Optimal Control of Feedback Control Systems Governed by Systems of Evolution Hemivariational Inequalities

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Abstract. In this paper, we introduce and consider a feedback control system governed by the system of evolution hemivariational inequalities. Several sufficient conditions are formulated by virtue of the properties of multimaps and partial Clarke’s subdifferentials such that the existence result of feasible pairs of the feedback control systems is guaranteed. Moreover, an existence result of optimal control pairs for an optimal control system is also established.

1. Introduction

It is well-known that hemivariational inequalities have played an important role in many applications, such as mechanics and engineering, especially in nonsmooth analysis and optimization (see [1, 8–10, 17–20, 22, 24, 25]). Some existence theorems and well-posedness results for hemivariational inequalities have been obtained in the literature; see e.g., [2–6, 27, 30–38] and references therein. Recently, some researchers devoted to consider the optimal control problems for hemivariational inequalities. In [12], Haslinger and Panagiotopoulos proved the existence of optimal control pairs for a class of coercive hemivariational inequalities. In [21], Migorski and Ochal discussed the optimal control problems for the parabolic hemivariational inequalities. Park and Park [24, 25] extended the existence of optimal control pairs to the hyperbolic linear systems. Furthermore, Tolstonogov [28, 29] made efforts to probe into the optimal control problems for differential inclusions with subdifferential type.

Our main purpose here is to study the existence result of feasible pairs of feedback optimal control systems for systems of evolution hemivariational inequalities. To begin with, let us recall several existing results. Throughout, we assume that $H$ is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$, and $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a compact $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on $H$. Let $E$ be a reflexive Banach space, $u : [0, T] \to E$ a control function and $B : E \to H$ a bounded linear operator. Denote by $J'(t, \cdot)$ the Clarke’s generalized directional derivative ([7]) of a locally Lipschitz functional $J(t, \cdot) : H \to \mathbb{R}$. Denote by $P(E)$ the collection of all nonempty subsets of $E$. Define two symbols: $P_{f(c)}(E) := \{\Omega \subseteq E : \Omega$ is nonempty, closed (convex) and $p_{\text{weak}}(E) := \{\Omega \subseteq E : \Omega$ is nonempty, (weakly) compact (convex)].

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Now, we focus on the following evolution hemivariational inequality problem
\[
\begin{aligned}
(-\dot{x}(t) + Ax(t) + Bu(t), v_H(t) + J'(t, x(t), v)) \geq 0, \text{ a.e. } t \in [0, T], \forall v \in H,
\end{aligned}
\]
and the following feedback control problem
\[
\begin{aligned}
(-\dot{x}(t) + Ax(t) + Bu(t), v_H(t) + J'(t, x(t), v)) \geq 0, \text{ a.e. } t \in [0, T], \forall v \in H,
u(t) \in \mathcal{U}(t, x(t)),
x(0) = x_0 \in H,
\end{aligned}
\]
where \( \mathcal{U} : [0, T] \times H \to P(E) \) is a multimap.

In [14], Huang, Liu and Zeng proved the existence of solutions of the evolution hemivariational inequality problem (1) and the existence of feasible pairs of the feedback control problem (2). To the best of our knowledge, feedback control problems are ubiquitous around us, including trajectory planning of a robot manipulator, guidance of a tactical missile toward a moving target, regulation of room temperature, and control of string vibrations. Li and Yong [16] studied the optimal feedback control of semilinear evolution equations in Banach spaces. Huang, Liu and Zeng [14] studied the above feedback control problem (2) governed by evolution hemivariational inequality. By using the properties of multimaps and Clarke’s generalized subdifferential, they formulated some sufficient conditions to guarantee the existence result of feasible pairs of feedback control problem. Moreover, they also established an existence result of optimal control pairs for an optimal control problem.

However, there is little study for the optimal control of feedback control problems described by evolution hemivariational inequalities in the literature. It is worth pointing out that the study for the optimal control of feedback control systems described by systems of evolution hemivariational inequalities is still untreated topic in the literature and this fact is the motivation of the present work.

For our purpose, we assume that \( V_i \) is a separable Hilbert space with the inner product \( (\cdot, \cdot)_{V_i} \) and norm \( ||\cdot||_{V_i} \), and \( A_i : D(A_i) \subseteq V_i \to V_i \) is the infinitesimal generator of a compact \( C_0 \)-semigroup \( \{T_i(t)\}_{t \geq 0} \) on \( V_i \), where \( i \in \{1, 2\} \) and \( t = [0, T] \) for some \( 0 < T < \infty \). Let \( U_i \) be a reflexive Banach space, \( u_i : [0, T] \to U_i \) a control function and \( B_i : U_i \to V_i \) a bounded linear operator. For \( i, j = 1, 2 \) and \( i \neq j \), the notation \( J_i'(t, x_1, x_2; v_i) \) stands for the partial Clarke generalized directional derivative (cf. [33]) of a locally Lipschitz functional \( J(t, \cdot, \cdot) : V_1 \times V_2 \to \mathbb{R} \) with respect to the \( i \)-th variable at \( x_j \) in the direction \( v_i \) for the given \( x_j \).

In the present paper, we aim to study the existence of solutions of the following system of evolution hemivariational inequalities:
\[
\begin{aligned}
&(-x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1)_{V_1} + J_1'(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \forall v_1 \in V_1,
&(-x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2)_{V_2} + J_2'(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \forall v_2 \in V_2,
x_i(0) = x_i^0 \in V_i, \ i = 1, 2.
\end{aligned}
\]

In what follows we are concerned with the existence of feasible pairs of the following feedback control systems:
\[
\begin{aligned}
&(-x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1)_{V_1} + J_1'(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \forall v_1 \in V_1,
&(-x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2)_{V_2} + J_2'(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \forall v_2 \in V_2,
u_i(t) \in \mathcal{U}_i(t, x_i(t), x_2(t)), \ i = 1, 2,
x_i(0) = x_i^0 \in V_i, \ i = 1, 2,
\end{aligned}
\]
where \( \mathcal{U}_i : [0, T] \times V_1 \times V_2 \to P(U_i) \) is a multimap for \( i = 1, 2 \).

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries and physical models. In Section 3, some sufficient conditions and techniques are established for the existence of feasible pairs of system (4). We first study the existence of solutions of (3) by a fixed point theorem of multimaps. In Section 4, we will study the optimal control system (4).
2. Preliminaries and Physical Models

In this section, we first introduce some basic preliminaries which are used throughout this paper. Some terminologies are borrowed from [14]. Let \( i \in \{1, 2\} \) and \( I = [0, T] \). The norm of the Hilbert space \( V_i \) will be denoted by \( \| \cdot \|_V \). Let \( V = V_1 \times V_2 \). Enforced with the norm defined by \( \| x \|_V := \| x_1 \|_{V_1} + \| x_2 \|_{V_2} \) for all \( x = (x_1, x_2) \in V \), \( V \) is a reflexive Banach space ([6]). For a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \), there exist constants \( \omega_i \) and \( p_i > 0 \) such that \( \| T(t) \| \leq p_i e^{\omega_i t} \) for \( 0 \leq t < \infty \) and we set \( \sup_{t \geq 0} \| T(t) \| \leq \sup_{t \geq 0} p_i e^{\omega_i t} \leq M_t \) with \( M_t > 0 \) ([26]). Let \( C(I, V) \) denote the Banach space of all continuous functions from \( I \) into \( V \); with the norm \( \| x \|_{C(I, V)} := \sup_{t \in I} \| x(t) \|_{V} \).

Now, let us proceed to the definition of the Clarke’s subdifferential for a locally Lipschitz functional \( h : X \to \mathbb{R} \), where \( X \) is a Banach space, \( X^* \) is the dual space of \( X \) and \( \langle \cdot, \cdot \rangle_X \) is the duality pairing between \( X^* \) and \( X \). We denote by \( h^\prime(x; v) \) the Clarke’s generalized directional derivative of \( h \) at the point \( x \in X \) in the direction \( v \in X \), that is,

\[
h^\prime(x; v) := \limsup_{\lambda \to 0^+, y \to x} \frac{h(y + \lambda v) - h(y)}{\lambda}.
\]

Recall also that the Clarke’s subdifferential or generalized gradient of \( h \) at \( x \in X \), denoted by \( \partial h(x) \), is a subset of \( X^* \) given by \( \partial h(x) := \{ x' \in X^* : h^\prime(x; v) \geq \langle x', v \rangle_X, \forall v \in X \} \).

**Lemma 2.1.** ([16]). Let \( X \) and \( Y \) be two topological vector spaces. Denote by \( \mathcal{P}(Y) \) \( \{C(Y), K(Y), K_c(Y)\} \) the collection of all nonempty, respectively, nonempty closed, nonempty compact, nonempty compact convex subsets of \( Y \). A multimap \( F : I \to C(X) \) is said to be measurable, if \( F^{-1}(Q) := \{ t \in I : F(t) \cap Q = \emptyset \} \in \Sigma \) for every closed set \( Q \subset X \), where \( \Sigma \) denotes the o-field of Lebesgue measurable sets on \( I = [0, T] \). Every measurable multimap \( F \) admits a measurable selection \( f : I \to X \), i.e., \( f \) is measurable and \( f(t) \in F(t) \) for a.e. \( t \in I \). A multimap \( F : I \to C(Y) \) is said to be upper semicontinuous (for short, u.s.c.), if for every open subset \( D \subset Y \) the set \( F^{-1}(D) = \{ x \in X : F(x) \subset D \} \) is open in \( X \); weakly u.s.c., if \( F : X \to C(Y_w) \) is u.s.c., where \( Y_w \) is the space \( Y \) equipped with a weak topology. A multimap \( F : X \to C(Y) \) is said to be closed if its graph \( \text{Gr}(F) := \{(x, y) \in X \times Y : x \in X, y \in F(x)\} \) is a closed subset of \( X \times Y \); compact if \( F \) maps bounded sets of \( X \) into relatively compact sets in \( Y \).

