Selection Principles in Function Spaces
with the Compact-Open Topology

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Abstract. For a Tychonoff space $X$, we denote by $C_k(X)$ the space of all real-valued continuous functions on $X$ with the compact-open topology. A subset $A \subset X$ is said to be sequentially dense in $X$ if every point of $X$ is the limit of a convergent sequence in $A$. In this paper, the following properties for $C_k(X)$ are considered:

\[ S_1(S, S) \Rightarrow S_{fin}(S, S) \Rightarrow S_{fin}(S, D) \Rightarrow S_1(S, D) \]

For example, a space $C_k(X)$ satisfies $S_1(S, D)$ (resp., $S_{fin}(S, D)$) if whenever $(S_n : n \in \mathbb{N})$ is a sequence of sequentially dense subsets of $C_k(X)$, one can take points $f_n \in S_n$ (resp., finite $F_n \subset S_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup \{F_n : n \in \mathbb{N}\}$) is dense in $C_k(X)$. Other properties are defined similarly.

In [22], we obtained characterizations these selection properties for $C_k(X)$. In this paper, we give characterizations for $C_k(X)$.

1. Introduction

For a Tychonoff space $X$, we denote by $C_k(X)$ the space of all real-valued continuous functions on $X$ with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A, U] = \{f \in C(X) : f(A) \subset U\}$, where $A$ is a compact subset of $X$ and $U$ is a non-empty open subset of $\mathbb{R}$. Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of $X$, we can represent a basic neighborhood of the point $f \in C(X)$ as $\langle f, A, \epsilon \rangle$ where $\langle f, A, \epsilon \rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \forall x \in A\}$, $A$ is a compact subset of $X$ and $\epsilon > 0$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set $X$. Then:

- $S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n \in \mathbb{N}}$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

- $S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n \in \mathbb{N}}$ of finite sets such that for each $n$, $B_n \subset A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

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there are finite sets $F$ convergent sequences to $X$ separable, if $X$ is called sequentially separable if it has a countable sequentially dense set. Call $X$ is finite.

A cover $U$ of a space $X$ is called:

- an $\omega$-cover (a $k$-cover) if each finite (compact) subset $C$ of $X$ is contained in an element of $U$;
- a $\gamma$-cover (a $\gamma_k$-cover) if $U$ is infinite and for each finite (compact) subset $C$ of $X$ the set $\{U \in U : C \not\subseteq U\}$ is finite.

Note that a $\gamma_k$-cover is a $k$-cover, and a $k$-cover is infinite. A compact space has no $k$-covers.

A space $X$ is said to be a $\gamma_k$-set if each open $k$-cover $U$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$ [9].

In a series of papers it was demonstrated that $\gamma$-covers and $k$-covers play a key role in function spaces [8–10, 13, 16, 22–26, 28] and many others. We continue to investigate applications of $k$-covers in function spaces with the compact-open topology.

2. Main Definitions and Notation

If $X$ is a topological space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space $X$ is called sequentially separable if it has a countable sequentially dense set. Call $X$ strongly sequentially separable, if $X$ is separable and every countable dense subset of $X$ is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

For a topological space $X$ we denote:

- $\mathcal{O}$ — the family of open covers of $X$;
- $\Gamma$ — the family of open $\gamma$-covers of $X$;
- $\Gamma_k$ — the family of open $\gamma_k$-covers of $X$;
- $\Omega$ — the family of open $\omega$-covers of $X$;
- $\mathcal{K}$ — the family of open $k$-covers of $X$;
- $\mathcal{K}_c^\omega$ — the family of countable co-zero $k$-covers of $X$;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{D}^\omega$ — the family of countable dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathbb{K}(X)$ — the family of all non-empty compact subsets of $X$.

- A space $X$ is $R$-separable, if $X$ satisfies $S_1(D, D)$ (Def. 47, [2]).
- A space $X$ is $M$-separable (selective separability), if $X$ satisfies $S_{fin}(D, D)$.
- A space $X$ is selectively sequentially separable, if $X$ satisfies $S_{fin}(S, S)$ (Def. 1.2, [3]).

For a topological space $X$ we have the next relations of selectors for sequences of dense sets of $X$.

$$S_1(S, S) \Rightarrow S_{fin}(S, S) \Rightarrow S_{fin}(S, D) \equiv S_1(S, D)$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$S_1(D, S) \Rightarrow S_{fin}(D, S) \Rightarrow S_{fin}(D, D) \equiv S_1(D, D)$$

Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ converges to $x$, $x = \lim A$, if $A$ is infinite, $x \not\in A$, and for each neighborhood $U$ of $x$, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \\setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subseteq A$ converging to $x$. So, simply $\Gamma_x$ may be the set of non-trivial convergent sequences to $x$. 

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $U_1, U_2, \ldots \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $F_n \subseteq U_n, n \in \mathbb{N}$, such that $\bigcup \{F_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [29].

$\mathcal{A} \equiv \mathcal{B}$: For each $U \in \mathcal{A}$ there exists $V \subseteq U$ such that $V \in \mathcal{B}$. Then $S_{fin}(\mathcal{A}, \mathcal{B})$ implies $\mathcal{A} \equiv \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $U$ is a cover of $X$ if $X = \bigcup U$ and $x \not\in U$.
We write \( \Pi(\mathcal{A}_x, \mathcal{B}_x) \) without specifying \( x \), we mean \((\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)\).

So, we have three types of topological properties described through the selection principles:

- local properties of the form \( S_1(\Phi, \Psi) \);
- global properties of the form \( S_1(\Phi, \Psi) \);
- semi-local properties of the form \( S_1(\Phi, \Psi) \).

