sigma(T)$ such that $z_n^2 \to z^2$ as $n \to \infty$ by the Spectral mapping theorem.

1. If $z = 0$, then it is clear that $z_n \to 0$ as $n \to \infty$. Hence, 0 is not an isolated point. It’s a contradiction.
2. If $z \neq 0$. Since $(z_n + z)(z_n - z) \to 0$ as $n \to \infty$, we may assume that (i) $z_n \to z$ as $n \to \infty$ or (ii) $z_n \to -z$ as $n \to \infty$.

Since $z$ is an isolated point of $\sigma(T)$, (i) does not hold. In the case of (ii), since $z_n \in \sigma(T)$, $\lim_{n \to \infty} z_n = -z$ and $\sigma(T)$ is compact, we have $-z \in \sigma(T)$. Since $z \neq 0$ and $z \in \sigma(T) \cap (\sigma(T))^c$, it’s a contradiction to (**). Thus, we prove the lemma. □

Our main result is the following.

**Theorem 2.2.** Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (**). Then $T$ is isoloid.

*Proof.* We assume that $z$ is an isolated point of $\sigma(T)$. Since $T$ satisfies (**), $z^2$ is an isolated point of $\sigma(T^2)$ by Lemma 2.1. Since $T^2$ is normal by Theorem 1.2, $z^2$ is in the point spectrum of $T^2$ by Theorem 1.2. Hence there exists a non-zero vector $x \in \mathcal{H}$ such that $T^2x = z^2x$. If $z = 0$, then it is clear that 0 is an eigenvalue. If $z \neq 0$, then $(T + z)(T - z)x = 0$ and $-z \notin \sigma(T)$. Since $T + z$ is invertible, we have $(T - z)x = 0$. Thus, $z$ belongs to the point spectrum of $T$. □

**Theorem 2.3.** Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (**). Then $\sigma(T) = \sigma_a(T)$.

*Proof.* Since $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, it takes only to show $\sigma_a(T) \subset \sigma_a(T)$. Let $z \in \sigma_a(T)$. Then there exists a non-zero vector $x \in \mathcal{H}$ such that $T^2x = z^2x$. Since $T^2x = \mathbb{E}x$ and $T^2$ is normal, we have $T^2x = z^2x$.

1. If $z \neq 0$, then $(T + z)(T - z)x = 0$. Since $-z \notin \sigma(T)$, it holds $(T - z)x = 0$ and hence $z \in \sigma_p(T)$.
2. If $z = 0$, then $T^2x = 0$ and we have $0 \in \sigma_p(T)$.

Therefore, $\sigma(T) = \sigma_a(T)$. □
Theorem 2.4. Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (**).
(1) If $z$ and $w$ are distinct eigenvalues of $T$ and $x, y \in \mathcal{H}$ are corresponding eigenvectors, respectively, then $\langle x, y \rangle = 0$.
(2) If $z, w$ are distinct values of $\alpha(z)$ and $\{x_n\}, \{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ as $n \to \infty$, then $\lim_{n \to \infty} \langle x_n, y_n \rangle = 0$.

Proof. (1) follows from (2). So we will only show (2).
Since $(T^2 - z^2)x_n \to 0$ and $(T^2 - w^2)y_n \to 0$ and $T^2$ is normal, it holds $(T^2 - w^2)y_n \to 0$. Hence, it holds
$$
\lim_{n \to \infty} z^2 \langle x_n, y_n \rangle = \lim_{n \to \infty} \langle z^2 x_n, y_n \rangle = \lim_{n \to \infty} \langle T^2 x_n, y_n \rangle = \lim_{n \to \infty} \langle x_n, T^2 y_n \rangle = \lim_{n \to \infty} w^2 \langle x_n, y_n \rangle.
$$
If $z^2 = w^2$, then $(z + w)(z - w) = 0$. Since $z \neq w$, we have $z = -w$. By (**), this implies $z = w = 0$, which is impossible for distinct values. Hence, $\lim_{n \to \infty} \langle x_n, y_n \rangle = 0$. □

So, we have the following corollary.

