Biprojectivity and Biflatness of Generalized Module Extension Banach Algebras

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Abstract. We investigate biprojectivity and biflatness of generalized module extension Banach algebra \( A \bowtie B \), in which \( A \) and \( B \) are Banach algebras and \( B \) is an algebraic Banach \( A \)-bimodule, with multiplication:
\[
(a, b) \cdot (a', b') = (aa', ab' + ba' + bb').
\]

1. Introduction

Let \( A \) and \( B \) be Banach algebras and let \( B \) be a Banach \( A \)-bimodule. Then, we will say that \( B \) is an algebraic Banach \( A \)-bimodule if for all \( a \in A \) and \( b, b' \in B \)
\[
a(bb') = (ab)b', \quad (bb')a = b(b'a), \quad (ba)b' = b(ab').
\]
The Cartesian product \( A \times B \) with the multiplication
\[
(a, b) \cdot (a', b') = (aa', ab' + ba' + bb'),
\]
and with the norm \( ||(a, b)|| = ||a|| + ||b|| \), becomes a Banach algebra, which is called the “generalized module extension Banach algebra”, and it is denoted by \( A \bowtie B \). Also \( A \equiv A \times \{0\} \) is a closed subalgebra, while \( B \equiv \{0\} \times B \) is a closed ideal of \( A \bowtie B \), and \( A \bowtie B/B \equiv A \). The authors in [11] have studied some properties of this kind of algebra, such as bounded approximate identity, spectrum, topological centers and \( n \)-weak amenability. This algebra can be a generalization of the following known algebras:

(a) Let \( A \times B \) be the direct product of two Banach algebras \( A \) and \( B \), with multiplication
\[
(a, b) \cdot (a', b') = (aa', bb').
\]
If we define the \( A \)-bimodule actions on \( B \) by \( ab = ba = 0 \), for \( a \in A \) and \( b \in B \), then \( A \times B \equiv A \bowtie B \).

(b) Let \( A \oplus X \) be the module extension Banach algebra, in which \( X \) is a Banach \( A \)-bimodule, with multiplication
\[
(a, x) \cdot (a', x') = (aa', ax' + xa').
\]
If we define the multiplication on \( X \) by \( xx' = 0 \), then \( A \oplus X \equiv A \bowtie X \).

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Let $A \times \theta B$ be the $\theta$–Lau product of two Banach algebras $A$ and $B$ with $\theta \in \Delta(A)$ and the following multiplication
\[(a, b) \cdot (a', b') = (aa', \theta(a)b') + \theta(a')b + bb'.\]

These kinds of products have been investigated in two prior studies [7, 13]. If we define the $A$–bimodule actions on $B$ by $ab = ba =: \theta(a)b$, for $a \in A$ and $b \in B$, then $A \times \theta B = A \boxtimes B$.

Let $A \times_T B$ be the $T$–Lau product of two Banach algebras $A$ and $B$ with algebra homomorphism $T : A \to B$ with $\|T\| \leq 1$, and with multiplication
\[(a, b) \cdot (a', b') = (aa', T(a)b') + bT(a') + bb'.\]

These kinds of products were introduced by Lau [7], and studied by many authors such as [2, 5, 13]. If we define the $A$–bimodule actions on $B$ by $ab =: T(a)b$, for $a \in A$ and $b \in B$, then $A \times_T B = A \boxtimes B$.

Let $A \bowtie I$ be the amalgamation of $A$ with $B$ along $I$ with respect to $\theta$, in which $A$ and $B$ are Banach algebras, $I$ is a closed ideal in $B$, $\theta : A \to B$ is a continuous Banach algebra homomorphism, and with the following multiplication
\[(a, i)(a', i') = (aa', \theta(a)i' + i\theta(a') + ii'),\]

for $a, a' \in A$ and $i, i' \in I$. These kinds of Banach algebras have been studied in some other studies [9, 10]. Now if we define the $A$–bimodule actions on $I$ by $ai =: \theta(a)i$ and $ia = i\theta(a)$, for $a \in A$ and $i \in I$, then $A \bowtie I$ is biprojective [biflat], but for biprojectivity [biflatness] of $B$, we need some conditions. Also it will be shown that if $A$ and $B$ are biprojective [biflat], then under a mild condition on $B$, we conclude the biprojectivity [biflatness] of $L = A \bowtie B$. In section 5 our results will be applied in some examples.

