A New Characterization of Browder’s Theorem

Mohammed Karmouni\textsuperscript{a}, Abdelaziz Tajmouati\textsuperscript{b}

\textsuperscript{a}Cadi Ayyad University, Multidisciplinary Faculty, Safi, Morocco.
\textsuperscript{b}Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Laboratory of Mathematical Analysis and Applications Fez, Morocco.

Abstract. We give a new characterization of Browder’s theorem using spectra originated from Drazin-Fredholm theory.

1. Introduction and Preliminaries

Throughout, \(X\) denotes a complex Banach space, \(B(X)\) the Banach algebra of all bounded linear operators on \(X\), let \(I\) be the identity operator, and for \(T \in B(X)\) we denote by \(T^*, N(T), R(T), R_\infty(T) = \bigcap_{n \geq 0} R(T^n), \rho(T), \sigma(T)\) respectively the adjoint, the null space, the range, the hyper-range, the resolvent set and the spectrum of \(T\).

Let \(E\) be a subset of \(X\). \(E\) is said \(T\)-invariant if \(T(E) \subseteq E\). We say that \(T\) is completely reduced by the pair \((E,F)\) if \(E\) and \(F\) are two closed \(T\)-invariant subspaces of \(X\) such that \(X = E \oplus F\). In this case we write \(T = T_{|E} \oplus T_{|F}\) and we say that \(T\) is the direct sum of \(T_{|E}\) and \(T_{|F}\). An operator \(T \in B(X)\) is said to be semi-regular, if \(R(T)\) is closed and \(N(T) \subseteq R_\infty(T)\) ([1]).

In the other hand, recall that an operator \(T \in B(X)\) admits a generalized Kato decomposition, (GKD for short), if there exists \((X_1, X_2) \in \text{Red}(T)\) such that \(T_{|X_1}\) is semi-regular and \(T_{|X_2}\) is quasi-nilpotent, in this case \(T\) is said a pseudo Fredholm operator. If we assume in the definition above that \(T_{|X_1}\) is nilpotent, then \(T\) is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [17, 20] for more information about generalized Kato decomposition.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi Fredholm) if \(\dim N(T) < \infty\) and \(R(T)\) is closed (resp, \(\text{codim}R(T) < \infty\)). \(T\) is semi-Fredholm if it is a lower or upper semi-Fredholm operator. The index of a semi-Fredholm operator \(T\) is defined by \(\text{ind}(T) := \text{dim}N(T) - \text{codim}R(T)\). Also, \(T\) is a Fredholm operator if it is a lower and upper semi-Fredholm operator, and \(T\) is called a Weyl operator if it is a Fredholm of index zero.

The essential and Weyl spectra of \(T\) are closed and defined by:

\[
\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator} \};
\]

\[
\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator} \}.
\]
Recall that an operator $R \in \mathcal{B}(X)$ is said to be Riesz if $R - \mu I$ is Fredholm for every non-zero complex number $\mu$ ([1]). Of course compact and quasi-nilpotent operators are particular cases of Riesz operators.

In [26], Živković-Zlatanović SC and M D. Cvetković introduced and studied a new concept of Kato decomposition to extend the Mbekhta concept to “generalized Kato-Riesz decomposition”. In fact, an operator $T \in \mathcal{B}(X)$ admits a generalized Kato-Riesz decomposition, ( GKRD for short ), if there exists $(X_1, X_2) \in \text{Red}(T)$ such that $T_{X_1}$ is semi-regular and $T_{X_2}$ is Riesz. The generalized Kato-Riesz spectrum is defined by

$$\sigma_{\text{GRD}}(T) := \{ \lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato-Riesz decomposition} \}.$$

Let $T \in \mathcal{B}(X)$, the ascent of $T$ is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such $p$ does not exist we let $a(T) = \infty$. Analogously the descent of $T$ is defined by $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such $q$ does not exist we let $d(T) = \infty$ [19]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

An operator $T \in \mathcal{B}(X)$ is upper semi-Browder if $T$ is upper semi-Fredholm and $a(T) < \infty$. If $T \in \mathcal{B}(X)$ is lower semi-Fredholm and $d(T) < \infty$ then $T$ is lower semi-Browder. $T$ is called Browder operator if it is a lower and an upper Browder operator.