We have the following important property for multimaps.

**Lemma 2.2.** ([16]). Let \( X \) be a Banach space and \( Y \) be a metric space. Let \( F : X \to P(Y) \) be a multimap. We say that \( F \) possesses the Cesari property at \( x_0 \), if \( \bigcap_{i=0}^\infty \text{co}(O_{\delta_i}(x_0)) = F(x_0) \), where \( \text{co}D \) is the closed convex hull of \( D \), \( O_{\delta_i}(x_0) \) is the \( \delta_i \)-neighborhood of \( x_0 \). If \( F \) has the Cesari property at every point \( x \in Z \subset X \), we simply say that \( F \) has the Cesari property on \( Z \).

**Lemma 2.3.** ([16]). Let \( X \) be a Banach space and \( Y \) be a metric space. Let \( F : X \to P(Y) \) be u.s.c. with convex and closed values. Then \( F \) has the Cesari property on \( X \).

Now, let us proceed to the definition of the Clarke’s subdifferential for a locally Lipschitz functional \( h : X \to \mathbb{R} \), where \( X \) is a Banach space, \( X^* \) is the dual space of \( X \) and \( \langle \cdot, \cdot \rangle_X \) is the duality pairing between \( X^* \) and \( X \). We denote by \( h^\prime(x; v) \) the Clarke’s generalized directional derivative of \( h \) at the point \( x \in X \) in the direction \( v \in X \), that is,

\[
h^\prime(x; v) := \limsup_{\lambda \to 0^+, y \to x} \frac{h(y + \lambda v) - h(y)}{\lambda}.
\]

The key tool in one of our main results is the following fixed point theorem.
Theorem 2.5. ([11]). Let $X$ be a Banach space, $C$ a closed convex subset of $X$, $D$ an open subset of $C$ (relative to $C$) and $0 \in D$. Suppose that $\mathcal{F}: \overline{D} \to K_c(C)$ is an u.s.c. and compact multifunction. Then either (i) $\mathcal{F}$ has a fixed point in $D$, or (ii) there are $x \in \partial D$ and $\lambda \in (0, 1)$ with $x = \lambda \mathcal{F}(x)$.

In the sequel, we shall study the existence of mild solutions of the following system of semilinear evolutionary inclusions, which is specified as follows: Find $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$ such that

\[
\begin{aligned}
& x_1'(t) = A_1x_1(t) + B_1u_1(t) + \partial_1 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in [0, T], \\
& x_2'(t) = A_2x_2(t) + B_2u_2(t) + \partial_2 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in [0, T], \\
& x_i(0) = x_i^0 \in V_i, \quad i = 1, 2,
\end{aligned}
\]

where, for $i = 1, 2$, $A_i : D(A_i) \subseteq V_i \to V_i$ is the infinitesimal generator of a $C_0$-semigroup $T_i(t)$ ($t \geq 0$) on a separable Hilbert space $V_i$. For $i, j = 1, 2$ and $i \neq j$, the notation $\partial_i J(t, x_1, x_2)$ stands for the partial Clarke generalized gradient (cf. [33]) of a locally Lipschitz functional $J(t, \cdot, \cdot) : V_1 \times V_2 \to \mathbb{R}$ with respect to the $i$th variable at $x_i$ for the given $x_j$. The control function $u_i$ takes values in $L^2(I, U_i)$, the admissible controls set $U_i$ is a Hilbert space, and $B_i$ is a bounded linear operator from $U_i$ into $V_i$.

We say that $(x_1, x_2) \in W^{1,2}(I, V_1) \times W^{1,2}(I, V_2)$ is a solution of (5) if there exists a pair of functions $(f_1, f_2) \in L^2(I, V_1) \times L^2(I, V_2)$ such that $(f_1, f_2) \in \partial J(t, x_1(t), x_2(t)) \times \partial J(t, x_1(t), x_2(t))$ and

\[
\begin{aligned}
& x_1'(t) = A_1x_1(t) + B_1u_1(t) + f_1(t), \text{ a.e. } t \in I, \\
& x_2'(t) = A_2x_2(t) + B_2u_2(t) + f_2(t), \text{ a.e. } t \in I, \\
& x_i(0) = x_i^0 \in V_i, \quad i = 1, 2,
\end{aligned}
\]

which implies

\[
\begin{aligned}
& (-x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1)_{V_1} + (f_1(t), v_1)_{V_1} = 0, \text{ a.e. } t \in I, \quad \forall v_1 \in V_1, \\
& (-x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2)_{V_2} + (f_2(t), v_2)_{V_2} = 0, \text{ a.e. } t \in I, \quad \forall v_2 \in V_2, \\
& x_i(0) = x_i^0 \in V_i, \quad i = 1, 2.
\end{aligned}
\]

Since for $i = 1, 2$, $f_i(t) \in \partial J(t, x_1(t), x_2(t))$ and $(f_i(t), v_1)_{V_1} \leq J_i'(t, x_1(t), x_2(t); v_1)$, we obtain

\[
\begin{aligned}
& (-x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1)_{V_1} + J_1'(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \quad \forall v_1 \in V_1, \\
& (-x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2)_{V_2} + J_2'(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \quad \forall v_2 \in V_2, \\
& x_i(0) = x_i^0 \in V_i, \quad i = 1, 2.
\end{aligned}
\]

Hence, any solutions of system (5) are also solutions of system (3).

Similarly, the feedback control system (4) of evolution hemivariational inequalities can be reduced to the following feedback control system with partial Clarke’s subdifferentials:

\[
\begin{aligned}
& x_1'(t) = A_1x_1(t) + B_1u_1(t) + \partial_1 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I, \\
& x_2'(t) = A_2x_2(t) + B_2u_2(t) + \partial_2 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I, \\
& u_i(t) \in U_i(t, x_1(t), x_2(t)), \quad i = 1, 2, \\
& x_i(0) = x_i^0 \in V_i, \quad i = 1, 2.
\end{aligned}
\]

Therefore, in order to study the system (3) of evolution hemivariational inequalities and the feedback control system (4) of evolution hemivariational inequalities, we only need to deal with the system (5) of semilinear evolutionary inclusions, and the feedback control system (6) with partial Clarke’s subdifferentials.

We note that system (3) arises in many important models for distributed parameter control problems and that a large class of identification problems enter our formulation. Let us indicate a problem which is one of the motivations for the study of the system (3) of evolution hemivariational inequalities ([21]).
We consider the following system of heat initial-boundary value problems:

\[
\begin{aligned}
\frac{\partial}{\partial t}x_1(t,y) &= \frac{\partial^2}{\partial y^2}x_1(t,y) + B_1u_1(t,y) + f_1(x_1,x_2), \quad (t,y) \in (0,1) \times (0,\pi), \\
\frac{\partial}{\partial t}x_2(t,y) &= \frac{\partial^2}{\partial y^2}x_2(t,y) + B_2u_2(t,y) + f_2(x_1,x_2), \quad (t,y) \in (0,1) \times (0,\pi), \\
x_1(t,0) &= x_1(t,\pi) = 0, \quad t \in (0,1), \quad i = 1,2, \\
x_1(0,y) &= x_1^0(y), \quad y \in (0,\pi), \quad i = 1,2.
\end{aligned}
\]

This system represents the heat flows with temperature-dependent sources. Let \( i \in \{1,2\} \). \( x_i = x_i(t,y) \) represents the temperature at the time \( t \in (0,1) \) and point \( y \in (0,\pi) \). The temperatures of boundaries are zero and the initial temperature is \( x_i^0(y) \) \( (y \in (0,\pi)) \). \( u_i \) is a control function. \( f_i(x_1,x_2) \) is a heat source dependent of temperatures.

We suppose that for \( i = 1,2 \), the control \( u_i \) is a feedback control by the temperatures such that

\[ u_i \in U_i(t,x_1,x_2), \text{ a.e. } (t,y) \in (0,1) \times (0,\pi), \]

\( f_i \) is a known function of the temperatures of the following form

\[ f_i(t,y) \in \partial f_i(t,y,x_1(t,y),x_2(t,y)), \text{ a.e. } (t,y) \in (0,1) \times (0,\pi), \]

where for \( i,j = 1,2 \) and \( i \neq j \), \( \partial f_i(t,y,\eta_1,\eta_2) \) denotes the partial Clarke’s generalized gradient of a locally Lipschitz functional \( f_i(t,y,\eta) : \mathbb{R}^2 \to \mathbb{R} \) with respect to the \( \eta \) variable at \( \eta_i \) for the given \( \eta_j \). The multivalued mapping \( \partial f_i(t,y,\eta) : \mathbb{R}^2 \to 2^\mathbb{R} \) is generally nonmonotone and it includes the vertical jumps which means that the law is characterized by the partial Clarke’s generalized gradient of a nonsmooth potential \( f_i \).

Let \( i \in \{1,2\} \). Take \( V_i = L^2_1(0,1), \) \( x_i(t)(\cdot) = x_i(t,\cdot) \) and the operator \( A_i : D(A_i) \subset V_i \to V_i \) is defined by \( A_i x_i = x_i^{''}, \) where the domain \( D(A_i) \) is given by \( \{ x_i \in V_i : x_i^{''}, x_i^{'} \in V_i, \ x_i(0) = x_i(\pi) \} \). Then, \( A_i \) can be represented as \( A_i x_i = -\sum_{n=1}^{\infty} n^2(x_i, e_n)e_n, x_i \in D(A_i), \) where \( e_n(y) = \sqrt{2/\pi} \sin ny \) \( (n = 1,2,\ldots) \) is an orthonormal basis of \( V_i \). We knew that \( A_i \) generates a strongly continuous semigroup \( T_i(t) \) \( (t > 0) \) in \( V_i \), which is compact and analytic (see [26]), given by \( T_i(t)x_i = \sum_{n=1}^{\infty} e^{-nt^2}(x_i, e_n)e_n, x_i \in V_i, \) and \( \|T_i(t)\| \leq e^{-t} < 1 = M_i \).

