Hermite-Hadamard Type Inequalities for $F$-Convex Function Involving Fractional Integrals

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Abstract. In this study, we firstly give some properties the family $F$ and $F$–convex function which are defined by B. Samet. Then, we establish Hermite–Hadamard type inequalities involving fractional integrals via $F$–convex function. Some previous results are also recaptured as special cases.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. If $f$ is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [14]

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$  (1)

Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping $f$. Both inequalities hold in the reversed direction if $f$ is concave (1).

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [2, 3, 7, 8, 10, 13, 19, 20]) and the references therein.

Over the years, many type of convexity have been defined, such as quasi-convex [1], pseudo-convex [11], strongly convex [16], $\varepsilon$–convex [6], $s$–convex [5], $h$–convex [22] etc. Recently, Samet [17] have defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity, including $\varepsilon$–convex functions, $\alpha$–convex functions, $h$–convex functions, and many others.

Recall the family $F$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ satisfying the following axioms:

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2010 Mathematics Subject Classification. Primary 26D07; Secondary 26D10, 26D15, 26A33

Keywords. Hermite-Hadamard inequality, $F$–convex, fractional integral

Received: 28 May 2017; Revised: 27 September 2017; Accepted: 30 September 2017

Communicated by Ljubiša D.R. Kočinac

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(A1) If $u_i \in L^1(0, 1), i = 1, 2, 3$, then for every $\lambda \in [0, 1]$, we have
\[
\int_0^1 F(u_1(t), u_2(t), u_3(t), \lambda) dt = F \left( \int_0^1 u_1(t) dt, \int_0^1 u_2(t) dt, \int_0^1 u_3(t) dt, \lambda \right).
\]

(A2) For every $u \in L^1(0, 1), w \in L^\infty(0, 1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have
\[
\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2) dt = T_{F,w} \left( \int_0^1 w(t)u(t) dt, z_1, z_2 \right),
\]
where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on $(F, w)$, and it is nondecreasing with respect to the first variable.

(A3) For any $(w, u_1, u_2, u_3) \in \mathbb{R}^4, u_4 \in [0, 1]$, we have
\[
wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w
\]
where $L_w \in \mathbb{R}$ is a constant that depends only on $w$.

**Definition 1.1.** Let $f : [a, b] \to \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b$, be a given function. We say that $f$ is a convex function with respect to some $F \in \mathcal{F}$ (or $F$-convex function) if
\[
F(f(tx + (1 - t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]

**Remark 1.2.** 1) Let $\varepsilon \geq 0$, and let $f : [a, b] \to \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b$, be an $\varepsilon$-convex function, that is (see [6])
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by
\[
F(u_1, u_2, u_3, u_4) = u_1 - u_4u_2 - (1 - u_4)u_3 - \varepsilon
\]
and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by
\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 tw(t) dt \right)u_2 - \left( \int_0^1 (1 - t)w(t) dt \right)u_3 - \varepsilon.
\]
For
\[
L_w = (1 - w)\varepsilon,
\]
it is clear that $F \in \mathcal{F}$ and
\[
F(f(tx + (1 - t)y), f(x), f(y), t) = f(tx + (1 - t)y) - tf(x) - (1 - t)f(y) - \varepsilon \leq 0,
\]
that is $f$ is an $F$-convex function. Particularly, taking $\varepsilon = 0$, we show that if $f$ is a convex function then $f$ is an $F$-convex function with respect to $F$ defined above.

2) Let $f : [a, b] \to \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b$, be an $\alpha$-convex function, $\alpha \in (0, 1)$, that is
\[
f(tx + (1 - t)y) \leq t^\alpha f(x) + (1 - t^\alpha)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].
\]

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by
\[
F(u_1, u_2, u_3, u_4) = u_1 - u_4^\alpha u_2 - (1 - u_4^\alpha)u_3
\]
and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 t^w w(t) dt \right) u_2 - \left( \int_0^1 (1 - t^w) w(t) dt \right) u_3. \tag{6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - t^w f(x) - (1 - t^w) f(y) \leq 0,$$

that is $f$ is an $F$-convex function.

3) Let $h : J \rightarrow [0, \infty)$ be a given function which is not identical to 0, where $J$ is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \rightarrow [0, \infty), (a, b) \in \mathbb{R}^2, \ a < b$, be an $h$-convex function, that is (see [22])

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1 - u_4)u_3 \tag{7}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left( \int_0^1 h(t) w(t) dt \right) u_2 - \left( \int_0^1 h(1 - t) w(t) dt \right) u_3. \tag{8}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is $f$ is an $F$-convex function.