An operator $T \in \mathcal{B}(X)$ is said to be B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and $T_n$, the restriction of $T$ to $R(T^n)$ is a Fredholm operator. This class of operators, introduced and studied by Berkani et al. in a series of papers extends the class of semi-Fredholm operators ([11], [12]). $T$ is said to be a B-Weyl operator if $T_n$ is a Fredholm operator of index zero. The B-Fredholm and B-Weyl spectra are defined by

$$\sigma_{\text{BF}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm}\};$$

$$\sigma_{\text{BW}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl}\}.$$

Note that $T$ is a B-Fredholm operator if there exists $(X_1, X_2) \in \text{Red}(T)$ such that $T_{X_1}$ is Fredholm and $T_{X_2}$ is nilpotent, see [11, Theorem 2.7]. Also, $T$ is a B-Weyl operator if and only if $T_{X_1}$ is a Weyl operator and $T_{X_2}$ is a nilpotent operator.

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl, see [13] [22][23] [25], precisely, $T$ is a pseudo B-Fredholm operator, if there exists $(X_1, X_2) \in \text{Red}(T)$ such that $T_{X_1}$ is a Fredholm operator and $T_{X_2}$ is a quasi-nilpotent operator. $T$ is said to be pseudo B-Weyl operator if there exists $(X_1, X_2) \in \text{Red}(T)$ such that $T_{X_1}$ is a Weyl operator and $T_{X_2}$ is a quasi-nilpotent operator. The pseudo B-Fredholm and pseudo B-Weyl spectra are defined by:

$$\sigma_{\text{pBF}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Fredholm}\};$$

$$\sigma_{\text{pBW}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

Let $T \in \mathcal{B}(X)$, $T$ is said to be Drazin invertible if there exist a positive integer $k$ and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ T^{k+1}S = T^k \text{ and } S^2T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where $T_1$ is invertible and $T_2$ is nilpotent. The Drazin spectrum is defined by

$$\sigma_{\text{D}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

The concept of Drazin invertible operators has been generalized by Koliha [16]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin \text{acc}(\sigma(T))$, where $\text{acc}(\sigma(T))$ is the set of accumulation points of $\sigma(T)$. This is also equivalent to the fact that there exists $(X_1, X_2) \in \text{Red}(T)$ such that $T_{X_1}$ is invertible and $T_{X_2}$ is quasi-nilpotent. The generalized Drazin spectrum is defined by

$$\sigma_{\text{GD}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible}\}.$$

The concept of analytical core for an operator has been introduced by Vrbova in [24] and study by Mbekhta [20, 21], that is the following set:
The quasi-nilpotent part of $T$, $H_0(T)$ is given by:

$$H_0(T) := \{x \in X; r_T(x) = 0\} \text{ where } r_T(x) = \lim_{n \to +\infty} \|T^n x\|^\frac{1}{n}.$$

In [14], M. D. Cvetković and SC. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in B(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace $M$ in $X$ such that $(M, H_0(T)) \in \text{Red}(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in \text{Red}(T)$ such that $T_M$ is bounded below and $T_N$ is quasi-nilpotent, see [14, Theorem 3.6]. An operator $T \in B(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace $N$ in $X$ such that $N \subseteq H_0(T)$ and $(K(T), N) \in \text{Red}(T)$ which is equivalent to there exists $(M, N) \in \text{Red}(T)$ such that $T_M$ is surjective and $T_N$ is quasi-nilpotent, see [14, Theorem 3.7].

The generalized Drazin bounded below and surjective spectra of $T \in B(X)$ are defined respectively by:

$$\sigma_{gDM}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin bounded below}\};$$

$$\sigma_{gDQ}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin surjective}\}.$$

From [14], we have:

$$\sigma_{gD}(T) = \sigma_{gDM}(T) \cup \sigma_{gDQ}(T).$$

Recently, Živković-Zlatanović SC and M D. Cvetković [26] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept, generalized Drazin bounded below, and generalized Drazin surjective to “generalized Drazin-Riesz invertible”, “generalized Drazin-Riesz bounded below” and “generalized Drazin-Riesz surjective” respectively. In fact, an operator $T \in B(X)$ is said to be generalized Drazin-Riesz invertible, if there exists $S \in B(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and} \quad TST - T \text{ is Riesz}.$$
These classes of operators motivate the definition of several spectra. The generalized Drazin-Riesz lower(upper) semi-Weyl and generalized Drazin-Riesz Weyl spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

\[
\sigma_{gDRW-}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz lower semi-Weyl} \};
\]

\[
\sigma_{gDRW+}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz upper semi-Weyl} \};
\]

\[
\sigma_{gDRW}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz Weyl} \}.
\]