Now let the function \( J : (0,1) \times V_1 \times V_2 \to \mathbb{R} \) be given by

\[ J(t,x_1,x_2) = \int_0^1 (j(t,y,x_1(y)) + j(t,y,x_2(y)))dy, \quad t \in (0,1), \quad (x_1,x_2) \in V_1 \times V_2, \]

where \( j(t,y,z) = \int_0^t \phi(t,y,\theta)d\theta, \) \( (t,y) \in (0,1) \times (0,\pi), \) \( z \in \mathbb{R}. \)

Let \( \phi : (0,1) \times (0,\pi) \times \mathbb{R} \to \mathbb{R} \) be a function satisfying the following assumptions:

(i) for all \( y \in (0,\pi), z \in \mathbb{R}, \phi(\cdot, y, z) : (0,1) \to \mathbb{R} \) is measurable;
(ii) for all \( t \in (0,1), z \in \mathbb{R}, \phi(t,\cdot, z) : (0,\pi) \to \mathbb{R} \) is continuous;
(iii) for all \( z \in \mathbb{R} \) there exists a constant \( c_1 > 0 \) such that \( |\phi(\cdot, z)| \leq c_1(1 + |z|) \) for \( z \in \mathbb{R}; \)
(iv) for every \( z \in \mathbb{R}, \phi(\cdot, z, \pm 0) \) exists.

If \( \phi \) satisfies (iii), then \( \partial j(z) \subset [\phi(z), \bar{\phi}(z)] \) for \( z \in \mathbb{R} \) (we omit \( t,y \) here), where \( \phi(z) \) and \( \bar{\phi}(z) \) denote the essential supremum and essential infimum of \( \phi \) at \( z \) [7], respectively.

If \( \phi \) satisfies (i)-(iv), then the function \( j(\cdot, \cdot, \cdot) \) defined above has the following properties:

(i) for all \( y \in (0,\pi), z \in \mathbb{R}, \ j(\cdot, y, z) \) is measurable and \( j(\cdot, s, 0) \in L^2((0,1) \times (0,\pi)); \)
(ii) for all \( t \in (0,1), z \in \mathbb{R}, j(t,\cdot, z) : (0,\pi) \to \mathbb{R} \) is continuous;
(iii) for all \( t \in (0,1), z \in \mathbb{R}, j(t,\cdot, \cdot) : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz;
(iv) there exists a constant \( c_2 > 0 \) such that \( |s| \leq c_2(1 + |z|) \) for all \( z \in \mathbb{R} \), \( (t,y) \in (0,1) \times (0,\pi); \)
(v) there exists a constant \( c_3 > 0 \) such that \( j(t, y, z; \cdot) \leq c_3(1 + |z|) \) for all \( (t, y) \in (0,1) \times (0,\pi). \)
Assume that for \( i = 1, 2 \), \( U_i \) is a reflexive Banach space, \( u_i : (0, 1) \rightarrow U_i \) a control function and \( B_i : U_i \rightarrow \mathbb{R} \) a bounded linear operator. Thus, combining (8)-(9), system (7) turns to be system (6).

Therefore, the variational formulation of the above system leads to the system (3) of evolution hemivariational inequalities and is met, for example, in the nonmonotone nonconvex interior semipermeability problems. For the latter, Panagiotopoulos [23] considered a temperature control problem in which they regulated the temperature to deviate as little as possible from a given interval. We remark that the monotone semipermeability problems, leading to variational inequalities, have been studied by Duvaut and Lions under the assumption that \( f(t, y, \cdot) \) is a proper, lower semicontinuous, convex function which means that \( \partial f(t, y, \cdot) \) is a maximal monotone operator in \( \mathbb{R}^2 \).

3. The Existence of Feasible Pairs

In this section we study the existence of feasible pairs for system (6).

At the first, we study the existence of solutions of system (6). We will make the following conditions.

(HT): \( T_i(t) \ (t > 0) \) is a compact operator for \( i = 1, 2 \).

Let \( J : l \times V_1 \times V_2 \rightarrow \mathbb{R} \) be a functional satisfying the following conditions:

(HJ1) the function \( t \mapsto f(t, x_1, x_2) \) is measurable for all \( (x_1, x_2) \in V_1 \times V_2 \);

(HJ2) the function \( (x_1, x_2) \rightarrow f(t, x_1, x_2) \) is locally Lipschitz on \( V_1 \times V_2 \) for a.e. \( t \in I \);

(HJ3) for \( i = 1, 2 \), there exist a function \( \phi_i \in L^2(I, \mathbb{R}^+) \) and a constant \( L_i > 0 \) such that

\[
\| \partial_j f(t, x_1, x_2) \| = \sup \{ \| \zeta_j \|_{V_i} : \zeta_j \in \partial_j f(t, x_1, x_2) \} \leq \phi_i(t) + L_i \| x \|_{V_i},
\]

for a.e. \( t \in I \) and all \( (x_1, x_2) \in V_1 \times V_2 \);

(HJ4) \( f(t, x_1, x_2) + f(t, y_1, y_2) = f(t, x_1, y_2) + f(t, y_1, x_2) \) for a.e. \( t \in I \) and all \( (x_1, x_2), (y_1, y_2) \in V_1 \times V_2 \).

Lemma 3.1. ([30]). Suppose that the functional \( J : l \times V_1 \times V_2 \rightarrow \mathbb{R} \) satisfies the hypotheses (HJ2), (HJ4). Then, for any sequence \( x^n = (x^n_1, x^n_2) \in V = V_1 \times V_2 \) converging strongly to \( x = (x_1, x_2) \in V \) and \( y^n_2 \in V_1 \) converging strongly to \( y_2 \in V_2 \), one has lim inf_{n \to \infty} f^n_1(t, x^n_1, x^n_2, y^n_2) \leq f^1_1(t, x_1, x_2; y_2), \) a.e. \( t \in I \), where \( i = 1, 2 \).

Lemma 3.2. ([22]). Let \( E \) be a separable reflexive Banach space, \( 0 < T < \infty \) and \( h : (0, T) \times E \rightarrow \mathbb{R} \) be a function such that \( h(t, x) \) is measurable for all \( x \in E \) and \( h(t, \cdot) \) is locally Lipschitz on \( E \) for all \( t \in (0, T) \). Then the multifunction \( (t, x) \in (0, T) \times E \mapsto \partial h(t, x) \subseteq E^* \) is measurable, where \( \partial h \) denotes the Clarke generalized gradient of \( h(t, \cdot) \).

Lemma 3.3. ([22]). Let \( X \) and \( Y \) be two topological spaces, \( F : X \rightarrow 2^Y \) a multivalued mapping.

(i) If \( F \) is u.s.c. and closed-valued, then \( F \) is closed;

(ii) If \( F \) is compact-valued, then \( F \) is u.s.c. at \( x \in X \) if and only if for any net \( \{ x_n \} \subseteq X \) with \( x_n \to x \) and for any net \( \{ y_n \} \subseteq Y \) with \( y_n \in F(x_n) \) for all \( n \), there exist \( y \in F(x) \) and a subnet \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that \( y_{n_k} \to y \);

(iii) \( F \) is l.s.c. at \( x \in X \) if and only if for any \( y \in F(x) \) and for any net \( \{ x_n \} \) with \( x_n \to x \), there exists a net \( \{ y_n \} \) with \( y_n \in F(x_n) \) for all \( n \) such that \( y_n \to y \).

We shall make use of the following well-known results in this paper.

Theorem 3.4. ([13]). If \( (\Omega, \Sigma) \) is a measurable space, \( X \) is a Polish space (i.e., separable completely metric space) and \( F : \Omega \rightarrow P_1(X) \) is measurable, then \( F(\cdot) \) admits a measurable selection (i.e., there exists \( f : \Omega \rightarrow X \) measurable such that for every \( x \in \Omega \), \( f(x) \in F(x) \)).

Lemma 3.5. ([22]). Let \( (\Omega, \Sigma, \mu) \) be a \( \sigma \)-finite measure space, \( E \) be a Banach space and \( 1 \leq p < \infty \). If \( f_n, f \in L^p(\Omega, E) \), \( f_n \rightarrow f \) weakly in \( L^p(\Omega, E) \) and \( f_n(x) \in G(x) \) for \( \mu \)-a.e. \( x \in \Omega \) and all \( n \in \mathbb{N} \) where \( G(x) \in P_{\text{att}}(E) \) for \( \mu \)-a.e. \( x \in \Omega \), then \( f(x) \in \text{conv}(\omega \lim \sup f_n(x)) \) for \( \mu \)-a.e. on \( x \in \Omega \), where \( \text{conv} \) denotes the closed convex hull of a set.
By the symbol of $S^2_{\psi}$ we will denote the set of all Bochner $L^2$-integrable selections of the multimap $\Psi: I \to P(H)$, i.e., $S^2_{\psi} = \{ \psi \in L^2(I, H): \psi(t) \in \Psi(t) \text{ for a.e. } t \in I \}$. Next, for $i = 1, 2$, we define the superposition multiopter $\mathcal{P}_i: C(I, V_1) \times C(I, V_2) \to P(L^2(I, V_i))$ as follows

$$\mathcal{P}_i(x_1, x_2) = S^2_{\partial_J(t,x_1,x_2)} = \{ w_i \in L^2(I, V_i): w_i(t) \in \partial_J(t, x_1(t), x_2(t)) \text{ a.e. } t \in I \},$$

for all $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$.