### 2. Preliminaries

Throughout this paper, $A$ and $B$ are Banach algebras, $B$ is an algebraic Banach $A$–bimodule, and $L = A \bowtie B$ denotes the generalized module extension Banach algebra. Consider the $A$–bimodule and also $B$–bimodule actions on $L = A \bowtie B$ by

\[
\begin{align*}
(a', a) \cdot (a, b) := (a', 0) \cdot (a, b), \\
(a, b) \cdot (a', 0) := (a, b) \cdot (0, a'), \\
\{ b \cdot (a, b) = (0, b') \cdot (a, b), \\
\{ b \cdot (a, b) = (a, b) \cdot (0, b'),
\end{align*}
\]

for all $(a, b) \in L$, $a' \in A$ and $b' \in B$. Following [4], we say that $A$ is biprojective if there exists a bounded $A$–bimodule map $\rho_A : A \to A \hat{\otimes} A$ such that $\pi_A \rho_A = \text{id}_A$, in which $\pi_A : A \hat{\otimes} A \to A$ denotes the product map with $\pi_A(a \otimes a') = aa'$. Also $A$ is called biflat if there is a bounded $A$–bimodule map $\lambda_A : (A \hat{\otimes} A)' \to A'$, such that $\lambda_A \pi_A' = \text{id}_A$. For the basic properties of biprojectivity and biflatness, see [3, 12]. Finally, the following maps will be introduced and then used in our results. Let $p_A : L = A \bowtie B \to A$ and $p_B : L = A \bowtie B \to B$ be the projections defined by $p_A(a, b) = a$ and $p_B((a, b)) = b$ for all $(a, b) \in L$. Also let $q_A : A \to L = A \bowtie B$ and $q_B : B \to L = A \bowtie B$ be the injections, defined by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$, for all $a \in A$ and $b \in B$. Besides, suppose that $B$ is unital with unit $e_B$, and define the following bounded linear maps
\[
\begin{align*}
rb : L = A \bowtie B \to B & \text{ by } rb(a, b) = ace_B + b, \\
s_A : A \to L = A \bowtie B & \text{ by } s_A(a) = (a, -ae_B).
\end{align*}
\]
Note that, the mappings \( p_A, q_A \) are bounded \( A \)–bimodule maps, and \( q_B \) is a bounded \( B \)–bimodule map. We have the following lemma about relations between bimodule structures for \( r_B \) and \( s_A \).

**Lemma 2.1.** Let \( A \) and \( B \) be Banach algebras, and let \( B \) be an algebraic Banach \( A \)–bimodule with unit \( e_B \), such that \( ae_B = e_Ba \) for all \( a \in A \). Then the mappings \( r_B \) and \( s_A \) are \( B \)–bimodule map and \( A \)–bimodule map, respectively.

**Proof.** Let \( a, a' \in A \) and \( b, b' \in B \). By using the assumptions, we have

\[
\begin{align*}
    r_B(b' \cdot (a, b)) &= r_B((0, b') \cdot (a, b)) \\
    &= r_B(0, b'a + b'b) \\
    &= b'a + b'b \\
    &= b'e_Ba + b'b \\
    &= b'ae_B + b'b \\
    &= b' \cdot (ae_B + b) \\
    &= b' \cdot r_B(a, b).
\end{align*}
\]

Similarly, we have \( r_B((a, b) \cdot b') = r_B(a, b) \cdot b' \), and we conclude that \( r_B \) is a \( B \)–bimodule map. Also we have

\[
\begin{align*}
    s_A(aa') &= (aa', -aa'e_B) \\
    &= (a, 0) \cdot (a', -a'e_B) \\
    &= a \cdot (a', -a'e_B) \\
    &= a \cdot s_A(a') ,
\end{align*}
\]

and similarly, by the assumptions

\[
\begin{align*}
    s_A(aa) &= (aa', -aa'e_B) \\
    &= (a, -ae_Ba) \cdot (a', 0) \\
    &= (a, -ae_Ba \cdot a') \\
    &= s_A(a) \cdot a' ,
\end{align*}
\]

and so \( s_A \) is an \( A \)–bimodule map. \( \square \)

3. Results on Biprojectivity

This section deals with relations between biprojectivity of \( L = A \otimes B \) and biprojectivity of \( A \) and \( B \).