Our main goal is to describe the topological properties for sequences of dense sets of \( C(X) \) in terms of selection principles of \( X \).

3. \( S_1(\mathcal{D}, \mathcal{S}) \)

Recall that \( X \) a \( \gamma_k \)-set if it satisfies the selection hypothesis \( S_1(\mathcal{K}, \Gamma_k) \) \([9]\).

**Theorem 3.1.** \([111]\) For a Tychonoff space \( X \) the following statements are equivalent:

1. \( C(X) \) satisfies \( S_1(\Omega_0, \Gamma_0) \) (i.e., \( C(X) \) is strongly Fréchet-Urysohn);
2. \( X \) is a \( \gamma_k \)-set.

Recall that the \( i \)-weight \( iw(X) \) of a space \( X \) is the smallest infinite cardinal number \( \tau \) such that \( X \) can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than \( \tau \).

**Theorem 3.2.** (Noble \([19]\)) A space \( C(X) \) is separable if and only if \( iw(X) = \aleph_0 \).

**Theorem 3.3.** For a Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{S}) \);
2. Every dense subset of \( C(X) \) is sequentially dense;
3. \( X \) satisfies \( S_1(\mathcal{K}, \Gamma_1) \) (\( X \) is a \( \gamma_k \)-set);
4. \( X \) is a \( \gamma_k \)-set;
5. \( C(X) \) is Fréchet-Urysohn;
6. \( C(X) \) satisfies \( S_{fin}(\mathcal{D}, \mathcal{S}) \);
7. \( X \) satisfies \( S_{fin}(\mathcal{K}, \Gamma_1) \);
8. Each finite power of \( X \) satisfies \( S_{fin}(\mathcal{K}, \Gamma_k) \);
9. \( C(X) \) satisfies \( S_1(\Omega_0, \Gamma_0) \);
10. \( C(X) \) satisfies \( S_1(\mathcal{D}, \Gamma_0) \).

**Proof.** (1) \( \Rightarrow \) (6) is immediate.

(4) \( \Leftrightarrow \) (5) By Theorem 4.7.4 in \([17]\).

(3) \( \Leftrightarrow \) (4) By Theorem 18 in \([4]\).

(5) \( \Leftrightarrow \) (7) By Theorem 5 in \([9]\).

(3) \( \Leftrightarrow \) (8) By Theorem 7 in \([9]\).

(3) \( \Leftrightarrow \) (9) By Theorem 3.1.

(9) \( \Rightarrow \) (10) is immediate.

(6) \( \Rightarrow \) (2) Let \( D \) be a dense subset of \( C(X) \). By \( S_{fin}(\mathcal{D}, \mathcal{S}) \), for sequence \( (D_i : D_i = D \text{ and } i \in \mathbb{N}) \) there is a sequence \( (K_i : i \in \mathbb{N}) \) such that for each \( i \), \( K_i \) is finite, \( K_i \subset D_i \), and \( \bigcup_{i \in \mathbb{N}} K_i \) is a countable sequentially dense subset of \( C(X) \). It follows that \( D \) is a sequentially dense subset of \( C(X) \).

(2) \( \Rightarrow \) (4) Let \( \mathcal{U} \) be an open \( k \)-cover of \( X \). Note that the set \( \mathcal{D} := \{ f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1 \text{ for some } U \in \mathcal{U} \} \) is dense in \( C(X) \), hence, it is sequentially dense. Take \( f_n \in \mathcal{D} \) such that \( f_n \upharpoonright 0 \). Let \( f_n \upharpoonright (X \setminus U_n) \equiv 1 \) for some \( U_n \in \mathcal{U} \). Then \( \{ U_n : n \in \mathbb{N} \} \) is a \( \gamma_k \)-subcover of \( \mathcal{U} \), because of \( f_n \upharpoonright 0 \). Hence, \( X \) is a \( \gamma_k \)-set.

(3) \( \Rightarrow \) (1) Let \( (D_{i,j} : i, j \in \mathbb{N}) \) be a sequence of dense subsets of \( C(X) \) and let \( D = \{ f_i : i \in \mathbb{N} \} \) be a countable dense subset of \( C(X) \).

For every \( i, j \in \mathbb{N} \) consider \( \mathcal{U}_{i,j} = \{ U_{b_{i,j}} : U_{b_{i,j}} = (f_{i,j} - h)^{-1}(-\frac{1}{2}, \frac{1}{2}) \text{ for } h \in D_{i,j} \} \). Note that \( \mathcal{U}_{i,j} \) is an \( k \)-cover of \( X \) for every \( i, j \in \mathbb{N} \). Since \( X \) a \( \gamma_k \)-set, there is a sequence \( (U_{b_{i,j}} : i, j \in \mathbb{N}) \) such that \( U_{b_{i,j}} \in \mathcal{U}_{i,j} \) and \( \{ U_{b_{i,j}} : i, j \in \mathbb{N} \} \) is an element of \( \mathcal{K}_i \). Claim that \( \{ h_{i,j} : i, j \in \mathbb{N} \} \) is a dense subset of \( C(X) \). Fix \( g \in C(X) \)
and a base neighborhood \( W = \{ (g,A) : g \} \) of \( g \), where \( A \) is a compact subset of \( X \) and \( \epsilon > 0 \). There are \( f_i \in D \) and \( j \in \mathbb{N} \) such that \( \langle f_i, A, \frac{1}{j} \rangle \subseteq W \). Since \( \{ U_{\ell,i,j} : i,j \in \mathbb{N} \} \) is an element of \( \Gamma \), there is \( j' > j \) such that \( A \subseteq U_{\ell,i,j'} \), hence, \( h(i,j') \in \{ f_i, A, \frac{1}{j'} \} \subseteq W \).