In [17], the author established the following Hermite-Hadamard type inequalities using the new convexity concept:

**Theorem 1.3.** Let $f : [a, b] \rightarrow \mathbb{R}, (a, b) \in \mathbb{R}^2, a < b$, be an $F$-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L_1[a, b]$. Then

$$F \left( f \left( \frac{a + b}{2} \right), \frac{1}{b - a} \int_a^b f(x) dx, \frac{1}{b - a} \int_a^b f(x) dx, \frac{1}{2} \right) \leq 0,$$

$$T_{F,1} \left( \frac{1}{b - a} \int_a^b f(x) dx, f(a), f(b) \right) \leq 0.$$

**Theorem 1.4.** Let $f : I^* \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^*$, $(a, b) \in I^* \times I^*, \ a < b$. Suppose that

(i) $|f'|$ is $F$-convex on $[a, b]$, for some $F \in \mathcal{F}$

(ii) the function $t \in (0, 1) \rightarrow L_w(t)$ belongs to $L^1(0, 1)$, where $w(t) = |1 - 2t|$. Then,

$$T_{F,w} \left( \frac{2}{b - a} \left| f(a) + f(b) \right|, \frac{1}{b - a} \int_a^b f(x) dx, \left| f'(a) \right|, \left| f'(b) \right| \right) + \int_0^1 L_w(t) dt \leq 0.$$
Theorem 1.5. Let \( f : I' \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I' \), \((a, b) \in I' \times I'\), \(a < b\) and let \( p > 1\). Suppose that \( \left| f' \right|^{p/(p-1)} \) is \( F \)-convex on \([a, b]\), for some \( F \in \mathcal{F}' \) and \( \int f' \in \mathcal{L}^{p/(p-1)}(a, b)\). Then

\[
T_{E,1}(A(p, f), \left| f'(a) \right|^{p/(p-1)}, \left| f'(b) \right|^{p/(p-1)}) \leq 0
\]

where

\[
A(p, f) = \left( \frac{2}{b-a} \right)^{1/p} (p+1)^{1/p} \left| \frac{1}{2} \right| \int_a^b f(x)dx \right|^{1/p}.
\]

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [4, 9, 12, 15].

Definition 1.6. Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_{a+}^a f \) and \( J_{b-}^a f \) of order \( \alpha > 0 \) with \( x \geq a \) are defined by

\[
J_{a+}^a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
J_{b-}^a f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]

respectively. Here, \( \Gamma(\alpha) \) is the Gamma function and \( J_{a+}^a f(x) = J_{b-}^a f(x) = f(x) \).

It is remarkable that Sarikaya et al. [21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.7. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2^{\alpha}(b-a)^{\alpha}} \left( J_{a+}^a f(b) + J_{b-}^a f(a) \right) \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

Meanwhile, Sarikaya et al. [21] presented the following important integral identity including the first-order derivative of \( f \) to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order \( \alpha > 0 \).

Lemma 1.8. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f' \in L[a, b] \), then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} = \frac{\Gamma(\alpha+1)}{2^{\alpha}(b-a)^{\alpha}} \left( J_{a+}^a f(b) + J_{b-}^a f(a) \right) = \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha} - t^\alpha] f'(ta + (1-t)b) dt.
\]

2. Hermite-Hadamard Type Inequality Involving Fractional Integrals

In this section, we establish some inequalities of Hermite-Hadamard type including fractional integrals via \( F \)-convex functions.
Theorem 2.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I' \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on $I'$, $a, b \in I'$, $a < b$. If $f$ is $F$-convex on $[a, b]$, for some $F \in \mathcal{F}$, then we have the inequalities

$$F \left( f \left( \frac{a+b}{2} \right) \right), \frac{\Gamma(a+1)}{(b-a)^a} f(a) \cdot \frac{1}{2} + \int_0^1 L_{a(t)}dt \leq 0 \tag{11}$$

and

$$T_{F,a} \left( \frac{\Gamma(a+1)}{(b-a)^a} [f'(a), f'(b)] f(a) \right), f(a) + f(b), f(a) + f(b) \cdot \frac{1}{2} + \int_0^1 L_{a(t)}dt \leq 0 \tag{12}$$

where $w(t) = at^{a-1}$.

