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On (α, ψ)-K-Contractions in the Extended b-Metric Space

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Abstract. In this paper, we introduce a notion of (α, ψ)-K-contraction in the setting of extended b-metric spaces and investigate the existence of a fixed point. The presented results generalize and unify a number of well-known fixed point theorem mainly in two distinct aspects; in the sense of the contraction conditions and in the frame of abstract spaces.

1. Introduction and Preliminaries

In 1993, Czerwik [16] suggested a successful and proper generalization of the metric space notion by introducing the concepts of b-metric space. In this paper, the author examine the basic topological properties of this new space and investigate the existence and uniqueness of certain mappings in framework of b-metric space. Following this famous result in the setting of b-metric space, a number of authors have reported several interesting results in this direction (see e.g. [2, 6–8],[11]-[14] and related references therein). Very recently, Kamran *et al.* [18] extend the b-metric space and successfully prove the analog of Banach mapping principle in this new space.

In this paper, we shall define a general contraction condition by the help of some auxiliary functions and investigate the existence and uniqueness of a fixed point for such mappings.

Throughout the manuscript, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the positive integers. Further, $\mathbb{R}$ represent the real numbers and $\mathbb{R}_0^+ := [0, \infty)$.

We, first, recall the notion of b-metric.

Definition 1.1 (Czerwik [16]). *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

$(b1)$ $d(x, y) = 0$ *if and only if* $x = y$.
$(b2)$ $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
$(b3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$, *where* $s \geq 1$.

The function d is called a b-metric and the space (X, d) is called a b-metric space, in short, bMS.

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The immediate examples of b-metric are the following (see also [2, 6–8],[11]-[14].)

Example 1.2. *Let $X = \mathbb{R}^2$. Then, the functional $d : X \times X \to [0, \infty)$ defined by:*

$$d((x_1, y_1), (x_2, y_2)) := \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & ((x_1, y_1), (x_2, y_2)) \in [0, 1] \times [0, 1] \\ |x_1 - x_2|^2 + |y_1 - y_2|^2, & ((x_1, y_1), (x_2, y_2))(1, \infty) \times (1, \infty) \\ 0, & \text{otherwise.} \end{cases}$$

It is a b-metric on X with coefficient $s = 2$.

Example 1.3. *The space $L^p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional*

$$d(x, y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L^p[0, 1],$$

is a b-metric space. Notice that $s = 2^{1/p}$.

Example 1.4. *Let X=$\{a, b, c\}$ and $d : X \times X \to \mathbb{R}_+$ such that $d(a, b) = d(b, a) = d(a, c) = d(c, a) = b, d(b, c) = d(c, b) = \alpha \geq c, d(a, a) = d(b, b) = d(c, c) = a$. Then*

$$d(x, y) \leq \frac{\alpha}{2}\left[d(x, z) + d(z, y)\right], \text{ for } x, y, z \in X.$$

Then (X, d) is a b-metric space. If $\alpha > c$ the ordinary triangle inequality does not hold and (X, d) is not a metric space.

Remark 1.5. *It is clear that for $s = 1$, the b-metric becomes a usual metric.*

In what follows, we recollect the notion of extend the b-metric space that is defined by Kamran *et al.* [18]

Definition 1.6. *[18] Let X be a non empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$i is called an extended $b-$ metric if for all $x, y, z \in X$ is satisfies*

($d_\theta 1$) *$d_\theta(x, y) = 0$ if and only if $x = y$;*

($d_\theta 2$) *$d_\theta(x, y) = d_\theta(y, x)$;*

($d_\theta 3$) *$d_\theta(x, z) \leq \theta(x, z)\left[d_\theta(x, y) + d_\theta(y, z)\right]$.*

The pair (X, d_θ) is called an extended $b-$metric space, in short extended-bMS.