Since \( C_\mu(X) \) is Fréchet-Urysohn, every dense subset of \( C_\mu(X) \) is sequentially dense. It follows that \( \{ h(i,j) : i,j \in \mathbb{N} \} \) is sequentially dense.

(10) \( \Rightarrow \) (3) Let \( \{ U_i : i \in \mathbb{N} \} \subseteq \mathcal{K} \) and let \( D = \{ d_j : j \in \mathbb{N} \} \) be a countable dense subset of \( C_\mu(X) \). Consider \( D_1 = \{ f_{k,U_i,j} \in C(X) : f_{k,U_i,j} \upharpoonright K \equiv d_j, f_{k,U_i,j} \upharpoonright (X \setminus U) \equiv 1 \} \) where \( K \in \mathcal{K}(X) \), \( K \subseteq U \subseteq \mathcal{U} \) for every \( i \in \mathbb{N} \). Since \( D \) is a dense subset of \( C_\mu(X) \), then \( D_1 \) is a dense subset of \( C_\mu(X) \) for every \( i \in \mathbb{N} \). By (10), there is a set \( \{ f_{k(0,U(0,i,j),0)} : i \in \mathbb{N} \} \) such that \( f_{k(0,U(0,i,j),0)} \in D_1 \) and \( \{ f_{k(0,U(0,i,j),0)} : i \in \mathbb{N} \} \in \Gamma_\mu \). Claim that a set \( \{ U(i) : i \in \mathbb{N} \} \in \Gamma_\mu \). Let \( K \in \mathcal{K}(X) \) and let \( W = [K, (-1, \frac{1}{2})] \) be a base neighborhood of \( 0 \). Since \( \{ f_{k(0,U(0,i,j),0)} : i \in \mathbb{N} \} \in \Gamma_\mu \), there is \( i' \in \mathbb{N} \) such that \( f_{k(0,U(0,i',j),0)} \in W \) for every \( i > i' \). It follows that \( K \subseteq U(i) \) for every \( i > i' \) and, hence, \( \{ U(i) : i \in \mathbb{N} \} \in \Gamma_\mu \).

Let \( S \subseteq \mathcal{K}(X) \). An open cover \( \mathcal{U} \) of a space \( X \) is called:
- a \( s \)-cover if each \( C \in S \) is contained in an element of \( \mathcal{U} \);
- a \( \gamma_s \)-cover if \( \mathcal{U} \) is finite and for each \( C \in S \) the set \( \{ U \in \mathcal{U} : C \subseteq U \} \) is finite.

**Definition 3.4.** Let \( S \subseteq \mathcal{K}(X) \). A space \( X \) is called a \( \gamma_s \)-set if each \( s \)-cover of \( X \) contains a sequence which is a \( \gamma_s \)-cover of \( X \).

**Definition 3.5.** A space \( X \) is called a \( \gamma_{s^*} \)-set if each countable cozero \( k \)-cover \( \mathcal{U} \) of \( X \) contains a set \( \{ U_n : n \in \mathbb{N} \} \) which is a \( \gamma_{s^*} \)-cover of \( X \).

For a mapping \( f : X \to Y \) we will denote by \( f(k) = \{ f(K) : K \in \mathcal{K}(X) \} \).

**Theorem 3.6.** For a Tychonoff space \( X \) with \( \operatorname{iw}(X) = \mathcal{N}_{10} \), the following statements are equivalent:
1. \( C_\mu(X) \) satisfies \( S_1(\mathcal{D}_{s^*},S) \);
2. \( C_\mu(X) \) is strongly sequentially separable;
3. \( X \) is a \( \gamma_{s^*} \)-set;
4. \( X \) satisfies \( S_1(\mathcal{K}_{s^*},1) \);
5. for every a condensation (one-to-one continuous mapping) \( f : X \to Y \) from the space \( X \) on a separable metric space \( Y \), the space \( Y \) is a \( \gamma_{s^*} \)-set.

**Proof.** (3) \( \Rightarrow \) (5) Let \( f \) be a condensation \( f : X \to Y \) from the space \( X \) on a separable metric space \( Y \). If \( \mu \) is a \( (k) \)-cover of \( Y \), then there is \( \mu' \subseteq \mu \) such that \( \mu' \) is a \( (k) \)-cover of \( Y \) and \( |\mu'| = \mathcal{N}_{10} \). The family \( f^{-1}(\mu') = \{ f^{-1}(V) : V \subseteq \mu' \} \) is a countable co-zero \( k \)-cover of \( X \). By the argument that \( X \) is a \( \gamma_{s^*} \)-set, we have that \( Y \) is a \( \gamma_{s^*} \)-set.

The remaining implications follow from the proofs of Theorem 3.3 and Theorem 18 in [4]. □

**Corollary 3.7.** For a separable metrizable space \( X \), the following statements are equivalent:
1. \( C_\mu(X) \) satisfies \( S_1(\mathcal{D},S) \);
2. Every dense subset of \( C_\mu(X) \) is sequentially dense;
3. \( C_\mu(X) \) is strongly sequentially separable;
4. \( C_\mu(X) \) is a Fréchet-Urysohn;
5. \( C_\mu(X) \) is metrizable and separable;
6. \( X \) satisfies \( S_1(\mathcal{K},1) \);
7. \( X \) satisfies \( S_1(\mathcal{K},\mathcal{K}) \);
8. \( X \) satisfies \( S_{fin}(\mathcal{K},\mathcal{K}) \);
9. \( X \) is a hemicompact.