From [26], we have:

\[
\sigma_{gDRW}(T) = \sigma_{gDRW+}(T) \cup \sigma_{gDRW-}(T);
\]

The generalized Drazin-Riesz upper (lower) semi-Fredholm and generalized Drazin-Riesz Fredholm spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

\[
\sigma_{gDFR+}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz upper semi-Fredholm} \};
\]

\[
\sigma_{gDFR-}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz lower semi-Fredholm} \};
\]

\[
\sigma_{gDFR}(T) = \{ \lambda \in \mathbb{C}, \; T - \lambda I \text{ is not generalized Drazin-Riesz Fredholm} \}.
\]

Also, from [26], we have:

\[
\sigma_{gDFR}(T) = \sigma_{gDFR+}(T) \cup \sigma_{gDFR-}(T).
\]

\[
\sigma_{gKR}(T) \subset \sigma_{gDFR+}(T) \subset \sigma_{gDFR-}(T) \subset \sigma_{gDR}(T);
\]

\[
\sigma_{gKR}(T) \subset \sigma_{gDFR}(T) \subset \sigma_{gDRW-}(T) \subset \sigma_{gDR}(T) \subset \sigma_{gDR}(T)
\]

A Banach space operator satisfies “Browder’s theorem” if the Browder spectrum coincides with the Weyl spectrum. Browder’s theorem has been studied by several authors (see [4], [3], [5], [6]). In this paper we shall give some characterizations of operators satisfying Browder’s theorem. In particular, we shall see that Browder’s theorem for a bounded linear operator is equivalent to the equality between the generalized Drazin-Riesz Weyl spectrum and generalized Drazin-Riesz spectrum. Also, we will give several necessary and sufficient conditions for $T$ to have equality between the spectra originated from Drazin-Fredholm theory.

2. Main Results

Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of $\lambda_0$, the only analytic function $f : U \rightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator $T$ is said to have the SVEP if $T$ has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence $T$ and $T^*$ have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. Also, we have the implication

\[
a(T) < \infty \implies T \text{ has SVEP at } 0.
\]

\[
d(T) < \infty \implies T^* \text{ has SVEP at } 0.
\]

In [26], the authors gave some examples showing that $\sigma_{gDRW-}(T) \subset \sigma_{gDR}(T)$, $\sigma_{gDRW+}(T) \subset \sigma_{gDR}(T)$ and $\sigma_{gDRW}(T) \subset \sigma_{gDR}(T)$ can be proper. In the following results we give several necessary and sufficient conditions for $T$ to have equality.

**Proposition 2.1.** Let $T \in \mathcal{B}(X)$, then $\sigma_{gDR}(T) = \sigma_{gDRW+}(T)$ if and only if $T$ has SVEP at every $\lambda \notin \sigma_{gDRW+}(T)$.
Proof. Assume that $T$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T - \lambda I$ is generalized Drazin Riesz upper semi-Weyl, then there exists $(M,N) \in \text{Red}(T - \lambda I)$ such that $(T - \lambda I)_{M}$ is semi-regular and $(T - \lambda I)_{N}$ is Riesz. $T$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$, it follows that $(T - \lambda I)_{M}$ has the SVEP at 0, then $(T - \lambda I)_{M}$ is bounded below, see [18, Corollary 3.1.7]. Hence $T - \lambda I$ is generalized Drazin Riesz bounded below, $\lambda \notin \sigma_{\text{gDR}}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{\text{gDRW}}(T) = \sigma_{\text{gDRW}}(T)$. Conversely, suppose that $\sigma_{\text{gDRW}}(T) = \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$ then $T - \lambda I$ is generalized Drazin Riesz bounded below so, $T$ has SVEP at $\lambda$, by [26, Theorem 2.4]. □

We denote by $\sigma_{b}(T)$ and $\sigma_{w}(T)$ respectively the lower Browder and lower Weyl spectra. In the same way we have the following result.

**Proposition 2.2.** Let $T \in \mathcal{B}(X)$, then $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$ if and only if $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$.

