NEC Rings

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Abstract. A ring $R$ is called NEC if for any $a,b \in N(R)$, $ab = ba$. The class of NEC rings is a proper generalization of the class of CN rings. First, with the aid of NEC rings, some characterizations of CN rings and reduced rings are given. Next we extend many properties of CN rings to NEC rings such as we show that NEC rings are directly finite and left min-abel; NEC regular ring are strongly regular; a ring $R$ is NEC if and only if every Pierce stalk of $R$ is NEC; Also we discuss some properties of NEC exchange rings; Finally, we give some properties of MP-invertible elements.

1. Introduction

Throughout this article, all rings considered are associated with identity, the symbols $N(R)$, $J(R)$, $U(R)$, $E(R)$, $Z(R)$, $Z_{r}(R)$ and $Z_{l}(R)$ will stand respectively for the set of all nilpotent elements, the Jacobson radical, the set of all invertible elements, the set of all idempotent elements, the center, the left and right singular ideal of $R$. And $\mathbb{Z}$ represents the set of all integers.

In [1], it is shown that if a ring $R$ satisfies: (1) $N(R)$ is commutative, (2) for every $x \in R$ there exists an element $x'$ in the subring $(x)$ generated by $x$ such that $x - x^{2}x' \in N(R)$, (3) for all $a \in N(R)$ and $b \in R$, $ba - ab$ commutes with $b$, then $R$ is commutative.

In [2], it is shown that if $R$ satisfies: (1) $N(R)$ is commutative, (2) for every $x \in R$ there exists an element $x'$ in the subring $(x)$ generated by $x$ such that $x - x^{2}x' \in N(R)$, (3) for every $x, y \in R$, there exists a positive integer $n = n(x, y) \geq 1$ such that both $(xy)^{n} - (yx)^{n}$ and $(xy)^{n+1} - (yx)^{n+1}$ belong to $Z(R)$, then $R$ is a subdirect sum of local commutative rings and nil commutative rings.

Motivated by the two theorems, we consider the class of rings satisfying the following condition:

$$ab = ba \quad a,b \in N(R)$$

A ring $R$ is called nilpotent elements commutative (for short, NEC) if it satisfies the above condition. Clearly, a ring with $N(R)^{2} = 0$ is always NEC.

Following [12], a ring $R$ is called CN if $N(R) \subseteq Z(R)$. Clearly, CN rings are NEC, but the converse is not true because of the following example 2.2. Hence NEC rings are proper generalization of CN rings.

Following [22], a ring $R$ is called reduced if $N(R) = 0$. And $R$ is called left (right) quasi-duo if every maximal left (right) ideal of $R$ is an ideal. Recall that a ring $R$ is said to be directly finite [19] if $ab = 1$ implies $ba = 1$.

\textit{Keywords.} NEC ring; reduced ring; clean ring; exchange ring; quasi-normal ring; regular ring; Moore penrose inverse; EP element.

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In preparation for the paper, we first state the following definitions.

An element \( e \in E(R) \) is called left minimal idempotent if \( Re \) is a minimal left ideal of \( R \). Write \( ME_2(R) \) to denote the set of all left minimal idempotents of \( R \). A ring \( R \) is called left min-abel [23] if either \( ME_2(R) = \emptyset \) or each element \( e \) of \( ME_2(R) \) is left semicentral (that is, \( ae = cae \) for all \( a \in R \)). An element \( a \) of a ring \( R \) is called regular [14] if \( a \in aRa_a \); \( a \) is said to be strongly regular [22] if \( a \in a^2R \cap Ra^2 \); and \( a \) is unit – regular [13] if \( a = au \) for some \( u \in U(R) \). A ring \( R \) is called regular, strongly regular, unit – regular if every element of \( R \) is regular, strongly regular and unit – regular, respectively. Following [18], a ring \( R \) is called exchange if for every \( x \in R \) there exists \( e \in E(R) \) such that \( e \in xR \) and \( 1 - e \in (1 - x)R \), and \( R \) is said to be clean if every element of \( R \) is a sum of a unit and an idempotent.

In section 2, we give some examples of NEC rings and with the aid of NEC rings, some characterizations of CN rings and reduced rings are given.

In section 3, we discuss the properties of NEC rings. We mainly show that NEC rings are directly finite and left min-abel, also give some characterizations of strongly regular rings.

In section 4, we discuss some properties of NEC exchange rings such as NEC exchange rings are clean rings and quasi-duo rings.

In section 5, we discuss some properties of Moore Penrose invertibility of NEC ring. Especially, we give some characterizations of EP elements.

2. Examples of NEC Rings

**Definition 2.1.** A ring \( R \) is called nilpotent elements commutative (for short, NEC) if \( ab = ba \) for any \( a, b \in N(R) \).

The class of NEC rings is rather large, and contains all commutative rings, all CN rings and all rings \( R \) with \( N(R)^2 = 0 \). However, the following example illustrates that NEC rings need not be CN.

**Example 2.2.** Let \( F \) be a field and \( R = T_2(F) = \left\{ \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \). Then \( R \) is NEC because \( N(R)^2 = 0 \). Since \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin Z(R) \), \( R \) is not CN.