**Proof.** By Theorem 3.3 and Theorem 6 in [4]. □
A space $X$ is called a $k$-Lindelöf space if for each open $k$-cover $\mathcal{U}$ of $X$ there is a $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is countable and $\mathcal{V} \in \mathcal{K}$. Each $k$-Lindelöf space is Lindelöf, so normal, too.

**Lemma 3.8.** ([17]) $C_k(X)$ has countable tightness if and only if $X$ is $k$-Lindelöf.

By Theorem 3.3, Theorem 3.6 and Lemma 3.8 we have

**Theorem 3.9.** For a Tychonoff space $X$ with $iw(X) = N_0$ the following statements are equivalent:

1. $C_k(X)$ is Fréchet-Urysohn;
2. $C_k(X)$ is strongly sequentially separable and has countable tightness;
3. $X$ satisfies $S_1(\mathcal{K}^c, \Gamma_k)$ and is $k$-Lindelöf;
4. Every dense subset of $C_k(X)$ contains a countable sequentially dense subset of $C_k(X)$.

In Doctoral Dissertation, A.J. March considered the following problem (Problem 117 in [15]): Is it possible to find a space $X$ such that $C_k(X)$ is strongly sequentially separable but $C_k(X)^2$ is not strongly sequentially separable?

We get a negative answer to this question.

**Proposition 3.10.** Suppose $X$ has the property $S_1(\mathcal{K}^c, \Gamma_k)$. Then $X \sqcup X$ has the property $S_1(\mathcal{K}^c, \Gamma_k)$.

**Proof.** Let $\mathcal{K} = \{U_i : i \in \mathbb{N}\}$ be a countable $k$-cover of $X \sqcup X$ by cozero sets. Let $X_i \subseteq X = X_1 \sqcup X_2$ where $X_i = X$ for $i = 1, 2$. Consider $\mathcal{V}_1 = \{U^{1}_i \cap X_1 : X_1 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ and $\mathcal{V}_2 = \{U^{2}_i = U_i \cap X_2 : X_2 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ as families of subsets of the space $X$. Define $\mathcal{V} := \{U^{1}_i \cap U^{2}_i : U_i^{1} \in \mathcal{V}_1 \text{ and } U_i^{2} \in \mathcal{V}_2\}$. Note that $\mathcal{V}$ is a countable $k$-cover of $X$ by cozero sets. By Theorem 18 in [4], there is $\{U^{1}_i \cap U^{2}_i : n \in \mathbb{N}\} \subseteq \mathcal{V}$ such that $\{U^{1}_i \cap U^{2}_i : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X$. It follows that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X \sqcup X$. \hfill \Box

**Theorem 3.11.** For a Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ is strongly sequentially separable;
2. $(C_k(X))^n$ is strongly sequentially separable for each $n \in \mathbb{N}$.

**Proof.** By Theorem 3.6, Proposition 9.1 and the argument that $C_k(X \sqcup X) = C_k(X) \times C_k(X)$. \hfill \Box

A.J. March considered the problem (Problem 116 in [15]): Is it possible to find spaces $X, Y$ such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable but $C_k(X) \times C_k(Y)$ is not strongly sequentially separable?

A. Miller constructed the following example [18].

**Example 3.12.** There exist disjoint subsets of the plane $X$ and $Y$ such that both $X$ and $Y$ are $\gamma_k$-sets but $X \cup Y$ is not. Let $X$ be the open disk of radius one, i.e., $X = \{(x, y) : x^2 + y^2 < 1\}$, and $Y$ be any singleton on the boundary of $X$, e.g., $Y = \{(1, 0)\}$.

Thus, we have the example of the subsets of the plane $X$ and $Y$ such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable, but $C_k(X \cup Y)$ is not.

Note that (in contrast to the $C_p$-theory) $C_k(X \cup Y) \neq C_k(X) \times C_k(Y)$,

In [4], the authors considered the next problem (Problem 21 in [4]): Is the class of $\gamma_k$-sets closed for finite unions?

A particular answer to this problem and March’s problem is the following.

**Theorem 3.13.** Suppose that $X$ and $Y$ are $\gamma_k$-sets, $iw(X) = iw(Y) = N_0$ and $Y$ is first-countable. Then $X \sqcup Y$ is a $\gamma_k$-set.

**Proof.** By Theorem 3.6, $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable. Notice that each hemicompact space belong to the class $S_1(\mathcal{K}, \Gamma_k)$, and the converse holds for first countable spaces [16]. It follows that $C_k(Y)$ is a separable metrizable (first countable) space. By Theorem 9 in [6], $C_k(X) \times C_k(Y)$ is strongly sequentially separable. Since $C_k(X) \times C_k(Y) = C_k(X \sqcup Y)$ and, by Theorem 3.6, we have that $X \sqcup Y$ is a $\gamma_k$-set. \hfill \Box

**Corollary 3.14.** The product $C_k(X) \times C_k(Y)$ of strongly sequentially separable space $C_k(X)$ and strongly sequentially separable first-countable space $C_k(Y)$ belongs to the class of strongly sequentially separable spaces.
4. $S_1(D, D)$

In [10] it was shown that a Tychonoff space $X$ belongs to the class $S_1(K, K)$ if and only if $C_k(X)$ has countable strong fan tightness (i.e. for each $f \in C_k(X)$, $S_1(\Omega_f, \Omega_f)$ holds [27]).