Corollary 2.5. Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (**). If $z$ and $w$ are distinct eigenvalues of $T$, then $\ker(T - z) \perp \ker(T - w)$.

Let $M$ be a subspace of $\mathcal{H}$. Then $M$ is said to be a reducing subspace for $T$ if $T(M) \subseteq M$ and $T^*(M) \subseteq M$, that is, $M$ is an invariant subspace for $T$ and $T^*$. Then we have the following result.

Theorem 2.6. Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (**). If $z$ is a non-zero eigenvalue of $T$, then $\ker(T - z) = \ker(T^2 - z^2) = \ker(T^* - z^2)$ and hence $\ker(T - z)$ is a reducing subspace for $T$.

Proof. First we show $\ker(T - z) = \ker(T^2 - z^2)$. Since it is obvious that $\ker(T - z) \subseteq \ker(T^2 - z^2)$, we show $\ker(T^2 - z^2) \subseteq \ker(T - z)$. Let $x \in \ker(T^2 - z^2)$, i.e., $(T^2 - z^2)x = 0$. Then $(T + z)(T - z)x = 0$. Since $z \neq 0$, by (**), we have $-z \notin \sigma(T)$. Hence, it follows $(T - z)x = 0$ and $x \in \ker(T - z)$. Therefore, $\ker(T^2 - z^2) \subseteq \ker(T - z)$ and $\ker(T - z) = \ker(T^2 - z^2)$. Since $T^2$ is normal, it is clear $\ker(T^2 - z^2) = \ker(T^* - z^2)$. Since $-z \notin \sigma(T^*)$, we have $\ker(T^2 - z^2) = \ker(T^* - z^2)$. Finally by the above results, it is clear that $\ker(T - z)$ is a reducing subspace for $T$. □

Remark 2.7. In general, $\ker(T)$ is not a reducing subspace for a 2-normal operator $T$.
(1) Let $T$ be as in Example 2.3 of [1], that is, let $\mathcal{H} = \mathbb{C}^2$ and $\{e_j\}_{j=1}^\infty$ be the standard orthonormal basis of $\mathbb{C}^2$. Let $T$ be defined by
$$
Te_j = \begin{cases} e_1 & \text{if } j = 1 \\
e_{j+1} & \text{if } j = 2k \\
0 & \text{if } j = 2k + 1.
\end{cases}
$$
Then $T$ is a 2-normal operator and satisfies (**). Since $e_3 \in \ker(T)$ and $TT^*e_3 = e_3 \neq 0$, $\ker(T)$ does not reduce $T$.

Let $P$ be the orthogonal projection to the first coordinate. Since $T^2 = P$, it is clear $\ker(T) \subseteq \ker(T^2) = \ker(P)$.

(2) We give this, even more simple, example. Let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Then, since $S^2 = 0$ and $\sigma(S) = \{0\}$, $T$ is 2-normal and satisfies (**). Let $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $x \in \ker(S)$ and $SS^*x = x \neq 0$. Hence $\ker(S)$ does not reduce $S$ and $\ker(S) \not\subseteq \ker(S^2) = \mathbb{C}^2$. It is clear that $\sigma(S) = \{0\} \neq \{a + bi \in \mathbb{C} : |ab| \leq \frac{1}{2}\} = W(S)$, where $W(S) = \{Sx, x : \|x\| = 1\}$ (the numerical range of $S$). Hence $S$ is not convexoid, i.e., $\sigma(T) \not\subseteq W(T)$, where $\sigma(T)$ is the convex hull of $\sigma(T)$ and $W(T)$ is the closure of $W(T)$.