**Example 2.3.** Let \( R = \mathbb{Z}_8 \). Then \( R \) is NEC, while \( N(R)^2 = \{0, 4\} \neq 0 \). Hence there exists a NEC ring \( R \) with \( N(R)^2 \neq 0 \).

Let \( R \) be a ring and \( V_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\} \). Then with the usual matrix addition and multiplication, \( V_2(R) \) forms a ring.

**Proposition 2.4.** \( R \) is a CN ring if and only if \( V_2(R) \) is a NEC ring.

**Proof** (\( \Rightarrow \)) Assume that \( A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \ B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in N(V_2(R)) \), then \( a_1, b_1 \in N(R) \subseteq Z(R) \), it follows that \( AB = \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 \\ 0 & a_1b_1 \end{pmatrix} = BA \). Therefore \( V_2(R) \) is NEC.

(\( \Leftarrow \)) For each \( a \in N(R), b \in R \), write \( A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \). Then \( A, B \in N(V_2(R)) \). Since \( V_2(R) \) is NEC, \( AB = BA \), that is, \( \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} \), it follows that \( ab = ba \). Hence \( R \) is CN.

Let \( R \) be a ring and \( R \propto R = \{(a, b) | a, b \in R \} \). Then with componentwise addition and the following multiplication:

\[ (a, b)(x, y) = (ax, ay + bx) \]
Corollary 2.5. The following conditions are equivalent for a ring $R$:

1. $R$ is a commutative ring;
2. $R \cong R / (x^2)$ is NEC.

Let $R$ be a ring and set $V_3(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_1 \end{pmatrix} | a_1, a_2, a_3, a_4 \in R \right\}$ and $SV_3(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix} | a_1, a_2, a_3 \in R \right\}$. Then with the usual matrix addition and multiplication, $V_3(R)$ and $SV_3(R)$ form rings. Clearly, $SV_3(R)$ is a subring of $V_3(R)$. The following example illustrates $V_3(R)$ need not be NEC even if $R$ is a division ring.

Example 2.6. Let $R = D$ only be a division ring. Then there exist $a, b \in R$ and $ab \neq ba$. Choose $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$. Then $A, B \in N(SV_3(R))$. Since $AB = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 & ba \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $AB \neq BA$. Hence $SV_3(R)$ is not NEC. Since each subring of NEC rings is NEC, $V_3(R)$ is not NEC.

Observing Example 2.6, we can obtain the following proposition.

Proposition 2.7. The following conditions are equivalent for a ring $R$:

1. $R$ is a commutative ring;
2. $SV_3(R)$ is a commutative ring;
3. $SV_3(R)$ is a NEC ring.

Motivated by Example 2.2, we obtain the following theorem which gives a characterization of reduced rings.

Theorem 2.10. $R$ is a reduced ring if and only if the $2 \times 2$ upper triangular matrix ring $T_2(R)$ over $R$ is a NEC ring.
**Proof** (\(\Rightarrow\)) Assume that \(R\) is reduced, then \(N(T_2(R)) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}\), so \(T_2(R)\) is NEC because \(N(T_2(R))^2 = 0\).

(\(\Leftarrow\)) Assume that \(a \in R\) with \(a^2 = 0\). Choose \(A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}\). Then \(A, B \in N(T_2(R))\). Since \(T_2(R)\) is NEC, \(AB = BA\), one has \(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\), it follows that \(a = 0\). Therefore \(R\) is reduced. \(\square\)

Let \(R\) be a ring and write \(GT_2(R) = \begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\) \([a_1, a_2, a_3 \in R]\), \(WGT_2(R) = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\) \([a_1, a_2, a_3 \in R]\) and \(QGT_2(R) = \begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\) \([a_1, a_2, a_3 \in R]\). Then by the usual matrix addition and multiplication, \(GT_2(R), WGT_2(R)\) and \(QGT_2(R)\) form rings. Set \(\rho : T_2(R) \rightarrow GT_2(R)\) defined by \(\rho\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_1 & a_1 & a_2 - a_1 + a_3 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\) and \(\tau : WGT_2(R) \rightarrow QGT_2(R)\) defined by \(\tau\left(\begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\right) = \begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}\). Then \(\rho, \sigma\) and \(\tau\) are ring isomorphisms. Hence Theorem 2.10 implies the following corollary.

**Corollary 2.11.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is reduced;
2. \(GT_2(R)\) is NEC;
3. \(WGT_2(R)\) is NEC;
4. \(QGT_2(R)\) is NEC.

**Remark 2.12.** Example 2.9 illustrates the \(3 \times 3\) upper triangular matrix ring \(T_3(R)\) over a field \(R\) need not be NEC.

Let \(R\) be a ring and write \(M_2^{(0)}(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\}_{a_{ij} \in R, \ i, j = 1, 2}\). For any \(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\), \(B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2^{(0)}(R)\), we define new multiplication as follows:

\[
AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}
\]

Then with the usual matrix addition and the new multiplication, \(M_2^{(0)}(R)\) is a ring.