Lj.D.R. Kočinac proved the next

**Theorem 4.1.** ([4, Theorem 6]) For a first countable Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ is first countable;
2. $C_k(X)$ has countable strong fan tightness;
3. $C_k(X)$ has countable fan tightness;
4. $X$ is locally compact Lindelöf space;
5. $X$ satisfies $S_1(K, K)$;
6. $X$ satisfies $S_{\text{fin}}(K, K)$;

We consider the generalizations (Theorem 4.2 and Theorem 5.3) of the Theorem 4.1 to the class of Tychonoff spaces with $\omega(X) = \aleph_0$.

**Theorem 4.2.** For a Tychonoff space $X$ with $\omega(X) = \aleph_0$ the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(D, D)$;
2. $X$ satisfies $S_1(K, K)$;
3. Each finite power of $X$ satisfies $S_1(K, K)$;
4. $C_k(X)$ satisfies $S_1(\Omega_0, \Omega_0)$ [countable strong fan tightness];
5. $C_k(X)$ satisfies $S_{\text{fin}}(D, \Omega_0)$.

**Proof.** (2) $\iff$ (3) By Theorem 5 in [14].

(2) $\implies$ (4) By Theorem 2.2 in [10].

(1) $\implies$ (2) Let $K_i \subseteq K$ for every $i \in \mathbb{N}$ and let $D$ be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K_i,L} \in C(X) : f(X \setminus L) \equiv 1$ and $f|K = d$ where $K$ is a compact subset of $X$, $U \in K_i$ such that $K \subseteq U$ and $d \in D\}$. Since $D$ is a dense subset of $C_k(X)$, we have that $D_i$ is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (1), there is a sequence $\{f_{K_i,L}, \epsilon_i\}_{i \in \mathbb{N}}$ such that for each $i$, $f_{K_i,L_i} \in D_i$, and $\{f_{K_i,L_i} : i \in \mathbb{N}\}$ is a dense subset of $C_k(X)$. Note that $U_i \in K_i$ for each $i \in \mathbb{N}$ and $\{U_i : i \in \mathbb{N}\} \subseteq K$.

(2) $\implies$ (1) Let $(D_{ij} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ and let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. For every couple $(i, j)$, $i, j \in \mathbb{N}$ and $f \in D_{ij}$ consider $K_{i,f} = \{x \in X : |f(x) - d_j(x)| < \frac{1}{2}\}$ and $K_i = \{K_{i,f} : f \in D_{ij}\}$. We claim that $K_{i,j} \subseteq K$ for every couple $(i, j)$, $i, j \in \mathbb{N}$. Let $K \in K(X)$ and $(d_{ij}, K_j)$ a base neighborhood of $d_j$. Since $D_{ij}$ is a dense subset of $C_k(X)$, there is $f \in D_{ij}$ such that $f \in (d_j, K_j)$, hence, $K \subseteq K_{i,f}$. Fix $j \in \mathbb{N}$ by (2), there is a family $\{K_{i,f} : i \in \mathbb{N}\}$ such that $K_{i,f} \subseteq K_{i,j}$ and $\{K_{i,f} \subseteq K : i \in \mathbb{N}\} \subseteq K$. So $f(i,j) \in D_{ij}$ for $i, j \in \mathbb{N}$. Claim that $\{f(i,j) : i, j \in \mathbb{N}\}$ is dense in $C_k(X)$. Let $p \in C(X), K \in K(X), \epsilon > 0$ and let $(p, K, \epsilon)$ be a base neighborhood of $p$. There is $f \in \mathbb{N}$ such that $f \in (p, K, \frac{\epsilon}{3})$. Since $\{K_{i,f} : i \in \mathbb{N}\} \subseteq K$, there is $f \in \mathbb{N}$ such that $K \subseteq K_{i,f} \subseteq (p, K, \epsilon)$ and $\frac{1}{2} < \frac{\epsilon}{3}$. It follows that $|f(i', j')(x) - p(x)| < |f(i', j') - f(i, j)| + |d_j(x) - p(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ for every $x \in K$. Hence, $f(i', j') \in (p, K, \epsilon)$ and $(f(i,j) : i, j \in \mathbb{N})$ is dense in $C_k(X)$.

(4) $\implies$ (5) is immediate.

(5) $\implies$ (1) Let $(D_{ij} : i \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ for each $j \in \mathbb{N}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. By (5), for every $j \in \mathbb{N}$ there is a family $\{d_{ij} : i \in \mathbb{N}\}$ such that $d_{ij} \in D_{ij}$ and $\{d_{ij} : i \in \mathbb{N}\} \subseteq \Omega_{d_j}$. Note that $\{d_{ij} : i, j \in \mathbb{N}\} \subseteq D$. \qed
5. \( S_{\text{fin}}(D, D) \)

According to [11] \( X \) belongs to \( S_{\text{fin}}(K, K) \) if and only if \( C_\omega(X) \) has countable fan tightness (i.e., for each \( f \in C_\omega(X), S_{\text{fin}}(\Omega_f, \Omega_f) \) holds [1]).

**Theorem 5.1.** For a Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C_\omega(X) \) satisfies \( S_{\text{fin}}(D, D) \);
2. \( X \) satisfies \( S_{\text{fin}}(K, K) \);
3. Each finite power of \( X \) satisfies \( S_{\text{fin}}(K, K) \).
4. \( C_\omega(X) \) satisfies \( S_{\text{fin}}(\Omega_0, \Omega_0) \) (countable fan tightness);
5. \( C_\omega(X) \) satisfies \( S_{\text{fin}}(D, \Omega_0) \).

**Proof.** (2) \( \iff \) (3) By Theorem 6 in [14].

(2) \( \iff \) (4) see in [11].