Proof. Since $f$ is $F$-convex, we have

$$F \left( f \left( \frac{x+y}{2} \right) \right), f(x), f(y), \frac{1}{2} \leq 0, \; x, y \in [a, b]$$

For $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we have

$$F \left( f \left( \frac{a+b}{2} \right) \right), f(ta + (1-t)b), f(tb + (1-t)a), \frac{1}{2} \leq 0, \; t \in [0, 1]$$

Multiplying this inequality by $w(t) = at^{a-1}$ and using axiom (A3), we get

$$F \left( \frac{\alpha}{(b-a)^a} f(a) \right), at^{a-1} f(ta + (1-t)b), at^{a-1} f(tb + (1-t)a), \frac{1}{2} + L_{a(t)} \leq 0$$

for $t \in [0, 1]$. Integrating over $[0, 1]$ with respect to the variable $t$ and using axiom (A1), we obtain

$$F \left( f \left( \frac{a+b}{2} \right) \right) \alpha \int_0^1 t^{a-1}dt, \alpha \int_0^1 t^{a-1} f(ta + (1-t)b)dt, \alpha \int_0^1 t^{a-1} f(tb + (1-t)a)dt, \frac{1}{2} + \int_0^1 L_{a(t)}dt \leq 0$$

Using the facts that

$$\int_0^1 t^{a-1} f(ta + (1-t)b)dt = \frac{1}{(b-a)^a} \int_a^b (b-x)^{a-1} f(x)dx = \frac{\Gamma(a)}{(b-a)^a} f(b)$$

and

$$\int_0^1 t^{a-1} f(tb + (1-t)a)dt = \frac{1}{(b-a)^a} \int_a^b (x-a)^{a-1} f(x)dx = \frac{\Gamma(a)}{(b-a)^a} f(a)$$

we obtain

$$F \left( f \left( \frac{a+b}{2} \right) \right), \frac{\Gamma(a+1)}{(b-a)^a} f(b), \frac{\Gamma(a+1)}{(b-a)^a} f(a), \frac{1}{2} + \int_0^1 L_{a(t)}dt \leq 0$$

which gives (11).

On the other hand, since $f$ is $F$-convex, we have

$$F \left( f \left( ta + (1-t)b, f(a), f(b), t \right) \right) \leq 0, \; t \in [0, 1]$$

and

$$F \left( f \left( tb + (1-t)a, f(b), f(a), t \right) \right) \leq 0, \; t \in [0, 1]$$
Using the linearity of $F$, we get

$$F(f(ta + (1-t)b) + f((tb + (1-t)a), f(a) + f(b), f(a) + f(b), t) \leq 0, \quad t \in [0,1].$$

Applying the axiom (A3) for $\omega(t) = at^{\alpha-1}$, we obtain

$$F(at^{\alpha-1} [f(ta + (1-t)b) + f((tb + (1-t)a)], at^{\alpha-1} [f(a) + f(b)], at^{\alpha-1} [f(a) + f(b)], t] + L_{\omega(t)} \leq 0,$$

for $t \in [0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$T_{F,\omega} \left( \int_0^1 at^{\alpha-1} [f(ta + (1-t)b) + f((tb + (1-t)a)] dt, f(a) + f(b), f(a) + f(b) \right) \leq 0,$$

that is

$$T_{F,\omega} \left( \Gamma(a+1) (\frac{1}{b-a})^a f(b) + \Gamma(a+1) (\frac{1}{b-a})^b f(a), f(a) + f(b), f(a) + f(b) \right) \leq 0.$$

This completes the proof. □

**Corollary 2.2.** If we choose $F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1 - u_4) u_3 - \epsilon$ in Theorem 2.1, then the function $f$ is $\epsilon$-convex on $[a, b]$, $\epsilon \geq 0$ and we have the inequality

$$f\left(\frac{a+b}{2}\right) - \epsilon \leq \frac{\Gamma(a+1)}{2(b-a)^{\alpha}} \left[ f(a) + f(b) \right] \leq \frac{f(a) + f(b)}{2} + \frac{\epsilon}{2}.$$

**Proof.** Using (4) with $\omega(t) = at^{\alpha-1}$, we have

$$\int_0^1 L_{\omega(t)} dt = \epsilon \int_0^1 (1 - at^{\alpha-1}) dt = 0. \quad (13)$$

Using (2), (11) and (13), we get

$$0 \geq F\left(\frac{a+b}{2}\right) - \frac{\Gamma(a+1)}{2(b-a)^{\alpha}} \left[ f(a) + f(b) \right] \geq f\left(\frac{a+b}{2}\right) - \frac{\Gamma(a+1)}{2(b-a)^{\alpha}} \left[ f(a) + f(b) \right] - \epsilon,$$

that is

$$f\left(\frac{a+b}{2}\right) - \epsilon \leq \frac{\Gamma(a+1)}{2(b-a)^{\alpha}} \left[ f(a) + f(b) \right].$$