**Proposition 2.13.** \(R\) is a reduced ring if and only if \(M_2^{(0)}(R)\) is a NEC ring.

**Proof** (\(\Rightarrow\)) Assume \(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\), \(B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in N(M_2^{(0)}(R))\), then \(a_{11}, a_{22}, b_{11}, b_{22} \in N(R) = 0\), so \(A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}\). It follows that \(AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = BA\). Hence \(M_2^{(0)}(R)\) is NEC.

(\(\Leftarrow\)) Choose \(a \in R\) with \(a^2 = 0\) and \(A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}\). Then \(A, B \in N(M_2^{(0)}(R))\). Since \(M_2^{(0)}(R)\) is NEC, \(AB = BA\), that is, \(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). This gives \(a = 0\). Hence \(R\) is reduced ring. \(\square\)
Let $R$ be a ring and write $WT_3(R) = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix} | a_i \in R, i = 1, 2, \ldots, 5$. Then with the usual matrix addition and multiplication, $WT_3(R)$ forms a ring.

**Theorem 2.14.** $R$ is a reduced ring if and only if $WT_3(R)$ is a NEC ring.

**Proof** Assume that $R$ is reduced, then $N(WT_3(R)) = \begin{pmatrix} 0 & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, this gives $WT_2(R)$ is NEC because $N(WT_3(R))^2 = 0$.

Conversely, assume that $WT_3(R)$ is NEC and $a \in R$ with $a^2 = 0$. Choose $A = \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$, $B = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. Then, clearly, $A, B \in N(WT_3(R))$. Since $WT_3(R)$ is NEC, $AB = BA$, this gives $A = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$, one gets $a = 0$. Therefore $R$ is reduced. \(\Box\)

Let $R$ be a ring and write $SV_4(R) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_5 \end{pmatrix} | a_1, a_2, a_3, a_4, a_5 \in R$. Then with the usual matrix addition and multiplication, $SV_4(R)$ forms a ring.

**Theorem 2.15.** $R$ is a commutative reduced ring if and only if $SV_4(R)$ is a NEC ring.

**Proof** ($\implies$) Assume that $R$ is a commutative reduced ring, then $N(SV_4(R)) = \begin{pmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, since $R$ is commutative, we can easily to show that $AB = BA$ for all $A, B \in N(SV_4(R))$, one gets $SV_4(R)$ is NEC.

($\impliedby$) Assume that $x, y, a \in R$ with $a^2 = 0$. Choose $C = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $E = \begin{pmatrix} a & 1 & 1 & a \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $C, D, E, F \in N(SV_4(R))$. Since $SV_4(R)$ is NEC, $CD = DC$ and $EF = FE$, this gives $\begin{pmatrix} 0 & 0 & xy & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & yx & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

$\begin{pmatrix} 0 & a & a & a \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so $xy = yx$ and $a = 0$. Hence $R$ is commutative reduced. \(\Box\)

3. **Properties of NEC Rings**

Let $R$ be a ring and write $Max_l(R)$ to denote the set of all maximal left ideals of $R$. 


Theorem 3.1. Let $R$ be a NEC ring, $e \in E(R)$, $a \in R$ and $M \in \text{Max}_a(R)$. Then we have
(1) $e \in M$ or $(1-e)R \subseteq R$;
(2) $1-ae \in M$ if and only if $1-ea \in M$;
(3) $Ra + R(\{ae-1\}) = R$;
(4) $Me \subseteq M$.

**Proof**
(1) If $e \notin M$, then $Re + M = R$, so $(1-e)R \subseteq (1-e)Re + M$. Since $R$ is NEC and $eR(1-e) \subseteq N(R)$, we have $eR(1-e)Re = (1-e)ReR(1-e) = 0$, it follows that $(1-e)Re \subseteq M$. Hence $(1-e)R \subseteq M$.

(2) If $1-ae \in M$, then $ae \notin M$, so $e \notin M$. By (1), we have $(1-e)R \subseteq M$, so $(1-e)a(1-e) \in M$, this gives
$$1-a = 1-ae + ae-a = (1-ae) - a(1-e) \in M,$$
so $1-ea = 1-a + a - ea = (1-a) + (1-e)a \in M$.

Conversely, assume that $1-ea \in M$. If $e \in M$, then $1-e \notin M$. By (1), we have $eR \subseteq M$, it follows that
$$1 = (1-ea) + ea \in M,$$
which is a contradiction, hence $e \notin M$. By (1), we have $(1-e)R \subseteq M$, this implies that
$$R(1-e) \subseteq M,$$
one gets $1-a = 1-ae+ae-a = (1-ae)-a(1-e) \in M$ and then $1-ae = 1-a+a-ae = (1-a)+a(1-e) \in M$.

(3) If $Ra + R(\{ae-1\}) \neq R$, then there exists a maximal left ideal $K$ of $R$ such that $Ra + R(\{ae-1\}) \subseteq K$. Since $ae-1 \in K$, by (2), $1-ae \in K$. Since $a \in K$, $ea \in K$, this gives $1 \in K$, which is a contradiction. Hence $Ra + R(\{ae-1\}) = R$.