The remaining implications are proved similarly to the proof of Theorem 4.2. \( \Box \)

**Remark 5.2.** It is easy to see that every hemicompact space is in the class \( S_1(K, K) \) and, thus, in \( S_{\text{fin}}(K, K) \). By Proposition 5 in [4], the converse is also true in the class of first countable spaces.

**Corollary 5.3.** For a first countable Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C_\omega(X) \) satisfies \( S_1(D, D) \);
2. \( C_\omega(X) \) satisfies \( S_{\text{fin}}(D, D) \);
3. \( X \) satisfies \( S_1(K, K) \).

6. \( S_1(S, D) \)

**Definition 6.1.** A \( \gamma_k \)-cover \( \mathcal{U} \) of co-zero sets of \( X \) is \( \gamma_k \)-shrinkable if there exists a \( \gamma_k \)-cover \( \{ F(U) : U \in \mathcal{U} \} \) of zero-sets of \( X \) with \( F(U) \subseteq U \) for every \( U \in \mathcal{U} \).

For a topological space \( X \) we denote:

- \( \Gamma_k^{sh} \) — the family of \( \gamma_k \)-shrinkable covers of \( X \).

**Theorem 6.2.** For a Tychonoff space \( X \) the following statements are equivalent:

1. \( C_\omega(X) \) satisfies \( S_1(\Gamma_0, \Omega_0) \);
2. \( X \) satisfies \( S_1(\Gamma_0^{sh}, K) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( C_\omega(X) \) satisfies \( S_1(\Gamma_0, \Omega_0) \) and \( \{ F_i : i \in \mathbb{N} \} \subseteq \Gamma_0^{sh} \). For each \( i \in \mathbb{N} \) we consider a set \( D_i = \{ f_{(U_i)} : U_i \in C(X) : f_{(U_i)} \upharpoonright X \subseteq \{ X \backslash U = 1 \} \} \) for \( U \in \mathcal{F}_i \).

Since \( \{ F(U) : U \in \mathcal{F}_i \} \) is a \( \gamma_k \)-cover of \( X \), we have that \( D_i \) converges to \( f \equiv 0 \) for each \( i \in \mathbb{N} \).

Consider \( U \in \mathcal{F}_i \), \( \{ U_i : i \in \mathbb{N} \} \) is a \( k \)-cover of \( X \).

Let \( K \) be a non-empty compact subset of \( X \) and \( U = \langle f, K, \frac{1}{2} \rangle \) be a base neighborhood of \( f \), then there is \( f_{(U_i)} \subseteq U \). It follows that \( K \subseteq U_i \). We thus get \( X \) satisfies \( S_1(\Gamma_0^{sh}, K) \).

(2) \( \Rightarrow \) (1) Let \( \{ f_i : i \in \mathbb{N} \} \) be a sequence converging to \( f \) for each \( i \in \mathbb{N} \). Without loss of generality we can assume that \( f = 0 \), a set \( W^i_k = \{ x \in X : -\frac{1}{i} < f_k(x) < \frac{1}{i} \} \neq X \) for any \( i \in \mathbb{N} \) and \( S_{\text{sh}}^i_k = \{ x \in X : -\frac{1}{i} \leq f_k(x) \leq \frac{1}{i} \} \neq X \) for any \( i \in \mathbb{N} \).
Consider \( \mathcal{V}_i = \{W^i_k : k \in \mathbb{N}\} \) and \( S_i = \{S^i_k : k \in \mathbb{N}\} \) for each \( i \in \mathbb{N} \). We claim that \( \mathcal{V}_i \) is a \( \gamma_k \)-cover of \( X \). Since \( \{f_k\}_{k \in \mathbb{N}} \) converges to \( f \), for each compact subset \( K \subset X \) there is \( k_0 \in \mathbb{N} \) such that \( f_k \in \langle f, K, \frac{1}{2}\rangle \) for \( k > k_0 \). It follows that \( K \subset W^i_k \) for any \( k > k_0 \). Since \( \mathcal{V}_{i+1} \) is a \( \gamma_k \)-cover, \( S_{i+1} \) is a \( \gamma_k \)-cover, too. \( S_{i+1} \) is a refinement of the family \( \mathcal{V}_i \), hence, \( \mathcal{V}_i \in \Gamma_k^{v_i} \).

By \( X \) satisfies \( S_1(\Gamma_k^{v_i}, K) \), there is a sequence \( \langle W^i_{k(i)} : i \in \mathbb{N}\rangle \) such that \( W^i_{k(i)} \in \mathcal{V}_i \) for each \( i \), and \( \{W^i_{k(i)} : i \in \mathbb{N}\} \) is an element of \( \mathcal{K} \).

We claim that \( f \in \{f_{k(i)} : i \in \mathbb{N}\} \). Let \( U = \langle f, K, \epsilon \rangle \) be a base neighborhood of \( f \) where \( \epsilon > 0 \) and \( K \in \mathcal{K}(X) \), then there is \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \epsilon \) and \( W^i_{k(i)} \supset K \). It follows that \( f_{k(i)} \in \langle f, K, \epsilon \rangle \) and, hence, \( f \in \{f_{k(i)} : i \in \mathbb{N}\} \). \( \square \)

**Lemma 6.3.** Let \( \mathcal{U} = \{U_n : n \in \mathbb{N}\} \) be a \( \gamma_k \)-shrinking cover of a space \( X \). Then the set \( S = \{f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1 \ 	ext{for some } n \in \mathbb{N}\} \) is sequentially dense in \( C_k(X) \).