Remark 2.8. Let $p$ be $0 < p \leq 1$ and $k \in \mathbb{N}$. An operator $T \in B(\mathcal{H})$ is said to be $(p,k)$-quasihyponormal if $T^k (T^p T^* - (T^* T)^p) T^k \geq 0$. Let a non-zero $z$ be an isolated point of the spectrum of a $(p,k)$-quasihyponormal operator $T$. Then $\ker(T - z)$ reduces $T$. But when $0$ is an isolated point of the spectrum of a $(p,k)$-quasihyponormal operator $T$, in general, $\ker(T)$ does not reduce $T$. See [8] for details.
Next we study Weyl’s theorem. For \( T \in B(\mathcal{H}) \), the Weyl spectrum \( \omega(T) \) is defined by

\[
\omega(T) = \bigcap_{K \in \mathcal{C}(\mathcal{H})} \sigma(T + K),
\]

where \( \mathcal{C}(\mathcal{H}) \) is the set of all compact operators on \( \mathcal{H} \). Let \( \pi_{00}(T) \) denote the set of all isolated eigenvalues of finite multiplicity of \( T \). We say that Weyl’s theorem holds for \( T \) if \( \omega(T) = \sigma(T) - \pi_{00}(T) \). J.V. Baxley showed the following result.

**Theorem 2.9.** (Baxley [3], Lemma 3) Let \( T \in B(\mathcal{H}) \) satisfy the following condition C-1: “If \( \{z_n\} \) is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of \( T \) and \( \{x_n\} \) is any sequence of corresponding normalized eigenvectors, then the sequence \( \{x_n\} \) does not converge.” Then

\[
\sigma(T) - \pi_{00}(T) \subset \omega(T).
\]

If \( T \) is a 2-normal operator satisfying (**) then \( T \) satisfies the condition C-1 by Corollary 2.5. Hence we have the following result by Theorem 2.9.

**Theorem 2.10.** If \( T \in B(\mathcal{H}) \) is a 2-normal operator satisfying (**) then

\[
\sigma(T) - \pi_{00}(T) \subset \omega(T).
\]

For the converse inclusion, we show the following result.

**Theorem 2.11.** If \( T \in B(\mathcal{H}) \) is a 2-normal operator satisfying (**), then

\[
\omega(T) \subset \sigma(T) - \{\pi_{00}(T) - \{0\}\}.
\]

**Proof.** Let \( z \in \pi_{00}(T) - \{0\} \). By Theorem 2.6, \( \ker(T - z) \) is a reducing subspace of \( T \). Hence we have the decomposition \( T - z = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \) on \( \ker(T - z) \oplus \ker(T - z)^{\perp} \). Since \( T = \begin{pmatrix} z & 0 \\ 0 & S + z \end{pmatrix} \) and \( T|_{\ker(T - z)^{\perp}} \) is 2-normal by Theorem 1.3 (4), \( S + z \) is a 2-normal operator satisfying (***) on \( \ker(T - z)^{\perp} \). If \( z \in \sigma(S + z) \), then \( z \in \sigma_p(S + z) \) because \( z \) is an isolated point of \( \sigma(S + z) \). It’s a contradiction. Hence \( z \notin \sigma(S + z) \) and hence \( S \) is invertible.

Let \( K = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \). Then \( K \in \mathcal{C}(\mathcal{H}) \) and \( T + K - z = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \) is an invertible operator. Therefore, \( z \notin \omega(T) \). It completes the proof. \( \square \)

If \( T \) satisfies (**), then \( T \) is invertible and \( 0 \notin \sigma(T) \). Hence we have the following result by Theorems 2.10 and 2.11.

**Theorem 2.12.** If \( T \in B(\mathcal{H}) \) is a 2-normal operator satisfying (**), then

\[
\omega(T) = \sigma(T) - \pi_{00}(T),
\]

that is, Weyl’s theorem holds for \( T \).

3. \( n \)-normal operators

In this section, we show spectral properties of \( n \)-normal operators. Recall that, for \( n \in \mathbb{N} \), \( T \) is said to be \( n \)-normal if \( T^n T^n = TT^n \). First we extend Proposition 2.19 of [1] as follows:

**Theorem 3.1.** The following statements are equivalent:

1. \( T - t \) is \( n \)-normal for all \( t \geq 0 \).
2. \( T \) is normal.
3. \( T - z \) is \( n \)-normal for all \( z \in \mathbb{C} \).
If $z$ and $w$ are distinct eigenvalues of $T$ and $x_i \neq 0$, then $\langle x_i, y_i \rangle = 0$. Proof. If $z$ and $w$ are distinct eigenvalues of $T$ and $x$ is an eigenvector corresponding to $z$, then $\langle x, y \rangle = 0$. Hence, $\sigma(T) \cap \{0\} = \emptyset$. Hence, $\sigma(T) = \sigma_n(T)$.