(4) If $Me \notin M$, then $Me+M = R$. Write $1 = me+n$ for some $m, n \in M$. By (3), we have $R = Rm+R(me-1) = Rm+R(-(n)) \subseteq M$, so $R = M$, which is a contradiction. Hence $Me \subseteq M$.

Recall that a ring $R$ is said to be directly finite if $ab = 1$ implies $ba = 1$.

**Lemma 3.2.** Let $R$ be a ring satisfying either $e \in M$ or $(1-e)R \subseteq M$ for each $e \in E(R)$ and $M \in \text{Max}_a(R)$. Then $R$ is directly finite.

**Proof**
Assume that $ab = 1$. Write $e = ba$. Then $e \in E(R)$, $ae = a$ and $eb = b$. If $Re \neq R$, then there exists $M \in \text{Max}_a(R)$ such that $Re \subseteq M$. Since $1-e \notin M$, by hypothesis, $eR \subseteq M$, one gets $b = eb \in M$, it follows that
$$1 = ab \in M,$$
which is a contradiction. Hence $Re = R$, this implies $ba = e = 1$. Therefore $R$ is directly finite.

The following corollary follows from Theorem 3.1 and Lemma 3.2.

**Corollary 3.3.** NEC rings are directly finite.

An element $e \in E(R)$ is called left minimal idempotent if $Re$ is a minimal left ideal of $R$. Write $ME_i(R)$ to denote the set of all left minimal idempotents of $R$. A ring $R$ is called left min-abel if either $ME_i(R) = \emptyset$ or each element $e$ of $ME_i(R)$ is left semicentral (that is, $ae = eae$ for all $a \in R$).

**Lemma 3.4.** A ring $R$ is left min-abel if and only if $Me \subseteq M$ for each $e \in ME_i(R)$ and $M \in \text{Max}_a(R)$.

**Proof**
Suppose that $R$ is left min-abel. Choose $e \in ME_i(R)$ and $M \in \text{Max}_a(R)$. If $Me \notin M$, then $Me + M = R$.

Since $e$ is left semicentral, $1-e$ is right semicentral, so $(1-e)R \subseteq (1-e)Me + (1-e)M \subseteq M$, one gets $R(1-e) = M$, so $Me = 0$, which is a contradiction. Hence $Me \subseteq M$.

Conversely, let $e \in ME_i(R)$. If $(1-e)Re \neq 0$, then there exists $a \in R$ such that $(1-e)ae \neq 0$. Write
$$g = e + (1-e)ae,$$
then $g \in ME_i(R)$, $eg = e$ and $ge = g$. Since $R(1-g) \in \text{Max}_a(R)$, $R(1-g)e \subseteq R(1-g)$ by hypothesis, it follows that $(1-g)ge = 0$, one gets $e = g$, so $(1-e)ae = 0$ which is a contradiction. Therefore $(1-e)Re = 0$, which shows that $R$ is left min-abel.

Theorem 3.1 and Lemma 3.4 implies the following corollary.

**Corollary 3.5.** NEC rings are left min-abel.

The following example illustrates the converses of Corollary 3.3 and Corollary 3.5 are not true.

**Example 3.6.** Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in Z, a \equiv d \ (\text{mod } 2), b \equiv c \equiv 0 \ (\text{mod } 2)$. Then by the usual addition and multiplication of matrix, $R$ forms a ring. It is easy to show that $E(R) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \subseteq Z(R)$, so $R$ is
left min-abel and directly finite. We claim that \( R \) is not NEC. In fact, let \( A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \), we have \( A^2 = B^2 = 0 \), so \( A, B \in N(R) \). Since \( AB = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \), \( BA = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \), we have \( AB \neq BA \). Therefore \( R \) is not NEC.

A ring \( R \) is said to be \( n \)-regular [24] if every element of \( N(R) \) is regular. It is well known that a ring \( R \) is strongly regular if and only if \( x \in Rx^2 \) for each \( x \in R \).

**Lemma 3.7.** Let \( R \) be a NEC ring. If \( x \in R \) is regular, then \( x \) is strongly regular.

**Proof** Since \( x \) is regular, \( x = xyx \) for some \( y \in R \). Set \( e = xy \), then \( e \in E(R) \) and \( x = ex \), one gets \( x(1-e) \in N(R) \). Since \( R \) is NEC, \( x(1-e)ye = (1-e)yex(1-e) = 0 \), it follows that \( e = xey \), so \( x = ex = xeyx = xyx \in x^2R \). Similarly, we can show that \( x \in Rx^2 \). Hence \( x \) is strongly regular.

The following two theorems follow from Lemma 3.7.

**Theorem 3.8.** The following conditions are equivalent for a ring \( R \):
1. \( R \) is a strongly regular ring;
2. \( R \) is a unit-regular ring and NEC ring;
3. \( R \) is a regular ring and NEC ring.

**Theorem 3.9.** The following conditions are equivalent for a ring \( R \):
1. \( R \) is a reduced ring;
2. \( R \) is a NEC ring and \( n \)-regular ring.