**Theorem 3.1.** Let \( A \) and \( B \) be Banach algebras, and let \( B \) be an algebraic Banach \( A \)–bimodule.

(i) If \( L = A \otimes B \) is biprojective, then \( A \) is biprojective.

(ii) Suppose that \( B \) has unit \( e_B \), such that for all \( a \in A \), \( ae_B = e_Ba \). If \( L = A \otimes B \) is biprojective, then \( B \) is biprojective.

**Proof.** By the hypothesis, there exist a bounded \( L \)–bimodule map \( \rho_L : L \to L \otimes L \), such that \( \pi_L \rho_L = id_L \).

(i) Define \( \rho_A : A \to A \otimes A \) by \( \rho_A = (\rho_A \otimes \rho_A) \rho_L \circ \rho_A \). Clearly \( \rho_A \) is bounded. Since \( \rho_L \) is \( L \)–bimodule map, for \( a, a' \in A \), and \( b, b' \in B \) we have

\[
\begin{align*}
    \rho_L(a' \cdot (a, b)) &= \rho_L((a', 0) \cdot (a, b)) \\
    &= (a', 0)\rho_L((a, b)) \\
    &= a' \cdot \rho_L((a, b)).
\end{align*}
\]
Similarly, we have $\rho_L((a, b) \cdot a') = \rho_L((a, b)) \cdot a'$. We conclude that $\rho_L$ is an $A$-bimodule map. Then $\rho_A$ is a bounded $A$-bimodule map. Also for $(a, b) \otimes (a', b') \in L \otimes L$

\[
\begin{align*}
(\pi_A \circ (p_A \otimes p_A))(a, b) &\otimes (a', b') = \pi_A(a \otimes a') = aa', \\
(p_A \circ \pi_A)(a, b) &\otimes (a', b') = p_A((a, b) \cdot (a', b')) = aa',
\end{align*}
\]

this shows the identity $\pi_A \circ (p_A \otimes p_A) = p_A \circ \pi_A$. Now one can have the following

\[
\begin{align*}
\pi_A \circ p_A &= \pi_A \circ (p_A \otimes p_A) \circ \rho_L \circ \pi_A \\
&= p_A \circ \pi_A \circ \rho_L \circ \pi_A \\
&= p_A \circ \rho_L \circ \pi_A = \text{id}_A.
\end{align*}
\]

This shows that $A$ is biprojective.

(ii) Define $\rho_B := (r_B \otimes r_B) \circ \rho_L \circ \pi_B$. Since $\rho_L, \pi_B$ and $r_B$ are bounded $B$-bimodule maps, then $\rho_B$ is bounded $B$-bimodule map. Also for $(a, b)$ and $(a', b')$ in $L$ we have

\[
\begin{align*}
(\pi_B \circ (r_B \otimes r_B))(a, b) \otimes (a', b') &= \pi_B((ae_B + b) \otimes (a'e_B + b')) \\
&= (ae_B + b) \cdot (a'e_B + b') \\
&= ae_B \cdot e_B + ae_B \cdot b' + ba' \cdot e_B + b'b' \\
&= aa' \cdot e_B + ab' + ba' + b'b' \\
&= r_B(aa', ab') + ba' + b'b' \\
&= r_B((a, b) \cdot (a', b')) \\
&= (\pi_B \circ \rho_L)(a, b) \otimes (a', b').
\end{align*}
\]

We conclude that $\pi_B \circ (r_B \otimes r_B) = r_B \circ \pi_B$. Moreover it is easy to check that $r_B \circ \pi_B = \text{id}_B$. Then

\[
\begin{align*}
\pi_B \circ p_B &= \pi_B \circ (r_B \otimes r_B) \circ \rho_L \circ \pi_B \\
&= r_B \circ \pi_B \circ \rho_L \circ \pi_B \\
&= r_B \circ \rho_L \circ \pi_B = \text{id}_B,
\end{align*}
\]

and this shows the biprojectivity of $B$. \qed

**Theorem 3.2.** Let $A$ and $B$ be Banach algebras, and let $B$ be an algebraic Banach $A$-bimodule with unit $e_B$ such that for all $a \in A, ae_B = e_Ba$. If $A$ and $B$ are biprojective, then $L = A \otimes B$ is biprojective.