Let $C = C(I, V_1) \times C(I, V_2)$. Endowed with the norm defined by $\|x\|_C := \|x_1\|_{C(L^2, V_1)} + \|x_2\|_{C(L^2, V_2)}$ for all $x = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2)$, $C$ is a reflexive Banach space.

We have the following property for the operator $\mathcal{P}_i$ for $i = 1, 2$.

**Lemma 3.6.** Let $i \in \{1, 2\}$. If conditions (HJ1)-(HJ4) are satisfied, then for every $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$, the set $\mathcal{P}_i(x_1, x_2)$ has nonempty and weakly compact values.

**Proof.** First of all, for $i = 1, 2$, from the reflexivity of $V_i$ and Lemma 2.4(ii), we know that for every $(t, x_1, x_2) \in I \times V_1 \times V_2$, the set $\partial_J(t, x_1, x_2)$ is nonempty, convex and weakly compact in $V_i$ and the multifunction $\partial J_i$ is $P_{\text{w-c}}(V_i)$-valued. Therefore, it is not difficult to check that for $i = 1, 2$, $\mathcal{P}_i(x_1, x_2)$ has convex and weakly compact values. Next, we show that for $i = 1, 2$, $\mathcal{P}_i(x_1, x_2)$ is nonempty. Indeed, let $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$. Then for $i = 1, 2$, there exists a sequence $\{q_i^n\} \subseteq C(I, V_i)$ of step functions such that

$$q_i^n(t) \to x_i(t), \text{ in } C(I, V_i), \text{ a.e. } t \in I. \quad (10)$$

From hypotheses (HJ1), (HJ2) and Lemma 3.2, it follows that for $i = 1, 2$, the multifunction $(t, x_1, x_2) \mapsto \partial J_i(t, x_1, x_2)$ is measurable. Thus, for $i = 1, 2$, $t \mapsto \partial J_i(t, q_i^n(t), q_i^n(t))$ is measurable from $I$ into $P_{\text{w-c}}(V_i)$. For $i = 1, 2$, applying Theorem 3.4, for every $n \geq 1$, there exists a measurable function $\zeta_i^n: I \to V_i$ such that $\zeta_i^n(t) \in \partial J_i(t, q_i^n(t), q_i^n(t)) \text{ a.e. } t \in I$. Next, from hypothesis (HJ3), we obtain that for $i = 1, 2$,

$$\|\zeta_i^n\|_{L^2(I, V_i)} \leq \|\phi_i\|_{L^2(I, R^+)} + L_i\|q_i^n\|_{L^2(I, V_i)}. $$

Hence, for $i = 1, 2$, $\{\zeta_i^n\}$ remains in a bounded subset of $L^2(I, V_i)$. Thus, for $i = 1, 2$, by passing to a subsequence if necessary, we may suppose that $\zeta_i^n \to \zeta_i$ weakly in $L^2(I, V_i)$ with $\zeta_i \in L^2(I, V_i)$. Then it follows from Lemma 3.5 that for $i = 1, 2$,

$$\zeta_i(t) \in \overline{\partial J_i(t, x_1(t), x_2(t))}, \text{ a.e. } t \in I. \quad (11)$$

We claim that for a.e. $t \in I$, the multifunction $(x_1, x_2) \mapsto \partial J_i(t, x_1(t), x_2(t))$ is u.s.c. from $V_1 \times V_2$ into $(V_i)_{w-c}$, where for $i = 1, 2$, $(V_i)_{w-c}$ is the space endowed with the $w$-topology of $V_i$.

Indeed, for any sequence $\{(x_1^n, x_2^n)\} \subseteq V_1 \times V_2$ with $(x_1^n, x_2^n) \to (x_1, x_2)$ in $V = V_1 \times V_2$ and for any sequence $\{y_i^n\} \subseteq V_i$ with $y_i^n \in \partial J_i(t, x_1^n, x_2^n)$ for all $n \geq 1$, we know by the definition of the partial Clarke generalized gradient $\partial J_i(t, x_1^n, x_2^n)$ of a locally Lipschitz functional $J(t, \cdot, \cdot): V_1 \times V_2 \to R$, that $\{y_i^n\}_{V_i}, V_i \subseteq I_i^n(t, x_1^n, x_2^n), \forall V_i \in V_i$. Since $(x_1^n, x_2^n) \to (x_1, x_2)$ in $V_1 \times V_2$, from Hypotheses (HJ2), (HJ4) and Lemma 3.1 it follows that for $i = 1, 2$,

$$\limsup_{n \to \infty} y_i^n(t) \leq \limsup_{n \to \infty} I_i^n(t, x_1^n, x_2^n; v_i) \leq I_i^n(t, x_1(t), x_2(t); v_i) , \forall v_i \in V_i. \quad (12)$$

Also, for $i = 1, 2$, from hypotheses (HJ3) and $y_i^n \in I_i^n(t, x_1^n, x_2^n)$, we get $\|y_i^n\|_{V_i} \leq \phi_i(t) + L_i\|x_i^n\|_{V_i}$, which together with the boundedness of $\{x_i^n\}$, implies that $\{y_i^n\}$ is bounded. Taking into account the reflexivity of $V_i$, we know that there exists a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ such that $y_i^{n_k} \to y_i$ weakly in $V_i$. So, it follows from (12) that

$$\langle y_i^n, v_i \rangle = \lim_{n \to \infty} \langle y_i^{n_k}, v_i \rangle \leq \limsup_{n \to \infty} \langle y_i^{n_k}, v_i \rangle \leq \int_{I_i^n(t, x_1^n, x_2^n; v_i)}, \forall v_i \in V_i, \text{ which yields } y_i \in \partial J_i(t, x_1(t), x_2(t)) \text{ in } V_i, \zeta_i^n \in C(I, V_i), \zeta_i^n \to \zeta_i \text{ in } C(I, V_i), \text{ then } f_i \in \partial J_i(t, \zeta_1, \zeta_2). \quad (i = 1, 2)$$
Therefore, by (12), we have
\[
\limsup_{t \to 0} \partial f(t, x(t)) \in \partial f(t, x(t), x'(t)) \quad \text{a.e. } t \in I.
\] (13)
where the Kuratowski limit superior (cf. Definition 3.14 of [22]) is given by
\[
\limsup_{t \to 0} \partial f(t, x(t)) = \{ \xi \in V_1 : \xi_t = \limsup_{t \to 0} \xi_t^{\cap}, \xi_t^{\cap} \in \partial f(t, x(t)) \}, \quad n_1 < \cdots < n_k < \cdots.
\]
So, from (11) and (13), we get for \( i = 1, 2 \),
\[
\zeta_i(t) = \limsup_{t \to 0} [\zeta_i(t)]_{n \geq 1} \subseteq \limsup_{t \to 0} [\zeta_i(t)]_{n \geq 1} \subseteq \partial f(t, x(t), x'(t)) \quad \text{a.e. } t \in I.
\]
Since for \( i = 1, 2 \), \( \zeta_i(t) \in \partial f(t, x(t), x'(t)) \), \( t \in I \), it is clear that \( \zeta_i \in \mathcal{P}_i(x_1, x_2) \).
Therefore, \( \mathcal{P}_i(x_1, x_2) \) is nonempty. The proof is completed. \( \square \)

**Lemma 3.7.** Let \( i \in \{ 1, 2 \} \). If conditions (H1)-(H4) are satisfied, then operator \( \mathcal{P}_i \) is closed in \( C \times L^2(I, V_i) \), where \( L^2(I, V_i) \) is the space furnished with the \( w \)-topology of \( L^2(I, V_i) \).

**Proof.** By Lemma 3.6, we know that for each \( x = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2) \), the set \( \mathcal{P}_i(x_1, x_2) \) has nonempty, convex and weakly compact values for \( i = 1, 2 \). Utilizing the similar arguments to those in the proof of Lemma 3.6, we can prove that for \( i = 1, 2 \) the operator \( \mathcal{P}_i : C = C(I, V_1) \times C(I, V_2) \to L^2(I, V_i) \) is u.s.c. from \( C = C(I, V_1) \times C(I, V_2) \) into \( L^2(I, V_i) \) where for \( i = 1, 2 \). So, it follows from Lemma 3.5(ii) that for \( i = 1, 2 \), the graph of the u.s.c. multifunction \( \mathcal{P}_i \) with closed values is closed (due to Lemma 3.3 (i)), which hence implies that for \( i = 1, 2 \), if \( x^n = (x^n_1, x^n_2) \to x = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2) \), \( w^n_i \to w_i \) weakly in \( L^2(I, V_i) \) and \( w^n_i \in \mathcal{P}_i(x^n_1, x^n_2) \), then \( w_i \in \mathcal{P}_i(x_1, x_2) \). The proof is completed. \( \square \)

**Lemma 3.8.** Let \( i \in \{ 1, 2 \} \). If the conditions (H1)-(H4) are satisfied, then for a.e. \( t \in I \), the multifmap \( \partial f_i(t, \cdot, \cdot) : V_1 \times V_2 \to P(V) \) has the Cesari property, i.e., \( \bigcap_{\delta > 0} \overline{\partial f_i}(t, V_0(x)) = \partial f_i(t, x) \) for all \( x = (x_1, x_2) \in V = V_1 \times V_2 \).

**Proof.** On one hand, it is clear that for any \( \delta > 0 \), \( \partial f_i(t, x) \subseteq \overline{\partial f_i}(t, V_0(x)) \), for all \( x = (x_1, x_2) \in V = V_1 \times V_2 \) and a.e. \( t \in I \). Therefore, \( \partial f_i(t, x) \subseteq \bigcap_{\delta > 0} \overline{\partial f_i}(t, V_0(x)) \), for all \( x = (x_1, x_2) \in V = V_1 \times V_2 \) and a.e. \( t \in I \).