On the other hand, using (3) with $\omega(t) = at^{\alpha-1}$, we have

$$T_{F,\omega}(u_1, u_2, u_3) = u_1 - \alpha \left( \int_0^1 t^\alpha dt \right) u_2 - \alpha \left( \int_0^1 (1-t)t^{\alpha-1} dt \right) u_3 - \epsilon = u_1 - \frac{\alpha u_2 + u_3}{\alpha + 1} - \epsilon \quad (14)$$
for $u_1, u_2, u_3 \in \mathbb{R}$. Hence, from (12) and (14), we obtain

$$0 \geq T_{F, \omega} \left( \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] + f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{\omega(t)} dt$$

$$= \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] - \frac{1}{\alpha + 1} \left[ \int_0^1 h(t) t^{\alpha-1} dt \right]$$

$$= \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] - \left( f(a) + f(b) \right) - \varepsilon.$$

This implies that

$$\frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] \leq f(a) + f(b) + \varepsilon$$

and thus the proof is completed. \(\square\)

**Remark 2.3.** If we take $\varepsilon = 0$ in Corollary 2.2, then $f$ is convex and we have the inequality (9).

**Corollary 2.4.** If we choose $F(u_1, u_2, u_3, u_4) = u_1 - h(u_4) u_2 - h(1 - u_4) u_3$ in Theorem 2.1, then the function $f$ is $h$-convex on $[a, b]$ and we have the inequality

$$\frac{1}{2h(\frac{1}{2})} \left( a + b \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] + f(a) + f(b) \leq \alpha \left( \int_0^1 \left[ h(t) + h(1-t) \right] t^{\alpha-1} dt \right) \frac{f(a) + f(b)}{2}.$$

**Proof.** Using (4) and (11) with $L_{\omega(t)} = 0$, we have

$$0 \geq F \left( \left( \frac{a + b}{2} \right), \quad \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] \right) + \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right],$$

that is

$$\frac{1}{2h(\frac{1}{2})} \left( a + b \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] + f(a) + f(b).$$

On the other hand, using (8) and (12) with $\omega(t) = at^{\alpha-1}$, we obtain

$$0 \geq T_{F, \omega} \left( \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] \right) + f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{\omega(t)} dt$$

$$= \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] - \alpha \left( \int_0^1 h(t) t^{\alpha-1} dt + \int_0^1 h(1-t) t^{\alpha-1} dt \right)$$

$$= \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] - \alpha \left( \int_0^1 \left[ h(t) + h(1-t) \right] t^{\alpha-1} dt \right)$$

$$= \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ \int_0^b f(b) \right] \leq \alpha \left( \int_0^1 \left[ h(t) + h(1-t) \right] t^{\alpha-1} dt \right) \frac{f(a) + f(b)}{2}.$$

and thus the proof is completed. \(\square\)
Theorem 2.5. Let $I \subseteq \mathbb{R}$ be an interval, $f : I^2 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^a$, $a, b \in I^a$, $a < b$. Suppose that $|f'|$ is $F$-convex on $[a, b]$, for some $F \in \mathcal{F}$ and the function $t \in [0, 1] \to L_{w(t)}$ belongs to $L_1[0, 1]$, where $w(t) = |(1 - t)^a - t^a|$. Then, we have the inequality

$$T_{F,w} \left( \frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \Gamma(\alpha + 1) \frac{1}{2(b-a)^\alpha} \left[ f'_0, f(b) + f'_0, f(a) \right] \right|, \left| f'(a) \right|, \left| f'(b) \right| \right) + \int_0^1 L_{w(t)}dt \leq 0.$$ 

Proof. Since $|f'|$ is $F$-convex, we have

$$F \left( \left| f'(ta + (1-t)b) \right|, \left| f'(a) \right|, \left| f'(b) \right|, t \right) \leq 0, \ t \in [0, 1].$$

Using axiom (A3) with $w(t) = |(1-t)^a - t^a|$, we get

$$F \left( w(t) \left| f'(ta + (1-t)b) \right|, w(t) \left| f'(a) \right|, w(t) \left| f'(b) \right|, t \right) + L_{w(t)} \leq 0, \ t \in [0, 1].$$

Integrating over $[0, 1]$ and using axiom (A2), we obtain

$$T_{F,w} \left( \int_0^1 w(t) \left| f'(ta + (1-t)b) \right| dt, \left| f'(a) \right|, \left| f'(b) \right| \right) + \int_0^1 L_{w(t)}dt \leq 0, \ t \in [0, 1].$$