Remark 1.7. *If $\theta(x, y) = s$, for $s \geq 1$ then we obtain the definition of bMS.*

Example 1.8. *Let $X = \{a, b, c\} \cup \mathbb{R}_0^+$ and $d : X \times X \to [0, \infty)$ be defined by*

Case 1. if $x, y \in \mathbb{R}_0^+$ then $d_\theta(x, y) = |x - y|^2$,
Case 2. if $x \in \{a, b, c\}$ and $y \in \mathbb{R}_0^+$ then $d_\theta(x, y) = 1 = d_\theta(y, x)$ and $d_\theta(x, x) = 0$,
Case 3. if $x, y \in \{a, b, c\}$

$$d_\theta(a, b) = 1, \quad d_\theta(a, c) = \frac{1}{2} \quad \text{and} \quad d_\theta(b, c) = 2,$$

with $d_\theta(x, x) = 0$ and $d_\theta(x, y) = d_\theta(y, x)$.

Notice that d is not a metric since $d_\theta(b, c) > d_\theta(b, a) + d_\theta(a, c)$. However, it is easy to see that d is a extended b-metric space. Indeed, for the following $\theta : X \times X \to [1, \infty)$, we conclude the desired result.

$$\theta(x, y) = \begin{cases} 2 & \text{if } x, y \in \mathbb{R}_0^+, \\ \frac{4}{3} & \text{if } x, y \in \{a, b, c\}, \\ 1 & \text{if } (x, y) \text{ or } (y, x) \in \{a, b, c\} \times \mathbb{R}_0^+. \end{cases} \tag{1}$$

Example 1.9. *Let $X = \{x, y, z\}$ and $\theta : X \times X \to [1, \infty)$, $\theta(x, y) = |x| + |y| + 2$. Define $d_\theta : X \times X \to [0, \infty)$ as*

$$d_\theta(x, y) = d_\theta(y, x) = 5,\ d_\theta(x, z) = d_\theta(z, x) = 3,\ d_\theta(y, z) = d_\theta(z, y) = 1,$$
$$d_\theta(x, x) = d_\theta(y, y) = d_\theta(z, z) = 0.$$

Obviously, $(d_\theta 1)$ and $(d_\theta 2)$ hold. For $(d_\theta 3)$, we have

$$5 = d_\theta(x, y) \leq \theta(x, y)(d_\theta(x, z) + d_\theta(z, y)) = (|x| + |y| + 2) \cdot 4,$$

$$3 = d_\theta(x, z) \leq \theta(x, z)(d_\theta(x, y) + d_\theta(y, z)) = (|x| + |z| + 2) \cdot 6,$$

$$1 = d_\theta(y, z) \leq \theta(y, z)(d_\theta(y, x) + d_\theta(x, z)) = (|y| + |z| + 2) \cdot 8,$$

In conclusion, for any $x, y, z \in X$,

$$d_\theta(x, z) \leq \theta(x, z) \left[d_\theta(x, y) + d_\theta(y, z)\right].$$

Hence, (X, d_θ) is an extended $b-$metric space. Notice also that

$$5 = d_\theta(x, y) > 4 = d_\theta(x, z) + d_\theta(z, y),$$

thus the standard triangle inequality does not hold in this case and (X, d) is not a metric space.

In what follows that we recollect some basic concepts, for instance, convergence, notion of the Cauchy sequence, and completeness in a extended-bMS. For more details, see e.g. [18].

Definition 1.10. *[18] Let (X, d_θ) be an extended-bMS.*

(*i*) *A sequence x_n in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.*

(*ii*) *A sequence x_n in X is said to be Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.*

Definition 1.11. *An extended-bmetric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.*

Lemma 1.12. *Let (X, d_θ) be an complete extended-bMS. If d_θ is continuous, then every convergent sequence has a unique limit.*

Theorem 1.13. *[18] Let (X, d_θ) be an extended-bMS such that d_θ is a continuous functional. Let $T : X \to X$ satisfy:*

$$d_\theta(Tx, Ty) \leq k d_\theta(x, y) \tag{2}$$

for all $x, y \in X$, where $k \in [0, 1)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1, 2, \ldots$. Then T has precisely one fixed point u. Moreover for each $y \in X$, $T^n y \to u$.

For our purposes, we need to recall the following definition of $\alpha-$orbital admissible mappings given by Popescu [29]

Definition 1.14. *Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is an $\alpha-$orbital admissible if for all $x, y \in X$ we have*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1. \tag{3}$$

Remark 1.15. *Each $\alpha-$admissible mapping is an $\alpha-$orbital admissible mapping.(see [29]).*

Let Φ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Φ_1) ϕ is nondecreasing;

(Φ_2) $\phi(t) < t$.