Recall that a ring \( R \) is left NPP [24] if for each \( a \in N(R) \), \( Ra \) is projective as left \( R \)-module. And \( R \) is said to be left idempotent reflexive if \( aRe = 0 \) implies \( eRa = 0 \) for each \( a \in R \) and \( e \in E(R) \). Clearly, \( R \) is a left NPP ring if and only if for each \( a \in N(R) \), \( l(a) = Re \) for some \( e \in E(R) \), where \( l(a) = \{ x \in R \mid xa = 0 \} \).

**Proposition 3.10.** Let \( R \) be a NEC left NPP ring. If \( R \) is left idempotent reflexive, then \( R \) is reduced.

**Proof** Assume that \( a \in R \) satisfying \( a^2 = 0 \). Then \( l(a) = Re \) for each \( e \in E(R) \) because \( R \) is left NPP. Hence \( a = ae \) and \( ea = 0 \). Since \( R \) is NEC, \( ax(1-x) = a(ex(1-x)) = ex(1-e)a = exa \) for each \( x \in R \), one gets \( ax(1-e) = 0 \), so \( aR(1-e) = 0 \). Since \( R \) is left idempotent reflexive, \( (1-e)Ra = 0 \), it follows that \( a = ea = 0 \). Therefore \( R \) is reduced.

Since semiprime rings are left idempotent reflexive, Proposition 3.10 implies the following corollary.

**Corollary 3.11.** \( R \) is a reduced ring if and only if \( R \) is a semiprime NEC left NPP ring.

**Lemma 3.12.** Let \( R \) be a NEC ring and \( I \) an ideal of \( R \). If \( I \subseteq N(R) \), then \( R/I \) is NEC.

**Proof** It is clear. Clearly, for a NEC ring \( R \), \( N(R) \) is only an addition subgroup of \( R \). If \( R/P(R) \) is a left NPP ring, then we can say more, where \( P(R) \) denotes the prime radical of \( R \).

**Theorem 3.13.** Let \( R \) be a NEC ring. If \( R/P(R) \) is left NPP, then \( N(R) = P(R) \).

**Proof** Since \( R \) is NEC, by Lemma 3.12, \( R/P(R) \) is NEC, Since \( R/P(R) \) is a semiprime left NPP ring, \( R/P(R) \) is reduced by Corollary 3.11, so \( N(R) \subseteq P(R) \). Therefore \( N(R) = P(R) \).

An ideal \( I \) of \( R \) is called reduced if \( I \cap N(R) = 0 \). Clearly, every ideal of reduced ring is reduced.

**Proposition 3.14.** Let \( R \) be a ring and \( I \) a reduced ideal of \( R \). If \( R/I \) is NEC, then so is \( R \).

**Proof** Suppose that \( a, b \in N(R) \), then in \( R = R/I, a, b \in N(R) \). Since \( R/I \) is NEC, \( ab - ba \in I \). Since \( a \in N(R) \), there exists \( n \geq 1 \) such that \( a^n = 0 \). If \( n = 1 \), then \( a = 0 \), so \( ab = ba \), we are done. Hence we assume that \( n \geq 2 \). Since \( (a^{n-1}(ab - ba)a)^2 = 0 \) and \( I \) is reduced, \( a^{n-1}(ab - ba)a = 0 \), this gives \( (a^{n-1}(ab - ba))^2 = 0 \), so \( a^{n-1}(ab - ba) = 0 \), again \( (a^{n-2}(ab - ba)a)^2 = 0 \) implies \( a^{n-2}(ab - ba)a = 0 \), further, we have \( a^{n-2}(ab - ba) = 0 \). Repeating this process, we can obtain that \( ab - ba = 0 \), this shows that \( R \) is NEC.
Lemma 3.15. Let \( R \) be a ring and \( I, J \) two ideals of \( R \). If \( R/I, R/J \) are NEC and \( I \cap J = 0 \), then \( R \) is NEC.

Proof. It is routine. \( \square \)

Theorem 3.16. Let \( R \) be a ring and \( I, J \) two ideals of \( R \). If \( R/I, R/J \) are NEC, then \( R/(I \cap J) \) is NEC.

Proof. It is an immediate result of Lemma 3.15. \( \square \)

Let \( R \) be a ring, \( B(R) \) be the set of all central idempotents of \( R \), and \( S(R) \) be the nonempty set of all proper ideals of \( R \) generated by central idempotents. An ideal \( P \in S(R) \) is a Pierce ideal of \( R \) if \( P \) is a maximal (with respect to inclusion) element of the set \( S(R) \). The set of all Pierce ideals of \( R \) is denoted by \( P(R) \). If \( P \) is a Pierce ideal of \( R \), then the factor ring \( R/P \) is called a Pierce stalk of \( R \).

Theorem 3.17. The following conditions are equivalent for a ring \( R \):
1. \( R \) is a NEC ring;
2. \( R/S \) is a NEC ring for every ideal \( S \) generated by central idempotents of \( R \);
3. All Pierce stalks of \( R \) are NEC rings.