From Lemma 1.8, we have

$$\frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \Gamma(\alpha + 1) \frac{1}{2(b-a)^\alpha} \left[ f'_0, f(b) + f'_0, f(a) \right] \right| \leq \int_0^1 w(t) \left| f'(ta + (1-t)b) \right| dt.$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we establish

$$\frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \Gamma(\alpha + 1) \frac{1}{2(b-a)^\alpha} \left[ f'_0, f(b) + f'_0, f(a) \right] \right| + \int_0^1 L_{w(t)}dt \leq 0.$$ 

The proof is completed. \qed

Corollary 2.6. Under assumptions of Theorem 2.5, if we choose $F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1 - u_4) u_3 - \varepsilon$, then the function $|f'|$ is $\varepsilon$-convex on $[a, b]$, $\varepsilon \geq 0$ and we have the inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ f'_0, f(b) + f'_0, f(a) \right] \right| \leq \frac{b-a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left[ \left| f'(a) \right| + \left| f'(b) \right| + 2\varepsilon \right].$$

Proof. From (4) with $w(t) = |(1-t)^a - t^a|$, we have

$$\int_0^1 L_{w(t)}dt = \varepsilon \int_0^1 (1 - |(1-t)^a - t^a|)dt$$

$$= \varepsilon \int_0^{1/2} (1 - (1-t)^a + t^a)dt + \int_{1/2}^1 (1 + (1-t)^a - t^a)dt$$

$$= \varepsilon \left( 1 - \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \right),$$

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ f'_0, f(b) + f'_0, f(a) \right] \right| \leq \frac{b-a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right).$$

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Using (3) with $w(t) = |(1 - t)^a - t^a|$

\[
T_{F,w}(u_1, u_2, u_3) = u_1 - \alpha \left( \int_0^1 t |(1 - t)^a - t^a| \, dt \right) u_2 - \alpha \left( \int_0^1 (1 - t) |(1 - t)^a - t^a| \, dt \right) u_3 - \varepsilon
\]

\[
= u_1 - \frac{1}{\alpha + 1} \left( 1 - \frac{1}{2^a} \right) (u_2 + u_3) - \varepsilon
\]

for $u_1, u_2, u_3 \in \mathbb{R}$. Then, by Theorem 2.5, we have

\[
0 \geq T_{F,w} \left( \frac{2}{b - a} \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^a} \left[ \int_{\alpha}^{\infty} f(b) + f'(a) \right] \gamma \right| \right) + \int_0^1 L_{\alpha(t)} \, dt
\]

\[
= \frac{2}{b - a} \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^a} \left[ \int_{\alpha}^{\infty} f(b) + f'(a) \right] \right|
\]

\[
- \frac{1}{\alpha + 1} \left( 1 - \frac{1}{2^a} \right) \left( |f'(a)| + |f'(b)| \right) - \varepsilon \left( 1 - \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^a} \right) \right).
\]

This completes the proof. 

**Remark 2.7.** If we choose $\varepsilon = 0$ in Corollary 2.6, then $|f'|$ is convex and we have the inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^a} \left[ \int_{\alpha}^{\infty} f(b) + f'(a) \right] \right| \leq \frac{b - a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^a} \right) \left( |f'(a)| + |f'(b)| \right)
\]

which is given by Sarikaya et al. in [21].

**Corollary 2.8.** Under assumption of Theorem 2.5, if we choose $F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1 - u_4)u_3$, then the function $|f'|$ is $h$-convex on $[a, b]$ and we have the inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^a} \left[ \int_{\alpha}^{\infty} f(b) + f'(a) \right] \right| \leq \frac{b - a}{2} \left( \int_0^1 h(t) |(1 - t)^a - t^a| \, dt \right) \left( |f'(a)| + |f'(b)| \right).
\]

**Proof.** From (8) with $w(t) = |(1 - t)^a - t^a|$, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^a} \left[ \int_{\alpha}^{\infty} f(b) + f'(a) \right] \right| \leq \frac{b - a}{2} \left( \int_0^1 h(t) |(1 - t)^a - t^a| \, dt \right) \left( |f'(a)| + |f'(b)| \right).
\]
for \( u_1, u_2, u_3 \in \mathbb{R} \). Then, by Theorem 2.5,

\[
T_{\frac{\alpha}{\alpha+1}} \left( \frac{2}{b-a} \left[ \int_a^b f'(t) \, dt \right] \right) = \frac{2}{b-a} \left[ \int_a^b f'(t) \, dt \right] = \int_0^1 (1-t^\alpha - t) \, dt \leq 0.
\]

This completes the proof. \( \Box \)

References