On the other hand, by the proof of Lemma 3.6, we know that for a.e. \( t \in I \), the multifunction \( (x_1, x_2) \to \partial f_i(t, x_1, x_2) \) is u.s.c. from \( V_1 \times V_2 \) into \( V_0(x) \). Let \( t \in I \), \( x = (x_1, x_2) \in V \) be fixed. For any neighborhood \( \Omega_1 \supset \partial f_i(t, x) \) (in the sense of weak topology of \( V_0(x) \)), there exists a \( \delta > 0 \) such that \( \partial f_i(t, V_0(x)) \subseteq \Omega_1 \). Since \( V_0(x) \) is locally convex, we can choose \( \Omega_1 \) to be convex. Therefore, \( \overline{\partial f_i}(t, V_0(x)) \subseteq \Omega_1 \).

Now, we show that \( \partial f_i(t, x) = \Omega_1 \) for all neighborhood \( \Omega_1 \) of \( \partial f_i(t, x) \). To the contrary, there exists \( y_1 \in \bigcap \Omega_1 \) and \( y_2 \notin \partial f_i(t, x) \). Then there exists a closed set \( \Omega_1 \supset \partial f_i(t, x) \), which contradicts the fact that \( \partial f_i(t, x) \subseteq \Omega_1 \). By the separation property, there exist a neighborhood \( N_1 \) of \( \partial f_i(t, x) \) and a neighborhood \( \Omega_2 \) of \( \partial f_i(t, x) \) such that \( N_1 \cap \Omega_2 = \emptyset \). This shows \( y_1 \notin \Omega_2 \), which is a contradiction. Therefore, \( \bigcap_{\delta > 0} \overline{\partial f_i}(t, V_0(x)) \subseteq \bigcap \Omega_1 = \partial f_i(t, x) \). The proof is complete. \( \square \)

**Definition 3.9.** A pair of functions \( x = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2) \) is said to be a mild solution of system (5) on the interval \( I = [0, T] \) if
\[
\begin{align*}
x_1(t) &= T_1(t)x_1^0 + \int_0^t T_1(t-s)(B_1u_1(s) + f_1(s))ds, \quad t \in I, \\
x_2(t) &= T_2(t)x_2^0 + \int_0^t T_2(t-s)(B_2u_2(s) + f_2(s))ds, \quad t \in I,
\end{align*}
\]
where \( (f_1, f_2) \in \mathcal{P}_{1,2}(x) \times \mathcal{P}_{1,2}(x) \).

**Lemma 3.10.** (16). Let \( i \in \{ 1, 2 \} \). If condition (HT) holds, then operator \( G_i : L^p(I, V_i) \to C(I, V_i) \) for some \( p > 1 \), given by \( (G_if)(t) = \int_0^t T_i(t-s)f_i(s)ds \), is compact for \( f_i \in L^p(I, V_i) \).
Now we can obtain the following result.

**Theorem 3.11.** If the conditions (HT) and (H1)-(H4) are satisfied, then for any $u = (u_1, u_2) \in L^2(I, U_1) \times L^2(I, U_2)$, system (5) has at least one mild solution in $C = C(I, V_1) \times C(I, V_2)$.

**Proof.** Consider the multimap $F : C \to K_0(C)$ defined by

$$F(x) = \{y = (y_1, y_2) \in C : \exists f = (f_1, f_2) \in P_1^x(x) \times P_2^x(x) \text{ s.t. } y_1(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)(B_1u_1(s) + f_1(s))ds,$$

$$y_2(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)(B_2u_2(s) + f_2(s))ds\}, \text{ for all } x = (x_1, x_2) \in C.$$

Now, we verify that $F$ has a fixed point in $C$. First, $F(x)$ is convex by the convexity of $P_i^x(x)$ which follows from Lemma 3.6. Next, we subdivide the proof into five steps.

**Step 1.** $F$ maps bounded sets into bounded sets in $C$.

For $\forall k_0 > 0$, let $B_{k_0} = \{x = (x_1, x_2) \in C : \|x\|_C \leq k_0\}$. Actually, it is enough to show that there exists a positive constant $\ell$ such that for each $\varphi = (\varphi_1, \varphi_2) \in F(x)$, $x \in B_{k_0}$, $\|\varphi\|_C \leq \ell$. If $\varphi = (\varphi_1, \varphi_2) \in F(x)$, then there exists $f = (f_1, f_2) \in P_1^x(x) \times P_2^x(x)$ such that for every $t \in I$,

$$\begin{cases}
    \varphi_1(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)f_1(s)ds + \int_0^t T_1(t-s)B_1u_1(s)ds,
    \\
    \varphi_2(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)f_2(s)ds + \int_0^t T_2(t-s)B_2u_2(s)ds.
\end{cases}$$

By (H3) and the Holder inequality, we obtain that for $i = 1, 2$ and every $t \in I$,

$$\|\varphi_i(t)\|_V \leq \|T_i(t)x_i^0\|_V + \int_0^t \|T_i(t-s)f_i(s)\|_V ds + \int_0^t \|T_i(t-s)B_iu_i(s)\|_V ds$$

$$\leq M_i\|x_i^0\|_V + M_i \int_0^t [\|\phi_i(s)\|_{L^2(U, R)} + \|B_i\|\|u_i(s)\|_{L^2(U, L)}] ds$$

$$\leq M_i\|x_i^0\|_V + M_i([\|\phi_i\|_{L^2(U, R)}] + \|B_i\|\|u_i\|_{L^2(U, L)}) T^{1/2} + M_iL_0k_0T := \ell_i.$$

Let $\ell = \ell_1 + \ell_2$. Then it is easy to see that for each $\varphi = (\varphi_1, \varphi_2) \in F(x)$, $x \in B_{k_0}$, $\|\varphi\|_C \leq \ell$. Thus, $\{F(x) : x = (x_1, x_2) \in B_{k_0}\}$ is bounded in $C$.

**Step 2.** $F$ maps bounded sets into equicontinuous sets of $C$.

In the following, we will show that $\{F(x) : x = (x_1, x_2) \in B_{k_0}\}$ is a family of equicontinuous functions.

Indeed, for any $x = (x_1, x_2) \in B_{k_0}$, $\varphi = (\varphi_1, \varphi_2) \in F(x)$, there exists $f = (f_1, f_2) \in P_1^x(x_1, x_2) \times P_2^x(x_1, x_2)$ such that for every $t \in I$,

$$\begin{cases}
    \varphi_1(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)f_1(s)ds + \int_0^t T_1(t-s)B_1u_1(s)ds,
    \\
    \varphi_2(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)f_2(s)ds + \int_0^t T_2(t-s)B_2u_2(s)ds.
\end{cases}$$

On one hand, for any $x = (x_1, x_2) \in B_{k_0}$, when $t_1 = 0$, $0 < t_2 \leq \delta_0$ and $\delta_0$ is small enough, we obtain that for $i = 1, 2$

$$\|\varphi_i(t_2) - \varphi_i(t_1)\|_V \leq \|T_i(t_2)x_i^0 - x_i^0\|_V + \int_0^{t_2} T_i(t_2-s)(B_iu_i(s) + f_i(s))ds|_V$$

$$\leq \|T_i(t_2)x_i^0 - x_i^0\|_V + M_i([\|\phi_i\|_{L^2(U, R)}] + \|B_i\|\|u_i\|_{L^2(U, L)}) \delta_0^{1/2} + M_iL_0k_0\delta_0.$$

Then, we can easily see that $\|\varphi_i(t_2) - \varphi_i(t_1)\|_V$ tends to zero independently of $x \in B_{k_0}$ as $\delta_0 \to 0$. 
On the other hand, for any \( x = (x_1, x_2) \in B_{b_0} \) and \( \frac{\delta_0}{2} \leq t_1 < t_2 \leq T \), we obtain

\[
\| 2^i (t_2) - 2^i (t_1) \|_{V^i} \leq \| T_i(t_2)x_0^i - T_i(t_1)x_0^i \|_{V^i} + \| \int_{0}^{t_1} T_i(t_2 - s)(B_i u_i(s) + f_i(s))ds \\
- \int_{0}^{t_1} T_i(t_1 - s)(B_i u_i(s) + f_i(s))ds \|_{V^i} \\
\leq \| T_i(t_2)x_0^i - T_i(t_1)x_0^i \|_{V^i} + \| \int_{t_1}^{t_2} T_i(t_2 - s)(B_i u_i(s) + f_i(s))ds \|_{V^i} \\
+ \| \int_{0}^{t_1} [T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds \|_{V^i} \\
:= Q_{i,1} + Q_{i,2} + Q_{i,3}.
\]

By (HJ3), we have \( Q_{i,2} \leq M_1(\| \phi_i \|_{L^2(\mathbb{R}^+)}) + \| B_i \| \| u_i \|_{L^2(\mathbb{R}^+)} \| (t_2 - t_1) \| + M_1L_k \| (t_2 - t_1) \| \). For \( t_1 \geq \frac{\delta_0}{2} > 0 \) and \( \delta > 0 \) being small enough, we obtain

\[
Q_{i,3} \leq \| \int_{0}^{t_1} [T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds \|_{V^i} + \| \int_{t_1}^{t_2} T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds \|_{V^i} \\
\leq (\| \phi_i \|_{L^2(\mathbb{R}^+)}) + \| B_i \| \| u_i \|_{L^2(\mathbb{R}^+)}(t_1 - \delta)^\frac{1}{2} + L_k \| (t_1 - \delta)^\frac{1}{2} + M_1L_k \| \delta \|.
\]

Since (HT) implies the continuity of \( T_i(t) \) (\( t > 0 \)) in \( t \) in the uniform operator topology, it is easily seen that \( Q_{i,3} \) tends to zero independently of \( x = (x_1, x_2) \in B_{b_0} \) as \( t_2 \rightarrow t_1 \), \( \delta \rightarrow 0 \). It is clear that \( Q_{i,1} \) and \( Q_{i,2} \) both tend to zero as \( t_2 \rightarrow t_1 \) does not depend on particular choice of \( x = (x_1, x_2) \). Thus, for \( i = 1, 2 \), we get that \( \| 2^i (t_2) - 2^i (t_1) \|_{V^i} \) tends to zero independently of \( x = (x_1, x_2) \in B_{b_0} \) as \( \delta_0 \rightarrow 0 \). Therefore, \( \{ F(x) : x = (x_1, x_2) \in B_{b_0} \} \) is equicontinuous.