Proof. Suppose that $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then by [26, Theorem 2.6], $T - \lambda I$ admits GKRD and $\lambda \notin \text{acc}_{\text{gDRW}}(T)$. $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$, and so $\sigma_{b}(T) = \sigma_{w}(T)$. Then $\lambda \notin \text{acc}_{b}(T)$. Therefore, $T - \lambda I$ is generalized Drazin Riesz surjective according to [26, Theorem 2.5], $\lambda \notin \sigma_{\text{gDRW}}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{\text{gDRW}}(T) = \sigma_{\text{gDRW}}(T)$. Conversely, suppose that $\sigma_{\text{gDRW}}(T) = \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T - \lambda I$ is generalized Riesz Drazin surjective so, $T$ has SVEP at $\lambda$, by [26, Theorem 2.5]. □

As a consequence of the two previous results we have the following corollary.

**Corollary 2.3.** Let $T \in \mathcal{B}(X)$, then $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$ if and only if $T$ and $T^{\ast}$ have the SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$.

Proof. Suppose that $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T - \lambda I$ is generalized Riesz Drazin invertible so, $T$ and $T^{\ast}$ have SVEP at $\lambda$, by [26, Theorem 2.3]. The “if” is an immediate consequence of Proposition 2.1 and Proposition 2.2. □

Moreover, we have the following result.

**Proposition 2.4.** Let $T \in \mathcal{B}(X)$, the following statements are equivalent:

1) $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$;
2) $T$ or $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$.

Proof. -If $T$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then by [26, Theorem 2.6], $T - \lambda I$ admits GKRD and $\lambda \notin \text{acc}_{\text{gDRW}}(T)$. $T$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$, and so $\sigma_{b}(T) = \sigma_{w}(T)$ [1, Theorem 4.23]. Thus $\lambda \notin \text{acc}_{b}(T)$. Therefore, $T - \lambda I$ is generalized Drazin Riesz invertible by [26, Theorem 2.3], $\lambda \notin \sigma_{\text{gDR}}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$.

-If $T^{\ast}$ has SVEP at every $\lambda \notin \sigma_{\text{gDRW}}(T)$. Since $\sigma_{b}(T) = \sigma_{b}(T^{\ast})$ and $\sigma_{w}(T) = \sigma_{w}(T^{\ast})$, we have $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$.

Conversely, suppose that $\sigma_{\text{gDR}}(T) = \sigma_{\text{gDRW}}(T)$. If $\lambda \notin \sigma_{\text{gDRW}}(T)$, then $T - \lambda I$ is generalized Riesz Drazin invertible so, $T$ and $T^{\ast}$ have SVEP at $\lambda$, by [26, Theorem 2.3]. □

We shall say that $T$ satisfies Browder’s theorem if $\sigma_{w}(T) = \sigma_{b}(T)$, or equivalently $\text{acc}(T) \subseteq \sigma_{w}(T)$, where $\sigma_{b}(T)$ is the Browder spectrum of $T$ ([15]).

It is known from [2] that a-Browder’s theorem holds for $T$ if $\sigma_{w}(T) = \sigma_{ab}(T)$, or equivalently $\text{acc}_{ab}(T) \subseteq \sigma_{w}(T)$, where $\sigma_{ab}(T)$ and $\sigma_{w}(T)$ are the upper semi-Browder and upper semi-Weyl spectra of $T$.

In the sequel, we characterize the equality between the generalized Drazin-Riesz invertible(surjective, bounded below) spectrum and generalized Drazin-Riesz Weyl(upper-lower Weyl) spectrum by means of the Browder’s theorem(a-Browder’s theorems), which give new characterizations for Browder’s and a-Browder’s theorems.
**Theorem 2.5.** Let $T \in \mathcal{B}(X)$, then
1) a-Browder’s theorem holds for $T$ if and only if $\sigma_{gD}(T) = \sigma_{gDRW^+}(T)$.
2) a-Browder’s theorem holds for $T^*$ if and only if $\sigma_{gD}(T) = \sigma_{gDRW^-}(T)$.
3) Browder’s theorem holds for $T$ if and only if $\sigma_{gD}(T) = \sigma_{gDRW}(T)$.