2. Main results

We start with the definition of (α,ψ)-K-contraction.

Definition 2.1. *Let (X,d) be an extended b-metric space $\alpha : X\times X \to [0,\infty)$ and $\theta : X\times X\to[1,\infty)$. A mapping $T : X \to X$ is called (α,ψ)-K-contraction if it satisfies*

$$\alpha(x,y)d_\theta(Tx,Ty) \le \phi(K(x,y)),\ \textit{for all } x,y\in X, \tag{4}$$

where $\phi\in\Phi$ and

$$K(x,y) = \max\{d_\theta(x,y), d_\theta(x,Tx), d_\theta(y,Ty), \frac{d_\theta(x,Tx)d_\theta(y,Ty)}{d_\theta(x,y)}, \frac{d_\theta(x,Ty)+d_\theta(y,Tx)}{2\max\{\theta(y,Tx),\theta(x,Ty)\}}\}. \tag{5}$$

The following is the first main result of this paper.

Theorem 2.2. *Let (X,d) be a complete extended b-metric space and $T : X\to X$ be a (α,ψ)-K-contraction mapping. Suppose that for each $x_0\in X$ and for each $t>0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n,x_m) < 1$$

where $x_n = T^n x_0$, $n\in\mathbb{N}$. Suppose also that

- (i) *T is α-orbital admissible,*
- (ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$*
- (iii) *T is continuous.*

Then the mappings T posses a fixed point u, that is, $Tu = u$.

Proof. By assumption, for a given $x_0\in X$, we have a constructive sequence $\{x_n\}$ that is defined by $x_n = T^n x_0$ for each $n\in\mathbb{N}$. If for some n_0, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T. From now on, we assume that $x_n\ne x_{n+1}$ for all $n\ge 0$. Since T is $\alpha-$admissible, we have

$$\alpha(x_0,x_1) = \alpha(x_0,Tx_0)\ge 1 \Rightarrow \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2)\ge 1.$$

Recursively, we find that

$$\alpha(x_n,x_{n+1})\ge 1,\ \text{for all } n = 0,1,\dots \tag{6}$$

On account of (6) and (4), we have

$$d_\theta(x_n,x_{n+1}) = d_\theta(Tx_{n-1},Tx_n) \le \phi(M(x_{n-1},x_n)),$$

where

$$\begin{aligned} K(x_{n-1},x_n) &= \max\Big\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},Tx_{n-1}), d_\theta(x_n,Tx_n),\\ &\qquad \frac{d_\theta(x_{n-1},Tx_{n-1})d_\theta(x_n,Tx_n)}{d_\theta(x_{n-1},x_n)}, \frac{d_\theta(x_n,Tx_{n-1})+d_\theta(x_{n-1},Tx_n)}{2\max\{\theta(y,Tx),\theta(x,Ty)\}}\Big\}\\ &= \max\Big\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1}),\\ &\qquad \frac{d_\theta(x_{n-1},x_n)d_\theta(x_n,x_{n+1})}{d_\theta(x_{n-1},x_n)}, \frac{d_\theta(x_{n-1},x_{n+1})}{2\max\{\theta(x_n,x_n),\theta(x_{n-1},x_{n+1})\}}\Big\}\\ &\le \max\left\{d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1}), \frac{d_\theta(x_n,x_{n+1})+d_\theta(x_{n-1},x_n)}{2}\right\}\\ &= \max\{d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1})\}. \end{aligned}$$

If for some n, we have $K(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) = d_\theta(x_n, x_{n+1})\}$, then

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),$$

a contradiction. Accordingly, we conclude, for all $n \geq 1$, that

$$K(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) = d_\theta(x_{n-1}, x_n).$$

We deduce that

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_{n-1}, x_n)) < d_\theta(x_{n-1}, x_n), \quad \forall\ n \geq 1. \tag{7}$$

We deduce

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi^n(d_\theta(x_0, x_1)), \quad \forall\ n \geq 0. \tag{8}$$