**Proof.** By the hypothesis, there exist bounded $A$-bimodule map $\rho_A : A \to A \otimes A$, and bounded $B$-bimodule map $\rho_B : B \to B \otimes B$, such that $\pi_A \circ \rho_A = \text{id}_A$ and $\pi_B \circ \rho_B = \text{id}_B$. For $(a \otimes a') \in A \otimes A$ we have

\[
\begin{align*}
(\pi_L \circ (S_A \otimes S_A))(a \otimes a') &= \pi_L((a, -ae_B) \otimes (a', -ae_B)) \\
&= (a, -ae_B) \cdot (a', -ae_B) \\
&= (aa', -aa' \cdot e_B + ae_Ba'e_B) \\
&= (aa', -aa' \cdot e_B + aa' \cdot e_B) \\
&= s_A(aa') \\
&= (S_A \circ \pi_A)(a \otimes a'),
\end{align*}
\]
and we conclude that $\pi_{L\circ}(a_i \otimes s_i) = s_i \circ \pi_A$. Also, it is easy to check that $\pi_{L\circ}(q_B \otimes q_B) = q_B \circ \pi_B$. Now define $\rho_L : L \to L \otimes L$ by

$$
\rho_L((a, b)) = L \big((s_A \otimes s_A) \circ \rho_A \circ \rho_A(a, b) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big).
$$

Clearly $\rho_L$ is bounded, we first show that $\rho_L$ is a left–$L$–module map. For all $(a, b), (c, d) \in L$, we have

$$
\rho_L((a, b) \cdot (c, d)) = L\big((s_A \otimes s_A)(\rho_A(a \cdot c, ad + bc + bd)) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big)
$$

$$
= L\big((s_A \otimes s_A)(\rho_A(a \cdot c)) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big)
$$

$$
= (a, 0) \cdot L\big((s_A \otimes s_A)(\rho_A(c)) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big)
$$

$$
= (a, 0) \cdot L\big((s_A \otimes s_A)(\rho_A(c) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big)
$$

$$
= (a, b) \cdot \left[L\big((s_A \otimes s_A)(\rho_A(c) + (a, b) \cdot \big((q_B \otimes q_B)(\rho_B(e_B))\big)\big)\right] - (0, b) \cdot \left(L\big((s_A \otimes s_A)(\rho_A(c))\big)\right).
$$

but $(0, b) \cdot (s_A \otimes s_A)(\rho_A(c)) = 0$, because for all $(a \cdot a, a, a) \in A \otimes A$, we can write

$$
(0, b) \cdot \left((s_A \otimes s_A)(a, a)\right) = (0, b) \cdot \left(s_A(a) \otimes s_A(a)\right)
$$

$$
= (0, b) \cdot \left\{(a, a, a) \otimes (a, a, a)\right\}
$$

$$
= (0, b) \cdot \left((0, 0) \otimes (a, a, a)\right)
$$

and we conclude that $(0, b) \cdot (s_A \otimes s_A)(\rho_A(c)) = 0$ for $\rho_A(c) = \sum_{i=1}^{\infty} a_i \otimes a_i$, in which $(a_i, a_i)$ are some sequences in $A$ with $\sum_{i=1}^{\infty} \|a_i\| \|a_i\| < \infty$.

Thus $\rho_L((a, b) \cdot (c, d)) = (a, b) \cdot \rho_L((c, d))$, and so $\rho_L$ is left–$L$–module map. To show that $\rho_L$ is right–$L$–module map, we note that for all $(b \cdot b, b) \in L \otimes B$

$$
(a, b) \cdot \left((q_B \otimes q_B)(b \cdot b)\right) = (q_B \otimes q_B)((b + ae_B) \cdot (b \cdot b)),
$$

$$
(q_B \otimes q_B)(b \cdot b) \cdot (a, b) = (q_B \otimes q_B)((b \cdot b) \cdot (b + ae_B)).
$$