**Proof.** 1) Suppose that a-Browder’s theorem holds for $T$ implies $\sigma_{ub}(T) = \sigma_{uw}(T)$. Using [26, Theorems 2.4 and 2.6], we conclude that

$$
\lambda \notin \sigma_{gD}(T) \iff T - \lambda I \text{ is generalized Drazin Riesz bounded below}
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{ub}(T)
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{uw}(T)
\iff T - \lambda I \text{ is generalized Drazin Riesz upper semi-Weyl}
\iff \lambda \notin \sigma_{gDRW^+}(T).
$$

Hence $\sigma_{gD}(T) = \sigma_{gDRW^+}(T)$. Conversely, if $\sigma_{gD}(T) = \sigma_{gDRW^+}(T)$, from Proposition 2.1, $T$ has SVEP at every $\lambda \notin \sigma_{gDRW^+}(T)$. Since $\sigma_{gDRW^+}(T) \subseteq \sigma_{uw}(T)$, $T$ has SVEP at every $\lambda \notin \sigma_{uw}(T)$, so a-Browder’s theorem holds for $T$, see [2, Theorem 4.34].

2) Suppose that a-Browder’s theorem holds for $T^*$ then $\sigma_{ub}(T) = \sigma_{uw}(T)$. Using [26, Theorems 2.5 and 2.6] we have

$$
\lambda \notin \sigma_{gD}(T) \iff T - \lambda I \text{ is generalized Drazin Riesz surjective}
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{ub}(T)
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{uw}(T)
\iff T - \lambda I \text{ is generalized Drazin Riesz lower semi-Weyl}
\iff \lambda \notin \sigma_{gDRW^-}(T).
$$

Hence $\sigma_{gD}(T) = \sigma_{gDRW^-}(T)$. Conversely, if $\sigma_{gD}(T) = \sigma_{gDRW^-}(T)$, from Proposition 2.2, $T^*$ has SVEP at every $\lambda \notin \sigma_{gDRW^-}(T)$. Since $\sigma_{gDRW^-}(T) \subseteq \sigma_{lw}(T)$, $T^*$ has SVEP at every $\lambda \notin \sigma_{lw}(T)$, so a-Browder’s theorem holds for $T^*$, see [2, Theorem 4.34].

3) Suppose that Browder’s theorem holds for $T$ then $\sigma_{ub}(T) = \sigma_{uw}(T)$. Using [26, Theorems 2.6 and 2.3] we have

$$
\lambda \notin \sigma_{gD}(T) \iff T - \lambda I \text{ is generalized Drazin Riesz invertible}
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{ub}(T)
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc_{uw}(T)
\iff T - \lambda I \text{ is generalized Drazin Riesz Weyl}
\iff \lambda \notin \sigma_{gDRW}(T).
$$

Hence $\sigma_{gD}(T) = \sigma_{gDRW}(T)$. Conversely, if $\sigma_{gD}(T) = \sigma_{gDRW}(T)$, from Corollary 2.3, $T$ and $T^*$ has SVEP at every $\lambda \notin \sigma_{gDRW}(T)$. Since $\sigma_{gDRW}(T) \subseteq \sigma_{w}(T)$, $T$ has SVEP at every $\lambda \notin \sigma_{w}(T)$, so Browder’s theorem holds for $T$, see [2, Theorem 4.23].

\[]

It will be said that generalized Browder’s theorem holds for $T \in \mathcal{B}(X)$ if $\sigma_{by}(T) = \sigma(T) \setminus \Pi(T)$, equivalently, $\sigma_{by}(T) = \sigma_{w}(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of $T$ ([4]). A classical result of M. Amouch and H. Zguitti [9, Theorem 2.1] shows that Browder’s theorem and generalized Browder’s theorem are equivalent. According to the previous results, [6, Theorem 2.2], [3, Theorem 2.3] an the equivalent between Browder’s theorem and generalized Browder’s theorem [9, Theorem 2.1] [10][Proposition 2.2] we have the following theorem.
Theorem 2.6. Let $T \in \mathcal{B}(X)$. The statements are equivalent:
1) Browder’s theorem holds for $T$;
2) Browder’s theorem holds for $T^*$;
3) $T$ has SVEP at every $\lambda \notin \sigma_d(T)$;
4) $T^*$ has SVEP at every $\lambda \notin \sigma_d(T)$;
5) $T$ has SVEP at every $\lambda \notin \sigma_{BW}(T)$;
6) generalized Browder’s theorem holds for $T$;
7) $T$ or $T^*$ has SVEP at every $\lambda \notin \sigma_{aBW}(T)$;
8) $\sigma_{aDR}(T) = \sigma_{aDRW}(T)$;
9) $T$ or $T^*$ has SVEP at every $\lambda \notin \sigma_{aDRW}(T)$;
10) $\sigma_d(T) = \sigma_{BW}(T)$.