Proof. (1) \( \implies \) (2) Assume that \( x, y \in R \) such that \( x, y \in N(R/S) \), then there exist \( m, n \geq 1 \) such that \( x^m, y^n \in S \). Since \( S \) is generated by central idempotents of \( R \), there exists a central idempotent \( g \in S \) such that \( x^m, y^n \in S \). Clearly \( (x(1-g))^m = 0 = (y(1-g))^m \), one gets \( x(1-g)y(1-g) = y(1-g)x(1-g) \) because \( R \) is NEC. Hence \( xy = 0 \), this shows that \( R/S \) is NEC.

(2) \( \implies \) (3) It is trivial.

(3) \( \implies \) (1) Suppose that \( R \) is not a NEC ring, then there exist \( a, b \in N(R) \) such that \( ab \neq ba \). Put \( \Sigma = \{kl \mid k, l \in \mathbb{Z}\} \) is an ideal of \( R \) generated by central idempotents and in \( \bar{R} = R/\bar{I}, \bar{ab} \neq b\bar{a} \). Then \( \Sigma \) is not an empty set because \( 0 \in \Sigma \). One can easily show that there exists a maximal element \( P \) in \( \Sigma \) by Zorn’s Lemma. If \( P \) is not a Pierce ideal of \( R \), then there is a central idempotent \( e \) of \( R \) such that \( P + eR \) and \( P + (1-e)R \) are proper ideals of \( R \) which properly contain the ideal \( P \). Hence \( P + eR \notin \Sigma \) and \( P + (1-e)R \notin \Sigma \), it follows that \( ab - ba \in (P + eR) \cap (P + (1-e)R) = P \), which is a contradiction. Thus \( P \) is a Pierce ideal of \( R \), by (3), \( R/P \) is NEC, which is also a contradiction because \( ab - ba \notin P \). Therefore \( R \) is NEC. \( \square \)

4. NEC Exchange Ring

Recall a ring is Abelian [4] if \( E(R) \subseteq Z(R) \). It is well known that clean rings are always exchange [3]. And the converse is true when \( R \) is an Abelian ring by [26]. Example 3.6 illustrates that NEC ring need not be Abelian.

Theorem 4.1. Let \( R \) be a NEC ring. If \( R \) is exchange, then \( R \) is clean.

Proof. Since \( R \) is NEC, \( R/P(R) \) is NEC by Lemma 3.12. Since \( R/P(R) \) is semiprime, \( R/P(R) \) is Abel, this implies that \( R/P(R) \) is an Abel exchange ring, so \( R/P(R) \) is clean by [26]. Therefore \( R \) is clean. \( \square \)

It is well known that an exchange ring with only two idempotents is local.

Lemma 4.2. Let \( R \) be a NEC exchange ring. If \( P \) is a prime ideal of \( R \), then \( R/P \) is local.

Proof. Since \( R \) is a NEC exchange ring, \( R/P(R) \) is Abel. Assume that \( \hat{a} \) is any idempotent of \( \hat{R} = R/P \), then there exists \( e \in E(R) \) such that \( \hat{e} = \hat{a} \) because \( R \) is exchange. Clearly, in \( R = R/P(R) \), \( eR(1-e) = 0 \), so \( eR(1-e) \subseteq P(R) \subseteq P \). Since \( P \) is a prime ideal of \( R \), \( e \in P \) or \( 1-e \in P \), this gives \( \hat{a} = 0 \) or \( \hat{a} = 1 \). Therefore \( R/P \) is local. \( \square \)

The following corollary is an immediate result of Lemma 4.2.

Corollary 4.3. Let \( R \) be a NEC exchange ring. If \( P \) is a left (right) primitive ideal of \( R \), then \( R/P \) is a division ring.

Theorem 4.4. Let \( R \) be a NEC exchange ring. Then \( R \) is a left and right quasi-duo ring.
Corollary 4.5. Let $R$ be a NEC ring with $\text{isr}(R) = 1$. Then $R$ is a left and right quasi-duo ring and $R$ has right square stable range one.

Proof. For any $a \in R$, the equation $aR + (-1)R = R$ gives $a + (-1)e \in U(R)$ for some $e \in E(R)$ because $\text{isr}(R) = 1$. Thus $a$ is a clean element and $R$ is a clean ring. Hence $R$ is an exchange ring, by Theorem 4.4, $R$ is a left and right quasi-duo ring.

Now let $xR + yR = R$. If $x^2R + yR \neq R$, then there exists a maximal right ideal $M$ of $R$ containing $x^2R + yR$. Since $M$ is an ideal of $R$, $R/M$ is a division ring. Clearly $xR + yR = R$ implies $xR = x^2R + xyR \subseteq M$, so $R = xR + yR \subseteq M$, which is a contradiction. Hence $x^2R + yR = R$, this leads to $x^2 + yg \in U(R)$ for some $g \in E(R)$. This shows that $R$ has right square stable range one.

Theorem 4.6. Let $R$ be a NEC exchange ring. Then $R$ has left and right square stable range one.