Therefore, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = L.$$

Letting $n \to \infty$ in (7), we get

$$L \leq \phi(L),$$

which holds unless $l = 0$. Thus

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{9}$$

We claim that $\{x_n\}$ is a Cauchy sequence. By using the modified triangle inequality $(b3)$ together with (7) and (8), we find that

$$\begin{aligned}
d_\theta(x_n, x_{n+k}) &\leq \theta(x_n, x_{n+k})[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+k})] \\
&\leq \theta(x_n, x_{n+k})d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + ... + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})d_\theta(x_{n+k-1}, x_{n+k}) \\
&\leq \theta(x_n, x_{n+k})\phi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})\phi^{n+1}(d_\theta(x_0, x_1)) \\
&\quad + ... + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})\phi^{n+k-1}(d_\theta(x_0, x_1)) \\
&\leq \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_n, x_{n+k})\phi^n(d_\theta(x_0, x_1)) \\
&\quad + \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})\phi^{n+1}(d_\theta(x_0, x_1)) \\
&\quad + ... + \cdots \\
&\quad + \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})\phi^{n+k-1}(d_\theta(x_0, x_1)) \\
&= \sum_{j=n}^{n+k-1} \phi^j(d_\theta(x_0, x_1)) \prod_{i=1}^{j} \theta(x_i, x_{n+k}).
\end{aligned}$$

We deduce that

$$d_\theta(x_n, x_{n+k}) \leq S_{n+k-1} - S_n, \tag{10}$$

for the series

$$\sum_{j=1}^{\infty} \phi^j(d_\theta(x_0, x_1)) \prod_{i=1}^{j} \theta(x_i, x_{n+k}).$$

Put $a_n = \phi^n(d_\theta(x_0, x_1)) \prod_{i=1}^{n} \theta(x_i, x_{n+k})$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\phi^{n+1}(d_\theta(x_0, x_1))}{\phi^j(d_\theta(x_0, x_1))} \theta(x_{n+1}, x_{n+k}).$$

In view of the assumption,

$$\limsup_{n\to\infty} \frac{\phi^{n+1}(t)}{\phi^j(t)} \theta(x_{n+1}, x_{n+k}) < 1,$$

the above series converges by ratio test. Consequently, in view of (10), we get

$$\lim_{n,m\to\infty} d_\theta(x_n, x_{n+k}) = 0, \tag{11}$$

that is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete extended b-metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} d_\theta(x_n, z) = 0. \tag{12}$$

Since the mapping T and the extended b-metric are continuous, we derive that

$$\lim_{n\to\infty} d_\theta(Tx_n, Tz) = 0 = \lim_{n\to\infty} d_\theta(x_{n+1}, Tz) = d_\theta(z, Tz). \tag{13}$$

Hence, we conclude that $Tz = z$. $\square$

In what follows, we refine the definition of (α, ψ)-K-contraction as (α, ψ)-M-contraction to remove the heavy condition, continuity, on the given self-mapping.

Definition 2.3. *Let (X, d) be an extended b-metric space $\alpha : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called (α, ψ)-M-*contraction *if it satisfies*

$$\alpha(x, y) d_\theta(Tx, Ty) \le \phi(M(x, y)), \text{ for all } x, y \in X, \tag{14}$$

where $\phi \in \Phi$ and

$$M(x, y) = \max\{d_\theta(x, y), \frac{d_\theta(x, Tx) + d_\theta(y, Ty)}{2}, \frac{d_\theta(x, Ty) + d_\theta(y, Tx)}{2\max\{\theta(y, Tx), \theta(x, Ty)\}}\}. \tag{15}$$

This is the second main result in which the continuity of the mapping is removed.

Theorem 2.4. *Let (X, d) be a complete extended b-metric space and $T : X \to X$ be a (α, ψ)-M-contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)} \theta(x_n, x_m) < 1$$

where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that

(*i*) *T is α-orbital admissible,*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$*
(*iii*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.*

Then the mappings T posses a fixed point u, that is, $Tu = u$.