Hence

$$
(a, b) \cdot \left((q_B \otimes q_B)(\rho_B(e_B))\right) = (q_B \otimes q_B)((b + ae_B) \cdot \rho_B(e_B))
$$

$$
= (q_B \otimes q_B)(\rho_B(e_B) \cdot (b + ae_B))
$$

$$
= \left((q_B \otimes q_B)(\rho_B(e_B))\right) \cdot (a, b).
$$
It follows that \((q_b \otimes q_b)(\rho_b(e_b))\) commutes with the members of \(L\). Consequently,
\[
\rho_L((c, d) \cdot (a, b)) = (s_A \otimes s_A)(\rho_A(\lambda))((c, d) \cdot (a, b)) + ((c, d) \cdot (a, b))((q_b \otimes q_b)(\rho_b(e_b)))
\]
\[
= (s_A \otimes s_A)(\rho_A(\lambda)) + ((c, d) \cdot (a, b))((q_b \otimes q_b)(\rho_b(e_b)))
\]
\[
= (s_A \otimes s_A)(\rho_A(\lambda)) \cdot (a, 0) + (c, d) \cdot (q_b \otimes q_b)(\rho_b(e_b)) \cdot (a, b)
\]
\[
= [(s_A \otimes s_A)(\rho_A(\lambda)) + (c, d) \cdot (q_b \otimes q_b)(\rho_b(e_b))](a, b)
\]
\[
= (s_A \otimes s_A)(\rho_A(\lambda)) \cdot (0, b)
\]
\[
= \rho_L((c, d) \cdot (a, b) - (s_A \otimes s_A)(\rho_A(\lambda)) \cdot (0, b),
\]
but with similar reasoning for \((0, b) \cdot (s_A \otimes s_A)(\rho_A(\lambda)) = 0\) we have the identity \((s_A \otimes s_A)(\rho_A(\lambda)) \cdot (a, b) = 0\). Thus \(\rho_L((c, d) \cdot (a, b)) = \rho_L((c, d) \cdot (a, b))\), and so \(\rho_L\) is a right--\(L\)--module map. Finally, we have
\[
(\pi_L \circ \rho_L)(a, b) = (\pi_L \circ (s_A \otimes s_A)(\rho_A \circ s_A))(a, b) + (a, b) \cdot (\pi_L \circ (q_b \otimes q_b)(\rho_b(e_b)))
\]
\[
= (s_A \otimes s_A)(\rho_A \circ s_A)(a, b) + (a, b) \cdot (q_b \otimes q_b)(\rho_b(e_b))
\]
\[
= s_A(a) + (a, b) \cdot (0, e_b)
\]
\[
= (a, -ae_b) + (0, ae_b + b)
\]
\[
= (a, b),
\]
therefore \(\pi_L \circ \rho_L = id_L\), and hence \(L = A \otimes B\) is biprojective. \(\square\)

4. Results on Biflatness

This section is devoted to the relations between biflatness of \(L = A \otimes B\) and biflatness of \(A\) and \(B\).

**Theorem 4.1.** Let \(A\) and \(B\) be Banach algebras, and let \(B\) be an algebraic Banach \(A\)–bimodule.

(i) If \(L = A \otimes B\) is biflat, then \(A\) is biflat.

(ii) Suppose that \(B\) has unit \(e_B\), such that for all \(a \in A\), \(ae_B = e_Ba\). If \(L = A \otimes B\) is biflat, then \(B\) is biflat.

**Proof.** By the hypothesis, there exist a bounded \(L\)–bimodule map \(\lambda_L : (L \overline{\otimes} L)^* \to L^*\), such that \(\lambda_L \circ \pi_L^* = id_L^*\). The following identities have been shown in the proof of theorem (3.1)
\[
\pi_A \circ (p_A \otimes p_A) = p_A \circ \pi_L,
\]
\[
\pi_B \circ (e_B \otimes r_B) = r_B \circ \pi_L.
\]
(i) Define \(\Lambda_A : (A \overline{\otimes} A)^* \to A^*\) by \(\Lambda_A = q_A^* \circ \lambda_L \circ (p_A \otimes p_A)^*\), which is a bounded \(A\)–bimodule map and
\[
\lambda_A \circ \pi_A^* = q_A^* \circ \lambda_L \circ (p_A \otimes p_A)^* \circ \pi_A^* = (p_A \otimes p_A)^* \circ \pi_A^* = \Lambda_A.
\]
Hence $A$ is biflat.