**Step 3.** \( F \) is a compact multivalued map.

We show that for any \( t \in I \), \( \lambda (t) := \{ q(t) : q = (q_1, q_2) \in F(B_{b_0}) \} \) is relatively compact in \( V = V_1 \times V_2 \).

Clearly, \( \lambda (0) = \{ x^0 \} \) with \( x^0 = (x_1^0, x_2^0) \) is compact. So, it is only necessary to consider \( t > 0 \). Let \( 0 < t \leq T \) be fixed. For any \( x = (x_1, x_2) \in B_{b_0} \), \( q = (q_1, q_2) \in F(x) \), there exists \( f = (f_1, f_2) \in P_1^j(x) \times P_2^j(x) \) such that for every \( t \in I \),

\[
\begin{align*}
\begin{cases}
\phi_1(t) = T_1(t)x_1^0 + \int_{0}^{t_1} T_1(t - s)f_1(s)ds + \int_{0}^{t_1} T_1(t - s)B_1u_1(s)ds, \\
\phi_2(t) = T_2(t)x_2^0 + \int_{0}^{t_1} T_2(t - s)f_2(s)ds + \int_{0}^{t_1} T_2(t - s)B_2u_2(s)ds.
\end{cases}
\end{align*}
\]

Let \( 0 < t \leq T \) be fixed. For \( \varepsilon \in (0, t) \), define \( \phi^\varepsilon (t) = (\phi_1^\varepsilon (t), \phi_2^\varepsilon (t)) \), where for \( i = 1, 2 \),

\[
\begin{align*}
\phi_i^\varepsilon (t) &= T_i(t)x_i^0 + \int_{0}^{t_1 - \varepsilon} T_i(t - s)(B_i u_i(s) + f_i(s))ds \\
&= T_i(t)x_i^0 + \int_{0}^{t_1 - \varepsilon} T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds \\
&= T_i(t)z_i(t, \varepsilon) + T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds \\
&:= T_i(t)z_i(t, \varepsilon).
\end{align*}
\]

where \( x = (x_1, x_2) \in B_{b_0} \) and \( f = (f_1, f_2) \in P_1^j(x) \times P_2^j(x) \). For \( i = 1, 2 \), from the boundedness of \( \int_{0}^{t_1 - \varepsilon} T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds \) and the compactness of \( T_i(t) \) (\( t > 0 \)), we obtain that the set \( \{ \phi_i^\varepsilon (t) : q \in F(B_{b_0}) \} \) is relatively
compact in $V_i$ for each $\varepsilon \in (0, t)$. So, it follows that $\Lambda_i(t) := \{q_i(t) \in F(B_{\mathcal{H}}) \}$ is relatively compact in $V = V_1 \times V_2$ for each $\varepsilon \in (0, t)$. Moreover, for every $q = (q_{1}, q_{2}) \in F(x)$ with $x = (x_1, x_2)$, we have for $i = 1, 2$
\[ ||q_i(t) - q_i^*(t)||_{V_i} = \int_{0}^{t} T_i(t-s)(B_{1} u_i(s) + f_i(s))ds \leq M_i(||\phi_i||_{\mathcal{L}^{2}(\mathcal{H}, \mathcal{K})} + ||B_{1}||u_i||_{\mathcal{L}^{2}(\mathcal{H}, \mathcal{K})})\varepsilon + M_i L_k \varepsilon, \]
which implies that for $i = 1, 2$, the set $q_i(t) \in F(B_{\mathcal{H}})$ ($t > 0$) is totally bounded, i.e., relatively compact in $V_i$. So, it follows that $\Lambda_i(t)$ ($t > 0$) is totally bounded, i.e., relatively compact in $V = V_1 \times V_2$. Therefore, from Steps 1-3, $\{F(x) : x = (x_1, x_2) \in B_{\mathcal{H}}\}$ is relatively compact by the generalized Ascoli-Arzela theorem. Thus, $F$ is a compact multivalued map.

**Step 4.** $F$ has a closed graph.

Let $x^n = (x^n_1, x^n_2) \rightarrow x^* = (x^*_1, x^*_2)$ in $C$, $q^n = (q^n_1, q^n_2) \in F(x^n)$ and $q^m \rightarrow q^* = (q^*_1, q^*_2)$ in $C$. We will show that $q^* \in F(x^*)$. Indeed, $q^m \in F(x^n)$ means that there exists a $f^m = (f^n_1, f^n_2) \in P^1_i(x^n) \times P^2_i(x^n)$ such that for each $t \in I$,
\[ \begin{cases} q^n_1(t) = T_1(t)x^n_1 + \int_{0}^{t} T_1(t-s)(B_{1} u_1(s) + f^n_1(s))ds, \\ q^n_2(t) = T_2(t)x^n_2 + \int_{0}^{t} T_2(t-s)(B_{2} u_2(s) + f^n_2(s))ds. \end{cases} \]

For $i = 1, 2$, from (HJ3) it is not difficult to show that $\{f^n_i\}_{n \geq 1} \subseteq L^2(I, V_i)$ is bounded, and hence we may assume that $f^n_i \rightarrow f_i^*$ for some $f_i^* \in L^2(I, V_i)$. Define the continuous linear operator $G_i : L^2(I, V_i) \rightarrow C(I, V_i)$ as follows $G_i(f_i(t)) = \int_{I} T_i(-s)f_i(s)ds$ for $f_i \in L^2(I, V_i)$. Since $x^n \rightarrow x^*$ in $C$ with $x^n = (x^n_1, x^n_2)$ and $x^* = (x^*_1, x^*_2)$, it follows from Lemmas 3.7 and 3.10 that
\[ \begin{cases} q^*_1(t) = T_1(t)x^*_1 + \int_{0}^{t} T_1(t-s)(B_{1} u_1(s) + f_1^*(s))ds, \\ q^*_2(t) = T_2(t)x^*_2 + \int_{0}^{t} T_2(t-s)(B_{2} u_2(s) + f_2^*(s))ds. \end{cases} \]
and $f^* := (f^*_1, f^*_2) \in P^1_i(x^*) \times P^2_i(x^*)$, i.e., $F$ has a closed graph. Therefore, since $F$ takes compact values, from Lemma 2.1 we deduce that $F$ is u.s.c.

**Step 5.** According to Theorem 2.5, it is sufficient to show that there exists an open set $D \subset C$ such that there is no $x \in \partial D$ satisfying $x^* = \lambda F(x)$ for all $\lambda \in (0, 1)$. Indeed, let $x = (x_1, x_2) \in \lambda F(x)$ for some $\lambda \in (0, 1)$. Then, there exists $f = (f_1, f_2) \in P^1_1(x) \times P^2_1(x)$ such that for each $t \in I$,
\[ \begin{cases} x_1(t) = \lambda T_1(t)x^n_1 + \lambda \int_{0}^{t} T_1(t-s)(B_{1} u_1(s) + f_1(s))ds, \\ x_2(t) = \lambda T_2(t)x^n_2 + \lambda \int_{0}^{t} T_2(t-s)(B_{2} u_2(s) + f_2(s))ds. \end{cases} \]

Then by the assumptions, we get that for $i = 1, 2$ and each $t \in I$
\[ ||x_i(t)||_{V_i} \leq ||T_i(t)x^n_i||_{V_i} + ||\int_{0}^{t} T_i(t-s)f_i(s)ds||_{V_i} + ||\int_{0}^{t} T_i(t-s)B_{1} u_i(s)ds||_{V_i} \leq M_i||x^n_i||_{V_i} + M_i \int_{0}^{t} (||\phi_i|| + L_i||x_i(s)||_{V_i} + ||B_{1}||u_i||_{V_i})ds \leq \rho_i + M_i L_i \int_{0}^{t} ||x_i(s)||_{V_i}ds, \]
where $\rho_i = M_i||x^n_i||_{V_i} + (||\phi_i||_{\mathcal{L}^{2}(\mathcal{H}, \mathcal{K})} + ||B_{1}||u_i||_{\mathcal{L}^{2}(\mathcal{H}, \mathcal{K})})T^2 \Gamma$. It follows from the standard Gronwall inequality that for $i = 1, 2$ and each $t \in I$, $||x_i(t)||_{V_i} \leq \rho_i e^{\rho_i L_i T}$. Hence,
\[ ||x||_{C} = ||x_1||_{C(I, V_1)} + ||x_2||_{C(I, V_2)} = \sup_{t \in I} ||x_1(t)||_{V_1} + \sup_{t \in I} ||x_2(t)||_{V_2} \leq \rho_1 e^{\rho_1 L_1 T} + \rho_2 e^{\rho_2 L_2 T} := \ell, \]
Set \( D = \{ x = (x_1, x_2) \in C : \| x \|_C < \ell + 1 \} \). Clearly, \( D \) is an open subset of \( C \), \( \mathcal{F} : \overline{D} \rightarrow K_c(C) \) is u.s.c. and compact. From the choice of \( D \), there is no \( x \in \partial D \) satisfying \( x \in \lambda \mathcal{F}(x) \) for some \( \lambda \in (0, 1) \). Therefore, by Theorem 2.5 we deduce that \( \mathcal{F} \) has a fixed point \( x^* \) in \( C \). Consequently, the system (5) has at least one mild solution in \( C \). The proof is completed. \( \square \)

Now, we give the following definition.