Proof. Since $R$ is a NEC exchange ring, $R$ is a left and right quasi-duo ring by Theorem 4.4, so $R/J(R)$ is a left quasi-duo ring, by [25, Corollary 2.4], $R/J(R)$ is a reduced ring, hence $R/J(R)$ is an Abel exchange ring, one gets $R/J(R)$ has stable range one by [26, Theorem 6]. Therefore $R$ has stable range one. Similar to the proof of Corollary 4.5, we can show that $R$ has left and right square stable range one.

Corollary 4.7. If $R$ is a NEC exchange ring, then $\text{isr}(R) = 1$.

Proof. Let $\bar{R} = R/J(R)$. By Theorem 4.6, $R$ has right square stable range one and $\bar{R}$ is an Abel exchange ring. Follows from [7, Theorem 12], we have $\text{isr}(\bar{R}) = 1$. And from [7, Theorem 9], one obtains $\text{isr}(R) = 1$.

Proposition 4.8. Let $R$ be a NEC exchange ring. Then the following conditions are equivalent:

1. there exists an $u \in U(R)$ such that $1 \pm u \in U(R)$;
2. for any $a \in R$ there exists $u \in U(R)$ such that $a \pm u \in U(R)$.

Proof. (1) $\Rightarrow$ (2) Since $R$ is a NEC exchange ring, $R/J(R)$ is an Abel exchange ring by Theorem 4.6, and by [26, Theorem 6], $R/J(R)$ is an exchange ring of bounded index. By [8, Corollary 2.4], there exists a $u \in U(R/J(R))$ such that $a \pm u \in U(R/J(R))$. Since invertible elements can be lifted modulo $J(R)$, there exists an $u \in U(R)$ such that $a \pm u \in U(R)$.

(2) $\Rightarrow$ (1) is trivial.

We call a ring $R$ a left (right) $P$-exchange ring if every projective left (right) $R$-module has the exchange property. This definition is not left-right symmetric, for example, a left perfect ring which is not right perfect is a left but not a right $P$-regular ring.

Theorem 4.9. Let $R$ be a NEC left $P$-exchange ring. Then $R/J(R)$ is a strongly regular ring.

Proof. Since $R$ is a NEC left $P$-exchange ring, $R$ is a NEC exchange ring, it follows that $R/J(R)$ is an Abel ring by Theorem 4.6, by [6, Corollary 2.16], $R/J(R)$ is a weakly $\pi$-regular ring. Since $R$ is a left quasi-duo ring by Theorem 4.4, $R/J(R)$ is left quasi-duo, it follows that $R/J(R)$ is strongly regular.

The following corollary is an immediate result of Theorem 4.9 which gives a characterization of strongly regular rings.

Corollary 4.10. $R$ is a strongly regular ring if and only if $R$ is a NEC left $P$-exchange ring with $J(R) = 0$. 
Recall that an element $a$ in $R$ is uniquely clean if it has exactly one clean decomposition, and $a$ is said to be strongly clean if it has a clean decomposition $a = e + u$ in which $eu = ue$. Following [16], we let $ucn(R)$ denote the set of uniquely clean elements and $scn(R)$ is the set of strongly clean elements. Clearly, a ring $R$ is Abel if and only if $E(R) \subseteq ucn(R)$.

**Proposition 4.11.** Let $R$ be a NEC ring. Then $ucn(R) \subseteq scn(R)$.

**Proof** Assume that $a \in ucn(R)$, then $a$ has the uniquely clean decomposition $a = e + u$. Since $R$ is NEC, by the proof of Theorem 3.1(1), we know that $e(1 - e)Re = 0 = eR(1 - e)x$ for each $x \in R$. Since $J(R)$ is a semiprimal ideal of $R$, $e(1 - e) \in J(R)$ and $(1 - e)x \in J(R)$ for each $x \in R$, follows from the decomposition $a = e + u = (e + e(1 - e)) + (u - e(1 - e)) = e + e(1 - e)x + (u - (1 - e)x)$, we can see that $e + (1 - e)x = e = e + e(1 - e)$ and $u - (1 - e)x = u = u - e(1 - e)x$ for each $x \in R$. Thus $eR(1 - e) = 0 = (1 - e)Re$.

**Theorem 4.12.** Let $R$ be an exchange ring and $I$ a right ideal of $R$, which contains no nonzero idempotents. Then $R$ has stable range one if and only if for any regular element $a$ of $R$, there exists $u \in U(R)$, such that $a - auu \in I$.