In the same way we have the following result.

Theorem 2.7. Let $T \in \mathcal{B}(X)$. The statements are equivalent:
1) a-Browder’s theorem holds for $T$;
2) generalized a-Browder’s theorem holds for $T$;
3) $T$ has SVEP at every $\lambda \notin \sigma_{aDR}(T)$;
4) $\sigma_{aDR}(T) = \sigma_{aDRW}(T)$;
5) $T$ has SVEP at every $\lambda \notin \sigma_{aDRW}(T)$;
6) $\sigma_{aDM}(T) = \sigma_{aDRW}(T)$.

We denote by $\sigma_{uf}(T)$ and $\sigma_{uf}(T)$, $T \in \mathcal{B}(X)$, respectively the lower and upper semi-Fredholm spectra. Note that $\sigma_{aDR\Phi_+}(T) \subset \sigma_{aDM}(T)$, $\sigma_{aDR\Phi_-}(T) \subset \sigma_{aDRQ}(T)$ and $\sigma_{aDR\Phi}(T) \subset \sigma_{aDR}(T)$ are strict [26]. In this case we have the following theorems:

Theorem 2.8. Let $T \in \mathcal{B}(X)$. The statements are equivalent:
1) $\sigma_{uf}(T) = \sigma_{ab}(T)$;
2) $T$ has SVEP at every $\lambda \notin \sigma_{uf}(T)$;
3) $T$ has SVEP at every $\lambda \notin \sigma_{aDR\Phi}(T)$;
4) $\sigma_{aDR}(T) = \sigma_{aDR\Phi}(T)$.

Proof. 1) $\iff$ 2): Suppose that $T$ has SVEP at every $\lambda \notin \sigma_{uf}(T)$. $\lambda \notin \sigma_{uf}(T)$, $T - \lambda I$ is upper semi-Fredholm. $T$ has SVEP at $\lambda$, then $a(T - \lambda I) < \infty$, see [1, Theorem 3.16]. So $\lambda \notin \sigma_{ab}(T)$. Now, Suppose that $\sigma_{uf}(T) = \sigma_{ab}(T)$. Let $\lambda \notin \sigma_{uf}(T)$, $\lambda \notin \sigma_{ab}(T)$ then $a(T - \lambda I) < \infty$, hence $T$ has SVEP at $\lambda$ by [1].

3) $\iff$ 4): Suppose that $T$ has SVEP at every $\lambda \notin \sigma_{aDR\Phi}(T)$. If $\lambda \notin \sigma_{aDR\Phi}(T)$, $T - \lambda I$ is generalized Drazin Riesz upper Fredholm, then there exists $(M, N) \in \text{Red}(T)$ such that $(T - \lambda I)_M$ is semi-regular and $(T - \lambda I)_N$ is Riesz. $T$ has SVEP at every $\lambda \notin \sigma_{aDR\Phi}(T)$ implies $(T - \lambda I)_M$ has the SVEP at $0$, it follows that $(T - \lambda I)_M$ is bounded below, see [18, Corollary 3.1.7]. Hence $T - \lambda I$ is generalized Drazin Riesz bounded below, $\lambda \notin \sigma_{aDR}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{aDR}(T) = \sigma_{aDR\Phi}(T)$. Conversely, assume that $\sigma_{aDR}(T) = \sigma_{aDR\Phi}(T)$. If $\lambda \notin \sigma_{aDR\Phi}(T)$ then $T - \lambda I$ is generalized Drazin Riesz bounded below so $T$ has the SVEP at $\lambda$, by [26, Theorem 2.4].

1) $\iff$ 4): Suppose that $\sigma_{uf}(T) = \sigma_{ab}(T)$.