**Definition 3.12.** A pair \((x, u)\) is said to be feasible if \((x, u)\) satisfies system (6) for \( t \in I \).

For convenience, we denote \( U[0, T] = \{ u : [0, T] \rightarrow U \mid u(\cdot) \) is measurable\} where \( U = U_1 \times U_2 \) is the reflexive Banach space endowed by the norm \( \| u \|_{U} = \| u_1 \|_{U_1} + \| u_2 \|_{U_2} \) for all \( u = (u_1, u_2) \in U = U_1 \times U_2, \forall [0, T] = [(x, u) \in C \times U[0, T] \mid (x, u) \) is feasible\].

Now, we study the existence result of feasible pairs for system (6). We assume that for \( i = 1, 2 \), the feedback multimap \( \mathcal{U}_i : I \times V_1 \times V_2 \rightarrow P(U_i) \) satisfies the following conditions:

- \((U1)\) there exist a function \( \phi_i^1 \in L^2(I, R^+) \) and a constant \( L_i^1 > 0 \), such that
  \[ \| \mathcal{U}_i(t, x_1, x_2) \| = \sup_{z \in \mathcal{U}_i(t, x_1, x_2)} \| z \|_{U_i} \leq \phi_i^1(t) + L_i^1 \| x \|_{V_i} \]

for all \((t, x_1, x_2) \in I \times V_1 \times V_2\);

- \((U2)\) for a.e. \( t \in I \), \( x = (x_1, x_2) \in V = V_1 \times V_2 \), the set \( \mathcal{U}_i(t, x) \) satisfies the following
  \[ \bigcap_{U \in \mathcal{U}_i(t, x)} U = \emptyset \]

**Remark 3.13.** (\( \{14\} \)). Let \( i \in \{1, 2\} \). Then by Lemma 2.1, condition \((U2)\) is fulfilled if \( \mathcal{U}_i \) is u.s.c. with convex and closed values.

Now, we are in a position to present the main result of this section.

**Theorem 3.14.** If conditions \((HT), (HJ1)-(HJ4)\) and \((U1)-(U2)\) are satisfied, then the set \( V[0, T] \) is nonempty.

**Proof.** For any integer \( k > 0 \), let \( t_j = j \frac{T}{k}, 0 \leq j \leq k - 1 \). We define \( u^k = (u_1^k, u_2^k) \) where \( i = 1, 2 \),

\[ u_i^k(t) = \sum_{j=0}^{k-1} u_{i,j} \chi_{[t_j, t_{j+1})}(t), \quad t \in I, \]

where \( \chi_{[t_j, t_{j+1})} \) is the characteristic function of interval \([t_j, t_{j+1})\). The sequence \( \{u_{i,j}\}_{j=0}^{k-1} \) is constructed as follows.

Firstly, for \( i = 1, 2 \), take \( u_{i,0} \in \mathcal{U}_i(0, x^0) \) with \( x^0 = (x_1^0, x_2^0) \). By Theorem 3.11, there exists \( x^k(\cdot) = (x_1^k(\cdot), x_2^k(\cdot)) \) given by

\[
\begin{align*}
\left\{ x_1^k(t) = T_1(t)x_1^0 + \int_{0}^{t} T_1(t-s)(B_1u_{1,0}(s) + f_1^0(s))ds, \quad t \in [0, T], \right.

\left. x_2^k(t) = T_2(t)x_2^0 + \int_{0}^{t} T_2(t-s)(B_2u_{2,0}(s) + f_2^0(s))ds, \quad t \in [0, T]. \right.
\end{align*}
\]

where \( \Phi = (f_1^0, f_2^0) \in P_1^0(x^0) \times P_2^0(x^0) \). Then take \( u_{i,1} \in \mathcal{U}_i(\frac{T}{k}, x^k(\frac{T}{k})) \) for \( i = 1, 2 \). We can repeat this procedure to obtain \( x^k = (x_1^k, x_2^k) \) on \([\frac{T}{k}, \frac{2T}{k}], \) etc. By induction, we end up with the following: for \( i = 1, 2 \),

\[
\begin{align*}
\left\{ x_1^k(t) = T_1(t)x_1^0 + \int_{0}^{t} T_1(t-s)(B_1u_{1,j}(s) + f_1^j(s))ds, \quad t \in [0, T], \right.

\left. x_2^k(t) = T_2(t)x_2^0 + \int_{0}^{t} T_2(t-s)(B_2u_{2,j}(s) + f_2^j(s))ds, \quad t \in [0, T]. \right.
\end{align*}
\]

where \( \Phi = (f_1^j, f_2^j) \in P_1^j(x^k) \times P_2^j(x^k) \). From the proof of Theorem 3.11, it is easy to prove that for \( i = 1, 2 \), there exists \( r^0_i > 0 \) such that \( \| x_i^k \|_{U_i(V)} \leq r_i^0 \). Moreover, it follows from \((HJ3)\) and \((U1)\) that for \( i = 1, 2 \) there
exist \( r_1, r_2 > 0 \) such that \( \|u^i(t)\|_{L^2(U,V_i)} \leq r_1 \), \( \|f^i(t)\|_{L^2(U,V_i)} \leq r_2 \). Therefore, there are subsequences of \( \{u^i(t)\} \) and \( \{f^i(t)\} \), denoted by \( \{u^k(t)\} \) and \( \{f^k(t)\} \) again, such that for \( i = 1, 2 \),

\[
|u^k(t) - u^i(t)| \rightarrow 0 \quad \text{in} \quad L^2(U, V_i), \quad |f^k(t) - f^i(t)| \rightarrow 0 \quad \text{in} \quad L^2(U, V_i).
\]

(14)

From (HT), by Lemma 3.10 we have that for \( i = 1, 2 \) and any \( t \in I \)

\[
\int_0^t T_i(t-s)(B_i u^i(s) + f^i(s))ds \rightarrow \int_0^t T_i(t-s)(B_i u^i(s) + f^i(s))ds.
\]

For \( i = 1, 2 \), we let \( x^i(t) = T_i(t)x^0 + \int_0^t T_i(t-s)(B_i u^i(s) + f^i(s))ds \), \( t \in I \). Then, for \( i = 1, 2 \), \( x^i(t) \rightarrow x^i(t) \), uniformly in \( t \in I \), i.e., \( x^k(t) \rightarrow x^i(t) \) in \( C(I, V_i) \). So it follows that \( x^k(t) \rightarrow x^i(t) \) in \( C \).

Hence, for any \( \delta > 0 \), there exists an integer \( k_0 > 0 \) such that

\[
x^k(t) \in O_{\delta}(x^i(t)), \quad t \in I, \quad k \geq k_0.
\]

(15)

On the other hand, by the definition of \( u^k(t) \) for \( k \) large enough, we have

\[
u^k_i(t) \in U_i(t, x^k(t)) \subset U_i(O_{\delta}(x^i(t))), \quad \forall t \in \left[ \frac{jT}{k}, \frac{(j+1)T}{k} \right), \quad 0 \leq j \leq k - 1.
\]

(16)

Secondly, by (14) and Mazur Theorem [16], let \( a_{m,l}, b_{m,l} \geq 0 \) and \( \sum_{m \geq 1} a_{m,l} \equiv \sum_{m \geq 1} b_{m,l} = 1 \) such that for \( i = 1, 2 \),

\[
\phi^i_j(t) = \sum_{m \geq 1} a_{m,l} \phi^i_m(t) \rightarrow u^i(t) \quad \text{in} \quad L^2(I, U_i), \quad \psi^i_j(t) = \sum_{m \geq 1} b_{m,l} \psi^i_m(t) \rightarrow f^i(t) \quad \text{in} \quad L^2(I, V_i).
\]

Then, there are subsequences of \( \{\phi^i_j(t)\} \) and \( \{\psi^i_j(t)\} \) with \( \phi^i_j = (\phi^i_{j1}, \phi^i_{j2}) \) and \( \psi^i_j = (\psi^i_{j1}, \psi^i_{j2}) \), denoted by \( \{\phi^i_j\} \) and \( \{\psi^i_j\} \) again, such that for \( i = 1, 2 \), \( \phi^i_j(t) \rightarrow u^i(t) \) in \( U_i \), \( \psi^i_j(t) \rightarrow f^i(t) \) in \( V_i \), a.e. \( t \in I \). Hence, from (15) and (16), for \( l \) large enough, \( \phi^i_j(t) \in \text{co}U_i(O_{\delta}(x^i(t))) \), \( \psi^i_j(t) \in \text{co}U_i(O_{\delta}(x^i(t))) \), a.e. \( t \in I \). Thus, for any \( \delta > 0 \),

\[
u^i_j(t) \in \text{co}U_i(O_{\delta}(x^i(t))), \quad f^i_j(t) \in \text{co}U_i(O_{\delta}(x^i(t))), \quad a.e. \ t \in I.
\]

From (U2) and Lemma 3.8, we have \( \nu^i_j(t) \in U_i(t, x^i(t)), \quad f^i_j(t) \in \partial J_i(t, x^i(t)), \quad a.e. \ t \in I \). Therefore, \( (x^*, u^*) \) is a feasible pair on \( I \). The proof is completed.