**Proof.** Let \( h \in C(X) \). For each \( n \in \mathbb{N} \), take \( f_n \in C(X) \) such that \( f_n \upharpoonright F(U_n) = h \upharpoonright F(U_n) \) and \( f_n \upharpoonright (X \setminus U_n) \equiv 1 \). Then obviously \( f_n \in S \), and \( f_n \Rightarrow h \), because \( \{F(U_n) : n \in \mathbb{N}\} \) is a \( \gamma_k \)-cover. \( \square \)

**Theorem 6.4.** For a Tychonoff space \( X \) with \( iw(X) = \mathbb{N}_0 \) the following statements are equivalent:

1. \( C_k(X) \) satisfies \( S_1(S, \mathcal{D}) \);
2. \( C_k(X) \) satisfies \( S_1(S, \Omega_0) \);
3. \( C_k(X) \) satisfies \( S_1(\Gamma_0, \Omega_0) \);
4. \( X \) satisfies \( S_1(\Gamma_k^{v_i}, K) \).

**Proof.** (1) \( \Rightarrow \) (4) Let \( \{F_i : i \in \mathbb{N}\} \subset \Gamma_k^{v_i} \). By Lemma 6.3, \( S_i = \{f \in C(X) : f \upharpoonright (X \setminus F^i_n) \equiv 1 \ 	ext{for some } F^i_n \in F_i \} \) is a sequentially dense subset of \( C_k(X) \) for each \( i \in \mathbb{N} \).

By (1), there is \( \{f_i : i \in \mathbb{N}\} \) such that \( f_i \in S_i \) and \( \{f_i : i \in \mathbb{N}\} \subset \mathcal{D} \).

Consider the sequence \( \{F^i_{n(i)} : i \in \mathbb{N}\} \).

(a) \( F^i_{n(i)} \in F_i \) for \( i \in \mathbb{N} \).

(b) \( \{F^i_{n(i)} : i \in \mathbb{N}\} \) is a \( k \)-cover of \( X \).

Let \( K \in \mathcal{K}(X) \) and let \( U = \langle 0, K, \frac{1}{2}\rangle \) be a base neighborhood of \( 0 \), then there is \( f_n \in \{f_i : i \in \mathbb{N}\} \) such that \( f_n \notin U \). It follows that \( K \subset F^i_{n(i)} \).

(4) \( \Rightarrow \) (3) Let \( X \) satisfies \( S_1(\Gamma_k^{v_i}, K) \) and let \( \{f_{i,m} \}_{m \in \mathbb{N}} \) converges to \( 0 \) for each \( i \in \mathbb{N} \).

Consider \( F_i = \{F_{i,m} : m \in \mathbb{N}\} = \{f_{i-1,m} \langle 0, K, \frac{1}{2}\rangle : m \in \mathbb{N}\} \) for each \( i \in \mathbb{N} \). Without loss of generality we can assume that a set \( F_{i,m} \neq X \) for any \( i, m \in \mathbb{N} \). Otherwise there is a sequence \( \{f_{i,m} : k \in \mathbb{N}\} \) such that \( f_{i,m} \mid k \in \mathbb{N} \) uniformly converges to \( 0 \) and \( \{f_{i,m} : k \in \mathbb{N}\} \in \Omega_0 \).

Note that \( F_i \) is a \( \gamma_k \)-shrinking cover of \( X \) for each \( i \in \mathbb{N} \).

By (4), there is a sequence \( \{F_{i,m(i)} : i \in \mathbb{N}\} \) such that for each \( i \), \( F_{i,m(i)} \in F_i \), and \( \{F_{i,m(i)} : i \in \mathbb{N}\} \) is an element of \( \mathcal{K} \).

We claim that \( 0 \in \{f_{i,m(i)} : i \in \mathbb{N}\} \). Let \( W = \langle 0, K, \epsilon \rangle \) be a base neighborhood of \( 0 \) where \( \epsilon > 0 \) and \( K \in \mathcal{K}(X) \), then there is \( i_1 \in \mathbb{N} \) such that \( \frac{1}{i_1} < \epsilon \) and \( F_{i_1,m(i)} \supset K \). It follows that \( f_{i_1,m(i)} \in \langle 0, K, \epsilon \rangle \) and, hence, \( 0 \in \{f_{i,m(i)} : i \in \mathbb{N}\} \) and \( C_k(X) \) satisfies \( S_1(\Gamma_0, \Omega_0) \).

(3) \( \Rightarrow \) (2) is immediate.

(2) \( \Rightarrow \) (1) Suppose that \( C_k(X) \) satisfies \( S_1(S, \Omega_0) \). Let \( D = \{d_n : n \in \mathbb{N}\} \) be a dense subspace of \( C_k(X) \). Given a sequence of sequentially dense subspace of \( C_k(X) \), enumerate it as \( \{S_{n,m} : n, m \in \mathbb{N}\} \). For each \( n \in \mathbb{N} \), pick \( d_{n,m} \in S_{n,m} \) so that \( d_n \in \{d_{n,m} : m \in \mathbb{N}\} \). Then \( \{d_{n,m} : m, n \in \mathbb{N}\} \) is dense in \( C_k(X) \). \( \square \)

7. \( S_{fin} (S, \mathcal{D}) \)

The following theorems are proved similarly to Theorems 6.2 and 6.4.
Theorem 7.1. For a Tychonoff space $X$ the following statements are equivalent:
1. $C_k(X)$ satisfies $S_{fin}(\Gamma_0, \Omega_0)$;
2. $X$ satisfies $S_{fin}(\Gamma^h_\kappa, \mathcal{K})$.