Proof. Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (6) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (14), for all k, we get that

$$d_\theta(x_{n(k)+1}, Tu) = d_\theta(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u) d_\theta(Tx_{n(k)}, Tu) \leq \phi(M(x_{n(k)}, u)). \tag{16}$$

On the other hand, we have

$$M(x_{n(k)}, u) = \max\left\{d_\theta(x_{n(k)}, u), \frac{d_\theta(x_{n(k)}, x_{n(k)+1}) + d_\theta(u, Tu)}{2}, \frac{d_\theta(x_{n(k)}, Tu) + d_\theta(u, x_{n(k)+1})}{2\max\{\theta(y, Tx), \theta(x, Ty)\}}\right\}.$$

Letting $k \to \infty$ in the above equality, we get that

$$\lim_{k\to\infty} M(x_{n(k)}, u) = \frac{d_\theta(u, Tu)}{2}. \tag{17}$$

Suppose that $d_\theta(u, Tu) > 0$. From (17), for k large enough, we have $M(x_{n(k)}, u) > 0$, which implies that $\phi(M(x_{n(k)}, u)) < M(x_{n(k)}, u)$. Thus, from (16), we have

$$d_\theta(x_{n(k)+1}, Tu) < M(x_{n(k)}, u).$$

Letting $k \to \infty$ in the above inequality, using (17), we obtain that

$$d_\theta(u, Tu) \leq \frac{d_\theta(u, Tu)}{2},$$

which is a contradiction. Thus we have $d_\theta(u, Tu) = 0$, that is, $u = Tu$. □

For the uniqueness of a fixed point of a (α, ψ)-K-contractive mapping (respectively, (α, ψ)-M-contractive mapping), we shall suggest the following hypothesis.

(U) For all $x, y \in \mathrm{Fix}(T)$, either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.5. *Adding condition (U) to hypotheses of Theorem 2.2 (respectively, Theorem 2.4), we obtain uniqueness of the fixed point of T.*

Proof. Suppose, on the contrary, that u and v are two distinct fixed points of T. Then we have $K(u, v) = d(u, v)$ (respectively, $M(u, v) = d(u, v)$). On account of the hypothesis of (U), we employ the contraction condition (14)

$$\begin{aligned}
&\phi(d(u, v)) = \phi(d(Tu, Tv)) \\
&\leq \alpha(u, v)\phi(d(Tu, Tv)) \\
&\leq \phi(K(u, v)) \\
&< \phi(d(u, v)),
\end{aligned}$$

which is a contradiction. Hence, we conclude that the obtained fixed points are unique in Theorem 2.2 and Theorem 2.4. □

Definition 2.6. *Let (X, d) be an extended b-metric space $\alpha : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called α-*contraction *if it satisfies*

$$\alpha(x, y) d_\theta(Tx, Ty) \leq \phi(d_\theta(x, y)), \text{ for all } x, y \in X, \tag{18}$$

where $\phi \in \Phi$.

Corollary 2.7. *Let (X, d) be a complete extended b-metric space and $T : X \to X$ be a α-contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n, x_m) < 1$$

where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that

(i) *T is α-orbital admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
(iii) *T is continuous.*

or

(iii)∗ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then the mappings T posses a fixed point u, that is, $Tu = u$. If, additionally, we assume the condition (U), then u is the unique fixed point of T.

Definition 2.8. *Let (X, d) be an extended b-metric space $\alpha : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called* α-Jaggi-type contraction *if it satisfies*

$$\alpha(x, y)d_\theta(Tx, Ty) \leq \phi(\frac{d_\theta(x, Tx)d_\theta(y, Ty)}{d_\theta(x, y)}), \text{ for all } x, y \in X, \tag{19}$$

where $\phi \in \Phi$.

Corollary 2.9. *Let (X, d) be a complete extended b-metric space and $T : X \to X$ be a α-Jaggi-type contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n, x_m) < 1$$

where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that

(i) *T is α-orbital admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
(iii) *T is continuous.*

Then the mappings T posses a fixed point u, that is, $Tu = u$. If, additionally, we assume the condition (U), then u is the unique fixed point of T.

3. Conclusion

One can easily drive several consequences from the presented main results in this paper in different aspects. For example, letting $\theta(x, y) = s \geq 1$ yields the corresponding fixed point results in the context of b-metric space. Moreover, the standard versions of the given results are follows when we take $\theta(x, y) = 1$. As in Corollary 2.7 and Corollary 2.9, we can derive more results by replacing $K(x, y)$ with a proper one. On the other hand, as in [19], by assign $\alpha(x, y)$ in a proper way, we can conclude results in the frame of "partially ordered spaces" and for "cyclic contraction."

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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