(ii) Define $\lambda_B : (B \hat{\otimes} B)^* \rightarrow B^*$ by $\lambda_B =: q_B^o \circ \lambda_L (r_B \otimes r_B)^*$. Since $B$ is unital and $ae_B = e_B a$ ($a \in A$), then $r_B$ and hence $\lambda_B$ are bounded $B$–bimodule maps, and we have

$$\lambda_B o \pi'_B = q_B^o \circ \lambda_L (r_B \otimes r_B)^* o \pi'_B$$

$$= q_B^o \circ \lambda_L (r_B o \pi_B (r_B \otimes r_B))^*$$

$$= q_B^o \circ \lambda_L (r_B o \pi_B)$$

$$= q_B^o \circ \lambda_L o r_B$$

$$= (r_B o \pi_B)^*$$

$$= (id_B)^*$$

$$= id_B^*$$. 

This proves the biflatness of $B$. $\square$

For the converse of theorem (4.1) we should determine the $L$–bimodule structures on $L' = (A \bowtie B)^*$. We recall that the dual space $A^*$ of $A$ is a Banach $A$–bimodule by module operations

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle fa, b \rangle,$$

for $a, b \in A$ and $f \in A^*$. We remark that the dual space $L' = (A \bowtie B)^*$ can be identified with $A^* \times B^*$ by the following bounded linear map

$$\theta : A^* \times B^* \rightarrow (A \bowtie B)^* = L', \quad \{(\theta(f, g), (a, b)) = (fa + g(b))\}.$$  

Now suppose that $B$ has unit $e_B$ such that for all $a \in A$, $ae_B = e_B a$. Define $\varphi : A \rightarrow B$ by $\varphi(a) = e_B a$. For $(a, b), (a', b') \in L = A \bowtie B$ and $(f, g) \in L'$ we have

$$\left( (f, g) \cdot (a, b) \right) (a', b') = \left( (f, g) \left( (a, b) \cdot (a', b') \right) \right)$$

$$= \left( (f, g) \left( a a', a' b' + b a' + b b' \right) \right)$$

$$= f(a a') + g(ab') + g(b a') + g(b b')$$

$$= f(a) (a') + g(ae_B b') + g(be_B a') + (g \cdot b)(b')$$

$$= (f \cdot a)(a') + (a \cdot (ae_B)) b' + \left( (g \cdot b) \varphi \right)(a') + (g \cdot b)(b')$$

$$= \left( f \cdot a + (g \cdot b) \varphi \right)(a') + \left( g \cdot (ae_B) + g \cdot b \right)(b')$$

$$= \left( f \cdot a + (g \cdot b) \varphi , g \cdot (ae_B) + g \cdot b \right)(a', b'),$$

therefore

$$(f, g) \cdot (a, b) = \left( f \cdot a + (g \cdot b) \varphi , g \cdot (ae_B) + g \cdot b \right),$$

and similarly

$$(a, b) \cdot (f, g) = \left( a \cdot f + (b \cdot g) \varphi , e_B a \cdot g + b \cdot g \right).$$

**Theorem 4.2.** Let $A$ and $B$ be Banach algebras, and let $B$ be an algebraic Banach $A$–bimodule with unit $e_B$ such that for all $a \in A$, $ae_B = e_B a$. If $A$ and $B$ are biflat, then $L = A \bowtie B$ is biflat.
Proof. By the hypothesis, there exist a bounded $A$–bimodule map $\lambda_A : (A \otimes A)' \to A'$ and a bounded $B$–bimodule map $\lambda_B : (B \otimes B)' \to B'$, such that $\lambda_A \circ \pi_A^* = id_A$ and $\lambda_B \circ \pi_B^* = id_B$. Define $\lambda_L : (L \otimes L)' \to L' \subseteq A' \times B'$ by

$$\lambda_L(h) := \left( (\lambda_A \circ (s_A \otimes s_A)')(h) + \left( (q^* \circ \lambda_B \circ (q_B \otimes q_B))' \right)(h), \left( (\lambda_B \circ (q_B \otimes q_B))' \right)(h) \right),$$

for $h \in (L \otimes L)'$ and $\varphi : A \to B$ ($\varphi(a) = ae_B$). Clearly $\lambda_L$ is a bounded map. To see that $\lambda_L$ is an $L$–bimodule map we need the following identities for $h \in (L \otimes L)'$ and $(a, b) \in L$

\begin{enumerate}
  
  \item $(q_B \otimes q_B)'(h \cdot (a, b)) = (q_B \otimes q_B)'(h) \cdot (ae_B + b)$,
  