According to [26, Theorems 2.4 and 2.6] we have

\[
\lambda \notin \sigma_{aDM}(T) \iff T - \lambda I \text{ is generalized Drazin Riesz bounded below} \\
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc \sigma_{ab}(T) \\
\iff T - \lambda I \text{ admits a GKRD and } \lambda \notin acc(\sigma_{uf}(T)) \\
\iff T - \lambda I \text{ is generalized Drazin Riesz Fredholm} \\
\iff \lambda \notin \sigma_{aDR\Phi}(T).
\]

Hence $\sigma_{aDR}(T) = \sigma_{aDR\Phi}(T)$. Conversely, if $\sigma_{aDR}(T) = \sigma_{aDR\Phi}(T)$, then by 3) $\iff$ 4), $T$ has SVEP at every $\lambda \notin \sigma_{aDR\Phi}(T)$. Since $\sigma_{aDR\Phi}(T) \subseteq \sigma_{uf}(T)$, $T$ has SVEP at every $\lambda \notin \sigma_{uf}(T)$, 1) $\iff$ 2) gives the result.

\[\Box\]
Theorem 2.9. Let \( T \in \mathcal{B}(X) \). The statements are equivalent:

1) \( \sigma_f(T) = \sigma_{\text{R}}(T); \)
2) \( T^* \) has SVEP at every \( \lambda \notin \sigma_f(T); \)
3) \( T^* \) has SVEP at every \( \lambda \notin \sigma_{\text{gD}}(T); \)
4) \( \sigma_{\text{gD}}(T) = \sigma_{\text{gDRB}}(T). \)

Proof. 1) \( \iff \) 2): Suppose that \( T^* \) has SVEP at every \( \lambda \notin \sigma_f(T). \) \( \lambda \notin \sigma_f(T) \) implies that \( T - \lambda I \) is lower semi-Fredholm. \( T^* \) has SVEP at \( \lambda \), then \( d(T - \lambda I) < \infty \), see [1, Theorem 3.17]. So \( \lambda \notin \sigma_{\text{R}}(T). \) Now, Suppose that \( \sigma_f(T) = \sigma_{\text{R}}(T). \) Let \( \lambda \notin \sigma_f(T), \lambda \notin \sigma_{\text{R}}(T) \) then \( d(T - \lambda I) < \infty \), hence \( T^* \) has SVEP at \( \lambda \) by [1].

3) \( \iff \) 4): Suppose that \( T^* \) has SVEP at every \( \lambda \notin \sigma_{\text{gD}}(T). \) If \( \lambda \notin \sigma_{\text{gD}}(T), \) \( T - \lambda I \) admits GKRD and \( \lambda \notin \text{acc}_{\text{bf}}(T) \) by [26, Theorem 2.6]. \( T^* \) has SVEP at every \( \lambda \notin \sigma_{\text{gD}}(T), \) so \( \lambda \notin \text{acc}_{\text{bf}}(T). \) Therefore, \( T - \lambda I \) is generalized Drazin Riesz surjective [26, Theorem 2.5], \( \lambda \notin \sigma_{\text{gD}}(T) \) and since the reverse implication holds for every operator we conclude that \( \sigma_{\text{gD}}(T) = \sigma_{\text{gDRB}}(T). \) Conversely, suppose that \( \sigma_{\text{gD}}(T) = \sigma_{\text{gDRB}}(T), \) if \( \lambda \notin \sigma_{\text{gDRB}}(T) \) then \( T^* \) has SVEP at every \( \lambda \notin \sigma_f(T), \) \( T \) has SVEP at every \( \lambda \notin \sigma_f(T), \) according to 1) \( \iff \) 2) we obtain the result. \( \square \)

As a direct consequence of the Theorems 2.8, 2.9 and [6, Corollary 2.1] we have the following corollary.

Corollary 2.10. Let \( T \in \mathcal{B}(X) \). The statements are equivalent:

1) \( \sigma_f(T) = \sigma_{\text{R}}(T); \)
2) \( T \) and \( T^* \) have SVEP at every \( \lambda \notin \sigma_f(T); \)
3) \( \sigma_{\text{bf}}(T) = \sigma_{\text{D}}(T); \)
4) \( T \) and \( T^* \) have SVEP at every \( \lambda \notin \sigma_{\text{bf}}(T); \)
5) \( \sigma_{\text{gD}}(T) = \sigma_{\text{bf}}(T). \)
6) \( T \) and \( T^* \) have SVEP at every \( \lambda \notin \sigma_{\text{bf}}(T); \)
7) \( \sigma_{\text{gD}}(T) = \sigma_{\text{gDRB}}(T). \)
8) \( T \) and \( T^* \) have SVEP at every \( \lambda \notin \sigma_{\text{gDRB}}(T); \)

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References