Remark 3.15. It is worth pointing out that the system (5) of semilinear inclusions and the feedback control system (6) reduce to the semilinear inclusion (2.1) and the feedback control problem (2.2) in [14], respectively, via choosing appropriately the operators \( A_i, B_i, f, U_i \) and \( u_i, x_i \) for \( i = 1, 2 \). By introducing the operators \( \mathcal{P}_i : C = C(I, V_i) \times C(I, V_i) \rightarrow P(L^2(I, V_i)), i = 1, 2 \) and using Theorem 2.5 for the existence of fixed points of u.s.c. and compact multimap, we prove Theorem 3.11 for the existence of mild solutions of system (5), which generalizes and extends Theorem 3.5 in [14] from the semilinear inclusion (2.1) in [14] to the system (5) of semilinear inclusions. By using Theorem 3.11 and the conditions (U1)-(U2) on the feedback multimap \( \mathcal{U}_i : I \times V_1 \times V_2 \rightarrow P(U_i), i = 1, 2 \), we establish Theorem 3.14 for the existence of feasible pairs of system (6), which generalizes and extends Theorem 3.8 in [14] from the feedback control problem (2.2) in [14] to the feedback control system (6).

4. Existence of Optimal State-Control Pairs

In this section, we consider the optimal control system stated as follows.

System (\( \varphi \)): find a pair \( (x^*, u^*) \in V[0, T] \) such that

\[
\begin{cases}
\varphi_1(x^*_1, u^*_1) \leq \varphi_1(x_1, u_1), \\
\varphi_2(x^*_2, u^*_2) \leq \varphi_2(x_2, u_2),
\end{cases}
\]


for all \((x, u) \in V[0, T]\), where \(\varphi_i(x_i, u_i) = \int_0^T f_i(t_i, x_i(t), u_i(t)) dt\) for \(i = 1, 2\).

We make the following assumptions on \(P^0 = \{f_1^0, f_2^0\}^2\):

(\(P^0\)) the functional \(f_i^0 : I \times V_i \times U_i \rightarrow R \cup \{\pm \infty\}\) is Borel measurable in \((x_i, u_i, t)\) for \(i = 1, 2\);

(\(P^2\)) \(f_i^0(t, \cdot, \cdot, \cdot)\) is lower semicontinuous on \(V_i \times U_i\) a.e. \(t \in I\) for \(i = 1, 2\) (i.e., for all \(x_i \in V_i, u_i \in U_i, [x_i'] \subset V_i, [u_i'] \subset U_i\) such that \(x_i'' \rightarrow x_i, u_i'' \rightarrow u_i\) in \(V_i, U_i\), we have \(\liminf_{\delta \rightarrow 0} f_i^0(t, x_i', u_i') \geq f_i^0(t, x_i, u_i)\)) and there exists a constant \(M_{i} > 0\) such that \(f_i^0(t, x_i, u_i) \geq -M_{i}\).

For any \((t, x) \in I \times V\) with \(x = (x_1, x_2) \in V = V_1 \times V_2\), we set the set

\[
\mathcal{E}_i(t, x) = \{(z_0^i, z_1^i, z_2^i) \in R \times V_i \times U_i \mid z_i^0 \geq f_i^0(t, x_i, z_i^2), z_i^1 \in \partial f_i(t, x), z_i^2 \in \mathcal{U}_i(t, x)\}.
\]

To obtain the existence result of optimal state-control pairs for system \((\varphi)\), we assume that

(\((V_i)\)) for a.e. \(t \in I\), the map \(\bar{e}_i(t, \cdot) : V_1 \times V_2 \rightarrow \overline{P}(R \times V_1 \times U_1)\) has the Cesari property, i.e.,

\[
\bigcup_{\delta > 0} \bigcap_{\epsilon > 0} \mathcal{E}_i(t, \delta O(a(x))) = \emptyset, \quad \forall x \in V.
\]

**Theorem 4.1.** If conditions (HT), (H1)-(H4), (U1)-(U2), (P1)-(P2), \((V_i)\) are satisfied, then system \((\varphi)\) admits at least one optimal state-control pair.

**Proof.** For \(i = 1, 2\), without considering the situation \(\inf[\varphi_i(x_i, u_i) \mid (x, u) \in V[0, T]] = +\infty\), we assume that \(\inf[\varphi_i(x_i, u_i) \mid (x, u) \in V[0, T]] = m_i < +\infty\). By (\(P^2\)), we have \(\varphi_i(x_i, u_i) \geq m_i \geq -M_{i} > -\infty\) for \(i = 1, 2\). Then there exists a sequence \(\{(x_i^0, u_i^0)\}_{i \geq 1} \subset V[0, T]\) such that for \(i = 1, 2\)

\[
\varphi_i(x_i^0, u_i^0) \rightarrow m_i.
\]

From the proof of Theorem 3.14, without loss of generality, we obtain that for \(i = 1, 2\)

\[
x_i^0(\cdot) \rightarrow x_i^{\ast}(\cdot) \in C(I, V_i),
\]

and

\[
u_i^0(\cdot) \rightarrow \nu_i^{\ast}(\cdot) \in L^2(I, U_i), \quad f_i^0(\cdot) \rightarrow f_i^{\ast}(\cdot) \in L^2(I, V_i),
\]

where \(x_i^{\ast}(t) = T_i(t)x_i^0 + \int_0^t T_i(t-s)(B_i\nu_i(s) + f_i(s)) ds, \quad t \in I\).

By Mazur Theorem again, let \(a_{m,J}b_{m,J} \geq 0\) and \(\sum_{\ell \geq 1} a_{m,J} = \sum_{\ell \geq 1} b_{m,J} = 1\) such that for \(i = 1, 2\),

\[
\psi_i(t) = \sum_{m \geq 1} a_{m,J} u_i^{\ast m}(t) \rightarrow u_i^{\ast}(t) \text{ in } L^2(I, U_i), \quad \psi_i(t) = \sum_{m \geq 1} b_{m,J} f_i^{\ast m}(t) \rightarrow f_i^{\ast}(t) \text{ in } L^2(I, V_i).
\]

For \(i = 1, 2\), \(\psi_i(t) = \sum_{\ell \geq 1} b_{\ell,J} f_i^{\ast}(t, x_i^{\ast k}(t), u_i^{\ast k}(t))\), and \(f_i^{\ast}(t) = \lim_{\delta \rightarrow 0} \psi_i^{\ast}(t) \leq -M_{i}\), a.e. \(t \in I\). For any \(\delta > 0\) and \(I\) large enough, from (\(P^2\)) we have for \(i = 1, 2\),

\[
(\psi_i^{\ast}(t), \psi_i(t), \phi_i(t)) \in e_i(t, O_0(x_i^{\ast}(t))), \quad \text{a.e. } t \in I,
\]

with \(x^{\ast} = (x_1^{\ast}, x_2^{\ast})\). Then for \(i = 1, 2\), \((f_i^{\ast}(t), f_i^{\ast}(t), u_i^{\ast}(t)) \in \overline{e} \mathcal{C}_i(t, O_0(x_i^{\ast}(t)))\), a.e. \(t \in I\). From \((V_c)\), we have for

\[
(\psi_i^{\ast}(t), \phi_i(t), u_i^{\ast}(t)) \in e_i(t, x_i^{\ast}(t)), \quad \text{a.e. } t \in I,
\]

i.e.,

\[
\begin{align*}
&f_i^{\ast}(t) \geq f_i^{0}(t, x_i^{\ast}(t), u_i^{\ast}(t)), \quad t \in I, \\
&\phi_i^{\ast}(t) \in \partial f_i(t, x_i^{\ast}(t)), \quad t \in I, \\
&\nu_i^{\ast}(t) \in \mathcal{U}_i(t, x_i^{\ast}(t)), \quad t \in I.
\end{align*}
\]
Therefore, \((x', u') \in V[0, T]\). By Fatou’s Lemma, we obtain that for \(i = 1, 2\),
\[
\int_0^T f_i'(t) dt = \int_0^T \lim_{n \to +\infty} \psi_i(t) dt \leq \lim_{n \to +\infty} \int_0^T \psi_i(t) dt
\]
\[
= \lim_{n \to +\infty} \int_0^T \sum_{k \geq 1} b_{k,i} f_i^0(t, x_i^k(t), u_i^k(t)) dt
\]
\[
= \lim_{n \to +\infty} \sum_{k \geq 1} b_{k,i} \int_0^T f_i^0(t, x_i^k(t), u_i^k(t)) dt
\]
\[
= \sum_{k \geq 1} b_{k,i} \lim_{n \to +\infty} \int_0^T f_i^0(t, x_i^k(t), u_i^k(t)) dt
\]
\[
= m_i.
\]
Therefore, for \(i = 1, 2\),
\[
m_i \leq \varphi_i(x_i', u_i') = \int_0^T f_i^0(t, x_i'(t), u_i'(t)) dt \leq m_i,
\]
i.e.,
\[
\int_0^T f_i^0(t, x_i'(t), u_i'(t)) dt = m_i = \inf_{(x,u) \in V[0,T]} \varphi_i(x_i, u_i)
\]
with \(x = (x_1, x_2)\) and \(u = (u_1, u_2)\).
Thus, \((x', u')\) is an optimal state-control pair. The proof is completed. \(\square\)

**Remark 4.2.** It is worth pointing out that the optimal control system \((\varphi)\) reduce to the optimal control problem \((\varphi)\) in [14], via choosing appropriately the functions \(f_i^0, \varphi_i\) and \(u_i, x_i\) for \(i = 1, 2\). By introducing the operators \(\varepsilon_i : I \times V_1 \times V_2 \to P(R \times V_1 \times U_1), i = 1, 2\) and using Mazur Theorem and the proof of Theorem 3.14, we prove Theorem 4.1 for the existence of optimal state-control pairs of system \((\varphi)\), which generalizes and extends Theorem 4.1 in [14] from the optimal control problem \((\varphi)\) in [14] to the optimal control system \((\varphi)\).

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