  \item $(q_B \otimes q_B)'((a, b) \cdot h) = (ae_B + b) \cdot (q_B \otimes q_B)'(h)$,
  
  \item $(s_A \otimes s_A)'(h \cdot (a, b)) = (s_A \otimes s_A)'(h) \cdot a$,
  
  \item $(s_A \otimes s_A)'((a, b) \cdot h) = a \cdot (s_A \otimes s_A)'(h)$.
\end{enumerate}

To prove the equality (1), for $(b' \otimes b'') \in B \otimes B$ we can write

$$
\left( (q_B \otimes q_B)'(h \cdot (a, b)) \right) (b' \otimes b'') = \left( h \cdot (a, b) \right) (q_B \otimes q_B)(b' \otimes b'') = \left( h (a, b) \right) ((0, b') \otimes (0, b'')) = h(0, ab' + bb') \otimes (0, b'').$$

This proves the identity (1). Similarly, we can prove the identity in (2). To investigate the equality (3), for $(a' \otimes a'') \in A \otimes A$ we can write

\begin{align*}
\left( (s_A \otimes s_A)'(h \cdot (a, b)) \right) (a' \otimes a'') &= \left( h \cdot (a, b) \right) (s_A \otimes s_A)(a' \otimes a'') \\
&= \left( h (a, b) \right) (a' \otimes (-a' e_B)) \otimes (a'' \otimes (-a'' e_B)) \\
&= h((aa', -aa' e_B + ba' e_B) \otimes (a'', -a'' e_B)) \\
&= h((aa', -aa' e_B + ba' e_B) \otimes (a'', -a'' e_B)) \\
&= h((s_A \otimes s_A)(aa' \otimes a'')) \\
&= \left( (s_A \otimes s_A)'(h) \right) (a' \otimes a'') \\
&= \left( (s_A \otimes s_A)'(h) \right) (a' \otimes a''),
\end{align*}

this proves the identity in (3), and similarly one can proves the identity in (4). Now, using the identities
(1-4) we have
\[
\lambda_L(h \cdot (a, b)) = \left((\lambda_A \circ (s_A \otimes s_A))(h \cdot (a, b)) + (\varphi^o \circ \lambda_B \circ (q_B \otimes q_B))(h \cdot (a, b))\right)
\]
\[
\text{and for } (f, g) \in L' \text{ we have}
\]
\[
(\lambda_L \circ \pi_L^*)(f, g) = \lambda_L(\pi_L^*(f, g))
\]
\[
= \left((\lambda_A \circ (s_A \otimes s_A)) \circ \pi_L^*(f, g) + (\varphi^o \circ \lambda_B \circ (q_B \otimes q_B)) \circ \pi_L^*(f, g)\right)
\]
\[
\text{this proves that } \lambda_L \circ \pi_L^* = \text{id}_{L'}, \text{ and the proof is completed.} \]

5. Examples

This section includes some illustrative examples.
Example 5.1. Let \( L = A \times_0 B \) be the \( \theta \)-Lau product of Banach algebras \( A \) and \( B \) with \( \theta \in \Delta(A) \). If \( B \) is unital with unit \( e_B \) such that \( e_B a = ae_B \) for all \( a \in A \), then \( A \times_0 B \) is biprojective [biflat] if and only if \( A \) and \( B \) are biprojective [biflat].

Example 5.2. Let \( L = A \times_T B \) be the \( T \)-Lau product of Banach algebras \( A \) and \( B \) with algebra homomorphism \( T : A \rightarrow B \) (\( \| T \| \leq 1 \)). If \( B \) is unital with unit \( e_B \), then for all \( a \in A \) we have \( e_B T(a) = T(a) e_B = T(a) \). Hence \( A \times_T B \) is biprojective [biflat] if and only if \( A \) and \( B \) are biprojective [biflat].