Theorem 3.4. Let $T \in B(\mathcal{H})$ be a bounded operator. If $z$ and $w$ are distinct eigenvalues of $T$ and $x_i \neq 0$, then $\langle x_i, y_i \rangle = 0$. Proof. If $z$ and $w$ are distinct eigenvalues of $T$ and $x_i \neq 0$, then $\langle x_i, y_i \rangle = 0$. Hence, $\sigma(T) \cap \{0\} = \emptyset$. Hence, $\sigma(T) = \sigma_n(T)$.

Theorem 3.5. Let $T \in B(\mathcal{H})$ be a bounded operator. If $z$ and $w$ are distinct eigenvalues of $T$ and $x_i \neq 0$, then $\langle x_i, y_i \rangle = 0$. Proof. If $z$ and $w$ are distinct eigenvalues of $T$ and $x_i \neq 0$, then $\langle x_i, y_i \rangle = 0$. Hence, $\sigma(T) \cap \{0\} = \emptyset$. Hence, $\sigma(T) = \sigma_n(T)$.
Corollary 3.6. Let $T \in B(H)$ be $n$-normal and satisfy $(** *)$. If $z$ and $w$ are distinct eigenvalues of $T$, then $\ker(T - z) \perp \ker(T - w)$.

Theorem 3.7. Let $T \in B(H)$ be $n$-normal and satisfy $(** *)$. If $z$ is a non-zero eigenvalue of $T$, then $\ker(T - z) = \ker(T^n - z^n) = \ker(T^n - \overline{z}^n)$ and hence $\ker(T - z)$ is a reducing subspace for $T$.

Theorem 3.8. If $T \in B(H)$ is an $n$-normal operator satisfying $(** *)$, then

$$\sigma(T) - \pi_{00}(T) \subset \omega(T) \subset \sigma(T) - \left(\pi_{00}(T) - |0|\right).$$

Moreover, $T$ is invertible, and $\sigma(T) - \pi_{00}(T) = \omega(T)$, that is, Weyl’s theorem holds for $T$.

4. $(n, m)$-normal operators

We begin with the definition of $(n, m)$-normal operators.

Definition 4.1. For $T \in B(H)$ and $n, m \in \mathbb{N}$, $T$ is said to be $(n, m)$-normal if

$$T^n T^m = T^m T^n.$$

From the definition, it is clear that $T$ is $(n, m)$-normal if and only if $T$ is $(m, n)$-normal. Let $T \in B(H)$ be $(n, m)$-normal. Then the following hold clearly:

1. $T^*$ is $(n, m)$-normal.
2. If $T^{-1}$ exists, then $T^{-1}$ is $(n, m)$-normal.
3. If $S \in B(H)$ is unitary equivalent to $T$, then $S$ is $(n, m)$-normal.
4. If $M$ is a closed subspace of $H$ which reduces $T$, then $T_{|M}$ is $(n, m)$-normal on $M$.

Lemma 4.2. (1) If $T \in B(H)$ is $(n, m)$-normal, then $T^k$ is normal, where $k$ is the least common multiple of $n$ and $m$.

(2) If $T^n$ is normal, then $T$ is $(n, m)$-normal for every $m$.

Proof. (1) Let $k = n \cdot j$ and $k = m \cdot \ell$. If $T$ is $(n, m)$-normal, then

$$T^k T^k = \underbrace{T^n \ldots T^n}_{\ell} \cdot \underbrace{T^m \ldots T^m}_{j} = T^n \ldots T^n \cdot T^m \ldots T^m = T^k T^k.$$

Hence $T^k$ is normal.