Theorem 7.2. For a Tychonoff space $X$ with $\text{iw}(X) = \aleph_0$ the following statements are equivalent:
1. $C_k(X)$ satisfies $S_{fin}(S, D)$;
2. $C_k(X)$ satisfies $S_{fin}(S, \Omega_0)$;
3. $C_k(X)$ satisfies $S_{fin}(\Gamma_0, \Omega_0)$;
4. $X$ satisfies $S_{fin}(\Gamma^h_\kappa, \mathcal{K})$.

8. $S_1(S, S)$

In [22], we proved the following theorems.

Theorem 8.1. ([22, Theorem 3.3]) For a Tychonoff space $X$ the following statements are equivalent:
1. $C_k(X)$ satisfies $S_1(\Gamma_0, \Omega_0)$;
2. $X$ satisfies $S_1(\Gamma^h_\kappa, \Gamma_\kappa)$.

Theorem 8.2. ([22, Theorem 3.5]) For a Tychonoff space $X$ such that $C_k(X)$ is sequentially separable the following statements are equivalent:
1. $C_k(X)$ satisfies $S_1(S, S)$;
2. $C_k(X)$ satisfies $S_1(S, \Gamma_0)$;
3. $C_k(X)$ satisfies $S_1(\Gamma^h_\kappa, \Gamma_\kappa)$;
4. $X$ satisfies $S_1(\Gamma^h_\kappa, \Gamma_\kappa)$;
5. $C_k(X)$ satisfies $S_{fin}(S, \Omega_0)$;
6. $C_k(X)$ satisfies $S_{fin}(S, \Gamma_0)$;
7. $C_k(X)$ satisfies $S_{fin}(\Gamma_0, \Gamma_0)$;
8. $X$ satisfies $S_{fin}(\Gamma^h_\kappa, \Gamma_\kappa)$.

We can summarize the relationships between considered notions in next diagrams.

Diagram 1. The Diagram of selectors for sequences of dense sets of $C_k(X)$.

Diagram 2. The Diagram of selection principles for a space $X$ corresponding to selectors for sequences of dense sets of $C_k(X)$. 
9. On the Particular Solution to a Problem

Recall that Arens’ space $S_2$ is the set $\{(0,0), (\frac{1}{n},0), (\frac{1}{n}, \frac{1}{m}) : n, m \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0,0), (\frac{1}{n},0) : n > 0 \}$ and $C_n = \{(\frac{1}{n},0), (\frac{1}{n}, \frac{1}{m}) : m > 0, n > 0 \}$. The sequential fan is the quotient space $S_{\omega} = S_2 / C_0$ obtained from the Arens’s space by identifying the points of the sequence $C_0$ [12].

**Proposition 9.1.** If $C_k(X)$ satisfies $S_{\text{fin}}(\Gamma_0, \Omega_0)$, then $S_{\omega}$ cannot be embedded into $C_k(X)$.

The following problem was posed in the paper [4].

**Problem 9.2.** Does a first countable (separable metrizable) space belong to the class $S_1(\Gamma_0, \mathcal{K})$ if and only if it is hemicompact?

A particular answer to this problem is the following

**Theorem 9.3.** Suppose that $X$ is first countable stratifiable space and $iw(X) = \aleph_0$. Then following the statements are equivalent:

1. $X$ satisfies $S_{\text{fin}}(\Gamma^0_0, \mathcal{K})$;
2. $X$ satisfies $S_{\text{fin}}(\Gamma^0_\omega, \mathcal{K})$;
3. $X$ satisfies $S_1(\mathcal{K}, \Gamma_1)$;
4. $X$ is hemicompact.

**Proof.** ($1 \Rightarrow 4$) Since $X$ is first countable stratifiable space and, by Proposition 9.1, $S_{\omega}$ cannot be embedded into $C_k(X)$, then, by Theorem 2.2 (+ Remark) in [7], $X$ is a locally compact. A locally compact stratifiable space is metrizable [5]. It is well-known that a locally compact metrizable space can be represented as $X = \bigsqcup X_\alpha$ where $X_\alpha$ is a $\sigma$-compact for each $\alpha < \tau$. Since $iw(X) = \aleph_0$, then $\tau \leq \omega$. Claim that $\tau < \omega_1$.

Assume that $\tau \geq \omega_1$. Then there is a continuous mapping $f : X \to D$ $(f(X_\alpha) = d_\alpha)$ from $X$ onto a discrete space $D = \{d_\alpha : \alpha < \tau\}$. Note that $D$ satisfies $S_{\text{fin}}(\Gamma^0_0, \mathcal{K})$ $(S_{\text{fin}}(\Gamma, \Omega))$ and, hence, $D$ is Lindelöf, but $|D| > \aleph_0$, a contradiction.

It follows that $X$ is a locally compact and Lindelöf, and, hence, $X$ is a hemicompact.

($4 \Rightarrow 3$) Since $X$ is hemicompact and $iw(X) = \aleph_0$, then $C_k(X)$ is a separable metrizable space [17]. Hence, $C_k(X)$ satisfies $S_1(\mathcal{D}, S)$, and, by Theorem 3.3, $X$ satisfies $S_1(\mathcal{K}, \Gamma_1)$. □

**Corollary 9.4.** Suppose that $X$ is a separable metrizable space. Then $X$ satisfies $S_{\text{fin}}(\Gamma^0_\omega, \mathcal{K})$ if and only if $X$ is hemicompact.

**Remark 9.5.** In the class of first countable stratifiable spaces with $iw(X) = \aleph_0$ (in particular, in the class of separable metrizable spaces) all properties in Diagram 1 (and, hence, Diagram 2) coincide.

**References**