Example 5.3. Let \( L = A \rtimes^\theta I \) be the amalgamation of Banach algebras \( A \) and \( I \) along the closed ideal \( I \) in \( B \), with respect to continuous Banach algebra homomorphism \( \theta : A \rightarrow B \). If \( I \) has unit \( e_I \) such that \( \theta(a) e_I = e_I \theta(a) \), for all \( a \in A \), then \( A \rtimes^\theta I \) is biprojective [biflat] if and only if \( A \) and \( I \) are biprojective [biflat].

Example 5.4. Let \( G \) be a locally compact group and let \( L^1(G) \) and \( M(G) \) be its group algebra and measure algebra, respectively. It is known that \( L^1(G) \) is unital if and only if \( G \) is discrete, and \( L^1(G) \) is biprojective if and only if \( G \) is compact [4, 12]. Also, \( L^1(G) \) is biflat if and only if \( G \) is amenable [4]. Therefore we have the following results

i) If \( L^1(G) \cong L^1(G) \) is biprojective, then \( G \) is compact.

ii) If \( L^1(G) \cong L^1(G) \) is biflat, then \( G \) is amenable.

iii) If \( G \) is discrete group, then \( L^1(G) \cong L^1(G) \) is biprojective if and only if \( G \) is finite, and \( L^1(G) \cong L^1(G) \) is biflat if and only if \( G \) is amenable.

iv) \( M(G) \cong M(G) \) is biprojective [biflat] if and only if \( M(G) \) is biprojective [biflat].

v) If \( M(G) \cong L^1(G) \) is biprojective [biflat], then \( M(G) \) is biprojective [biflat].

vi) Suppose that \( G \) be discrete, and \( A \) be a Banach algebra, such that \( L^1(G) \) be an algebraic Banach \( A \)-bimodule.

If \( A \cong L^1(G) \) is biprojective, then \( L^1(G) \) and \( A \) are biprojective, and \( G \) is finite.

If \( A \cong L^1(G) \) is biflat, then \( L^1(G) \) and \( A \) are biflat, and \( G \) is amenable.

If \( G \) is finite and \( A \) is biprojective, then \( A \cong L^1(G) \) is biprojective.

If \( G \) is amenable and \( A \) is biflat, then \( A \cong L^1(G) \) is biflat.

vii) If \( C_0(G) \cong M(G) \) is biprojective [biflat], then \( C_0(G) \) and \( M(G) \) are biprojective [biflat].

viii) If \( G \) is finite, then \( C_0(G) \) and \( C_0(G) \cong M(G) \) are biprojective.

Example 5.5. Let \( A'' \) be the second dual of a Banach algebra \( A \) with first Arens product \( \Delta \). Then \( A'' \) can be an \( A \)-bimodule by \( a F = \Delta \omega F \) and \( F a = F \Delta \omega \), for all \( a \in A \) and \( F \in A' \), and with natural embedding of \( A \) into \( A'' \) \( (a \mapsto \delta) \). Also it is known that if \( A \) is Arens regular, then \( A'' \) is unital if and only if \( A \) has bounded approximate identity, [3]. By theorems (3.1) and (4.1), if \( L = A \cong A'' \) is biprojective [biflat], then \( A \) is biprojective [biflat]. Also we can apply part (ii) of theorems (3.1) and (4.1) and theorems (3.2) and (4.2) for Arens regular Banach algebras \( A \) with bounded approximate identity and for \( L = A \cong A'' \).

On the other hand, by using the results in [8], if \( A \) is Arens regular with bounded approximate identity, then \( L = A \cong A'' \) is biflat if and only if \( A'' \) is biflat. Besides if \( A \cong A'' \), then \( L = A \cong A'' \) is biprojective if and only if \( A'' \) is biprojective.

One can use this example for a \( C^* \)-algebra, which is Arens regular and has bounded approximate identity. Also, for \( A = L^1(G) \), in which \( G \) is compact, we will have \( L^1(G) \cong L^1(G)'' \).
References