(2) Since $T^n$ is normal and $T^n \cdot T^n = T^n \cdot T^n$, it follows from Fuglede’s theorem that $T^n \cdot T^n = T^n \cdot T^n$. Hence, $T$ is $(n, m)$-normal. □

Theorem 4.3. If $T \in B(H)$ is quasi-nilpotent and $(n, m)$-normal, then $T$ is nilpotent.

Proof. Since $\sigma(T) = \{0\}$, we have $\sigma(T^k) = \{0\}$ for every $k \in \mathbb{N}$. Let $k$ be the least common multiple of $n$ and $m$. Then, by Lemma 4.2, $T^k$ is normal. Hence $T^k = 0$. □

Theorem 4.4. If $T, S \in B(H)$ are commuting $(n, m)$-normal operators, then $TS$ is $(k, j)$-normal for every $j \in \mathbb{N}$, where $k$ is the least common multiple of $n$ and $m$.

Proof. Since $k$ is the least common multiple of $n$ and $m$, by Lemma 4.2, $(TS)^k$ is normal. Since $(TS)^k$ commutes with $(TS)^j$ for every $j \in \mathbb{N}$. By Fuglede’s theorem, it holds $(TS)^j(TS)^k = (TS)^k(TS)^j$. Hence $TS$ is $(k, j)$-normal for every $j \in \mathbb{N}$. □
Theorem 4.5. Let $T \in B(\mathcal{H})$ be $(n, m)$-normal and $(n + 1, m)$-normal. If either $T$ or $T^*$ is injective, then $T$ is $m$-normal.

Proof. (1) Let $T$ be injective. Since $T$ is $(n, m)$-normal and $(n + 1, m)$-normal, it holds
\[ T^{m+1}T^m = T^mT^{m+1} = (T^mT^n)T = T^nT^mT. \]

Hence, we have $T^m(TT^m - T^mT) = 0$. Since $T$ is injective, we have $TT^m = T^mT$ and $TT^m = T^mT$. Hence, $T$ is $m$-normal.

(2) Let $T^*$ be injective. Since it holds that $T^*$ is $(n, m)$-normal and $(n + 1, m)$-normal, we have $T^*$ is $m$-normal by (1) and $T$ is $m$-normal. □

Theorem 4.6. For $T \in B(\mathcal{H})$, let $F = T^n + T^m$ and $G = T^n - T^m$. Then $T$ is $(n, m)$-normal if and only if $F$ commutes with $G$.

Proof. Since
\[ FG = T^{2n} - T^nT^m + T^mT^n - T^{2m} \quad \text{and} \quad GF = T^{2n} + T^nT^m - T^mT^n - T^{2m}, \]

Hence, $FG = GF$ if and only if $T^nT^m = T^mT^n$. It completes the proof. □

Theorem 4.7. For $T \in B(\mathcal{H})$, let $A = T^nT^m$, $F = T^n + T^m$ and $G = T^n - T^m$. If $T$ is $(n, m)$-normal, then $A$ commutes with $F$ and $G$.

Proof. Since $T$ is $(n, m)$-normal, we have
\[ AF = T^nT^m(T^n + T^m) = T^nT^nT^m + T^mT^nT^m = FA. \]

Similarly we have $AG = GA$. □

Theorem 4.8. Let $T \in B(\mathcal{H})$ be an invertible $(n, m)$-normal operator. Then $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace.

Proof. Let $k$ be the least common multiple of $n$ and $m$. Then by Lemma 4.2, $T^k$ is normal. Hence $T^{-k}$ is also normal. Hence, $T^k$ and $T^{-k}$ have no hypercyclic vector by Corollary 4.5 of [6]. Hence, $T$ and $T^{-1}$ have no hypercyclic vector by [2]. Therefore, $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace by [5]. □

Finally, we show results of the direct sum and the tensor product. The proof is easy. So we omit the proof.

Theorem 4.9. If $T, S \in B(\mathcal{H})$ are $(n, m)$-normal, then $T \oplus S$ and $T \otimes S$ are $(n, m)$-normal on $\mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{H}$, respectively.

References