**Proof** ($\Rightarrow$) Let $a, x \in R, e \in E(R)$ such that $ax + e = 1$. If $ea = 0$, then $a = axa$, so there exists $u \in U(R)$ such that $a - auu = y \in I$. We have $1 - e = ax = (aua + y)x = axa + yx = au(1 - e) + yx, (au - e)^2 = auau - au - eau + e = (a - u)u - eu + e = (au - e)u + e = 1 - e - yu - eu + e = 1 - y(1 + e)$. Since $R$ is an exchange ring, there exists $g^2 = g \in E(R) \subseteq I$ such that $1 - g = (1 - y(1 + e))R$. Since $I$ contains no nonzero idempotents, one gets $g = 0$, so $1 \in (1 - y(1 + e))R$. Assume $1 = (1 - y(1 + e)x)$ for some $z \in R$, so that $(au - e)z = 1$. Let $v = (au - e)z$. Then $v = 0$, $v = 0$, and $v = 0$, so $v = 0$, and $v = 0$. Hence in any case, one has $u \in U(R), v \in R$ such that $au + ru - v = 1$. Since $fr = 0$, we have $(au + ru - v)w = 1$. Write $h = v(au + es)$. Then $h^2 = h$ and $(au + es)h = au + es$. Since $v(au + es) + 1 = 1$, the above proof, there exists $w \in U(R), t, q \in R$ such that $(uw + (1 - h))tq = 1, su + es = (au + es)(uw + (1 - h))tq = wq$, then $q = w^{-1}(au + es)$. Hence $(uw + (1 - h)t)w^{-1}(au + es) = 1$, this implies $au + es \in U(R)$, so $a + esu^{-1} \in U(R)$. Therefore $R$ has stable range one.

**Corollary 4.13.** [21, Proposition 5.3] An exchange ring $R$ has stable range one if and only if for each regular element $a$ of $R$, there exists $u \in U(R)$ such that $a - auu \in J(R)$.

**Corollary 4.14.** [27, Proposition 4.6] An exchange ring $R$ has stable range one if and only if for each regular element $a$ of $R$, there exists $u \in U(R)$ such that $a - auu \in Z(R)$.

**Corollary 4.15.** An exchange ring $R$ has stable range one if and only if for each regular element $a$ of $R$, there exists $u \in U(R)$ such that $a - auu \in Z_e(R)$.

5. Generalized Inverses

An involution $a \mapsto a^*$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*.$$ 

A ring $R$ with an involution $*$ is called $*$-ring. An element $a^+$ in a $*$-ring $R$ is called the Moore-Penrose inverse (or MP-inverse) of $a$, if [20]

$$aa^+a = a, aa^+a^+ = a^+, a^+a = (aa^+)^*.$$ 

In this case, we call a $a$ is MP-invertible in $R$. The set of all MP-invertible elements of $R$ is denoted by $R^+$.

An involution $*$ of $R$ is called proper if $x^*x = 0$ implies $x = 0$ for all $x \in R$.

Following [5], an element $a$ of a ring $R$ is called group invertible if there is $a^\dagger \in R$ such that
Theorem 5.2. Let R be a NEC ring. If a \in R^4, then a \in R^5.

Proof. Since R is a NEC ring and a \in R^4, by Lemma 5.1, a \in R^5, so \bar{a}^4 exists.

Write f = aa^4, g = a^4a and e = aa^4. Then f = f^2, g = g^2, e = e^2 and \bar{a} = a\bar{a} = fa = ea = ae. Noting that \bar{a}^4 = f\bar{a}^4 = a\bar{a}. Then (1 - f)(1 - g)\bar{a}^4 \in N(R), this gives that (1 - g)\bar{a}^4(a - f) = (1 - f)(1 - g)\bar{a}^4, so a(1 - f)(1 - g)\bar{a}^4 = 0. Noting that a = \bar{f}a. Then (1 - f)(1 - g)\bar{a}^4 = 0, which implies that

\begin{equation}
(1 - f)(1 - g)\bar{a}^4 = 0.
\end{equation}

Equation (5.1) gives that

\begin{equation}
\bar{a}^4 = a\bar{a}^4 = a\bar{a}^4a\bar{a}^4.
\end{equation}

Hence \bar{a} = (a\bar{a})^4 \in a\bar{a}a^4, (1) is completed.

Noting that \bar{a}^4 = a\bar{a}^4 = a\bar{a}^4a\bar{a}^4. Then (3) is completed.

Equation (5.2) gives that \bar{a}^4 = a\bar{a}^4 + a\bar{a}^4 + a\bar{a}^4 + a\bar{a}^4, hence (2) holds.

Since (1 - e)\bar{a}^4 = (1 - e)a\bar{a}^4 = ((1 - e)a\bar{a}^4)(1 - e) = (e\bar{a}^4(1 - e))(1 - e)a\bar{a}^4 = 0, we have

\begin{equation}
\bar{a}^4 = e\bar{a}^4 + a\bar{a}^4 + e\bar{a}^4 + a\bar{a}^4 - a\bar{a}^4, which implies that (4) holds.
\end{equation}

(5) Noting that \bar{a}^4(1 - f)(1 - f)\bar{a}^4 \in N(R) and \bar{a}^4 = a\bar{a}^4. Then (1 - f)(1 - f)\bar{a}^4 = (1 - f)a\bar{a}^4(1 - f) = 0, it follows that

\begin{equation}
\bar{a}^4 = a\bar{a}^4 + a\bar{a}^4 + a\bar{a}^4 + a\bar{a}^4.
\end{equation}

(6) Noting that (a - a\bar{a})^2 = 0 = (a\bar{a} - a\bar{a})^2. Then \bar{a}^4 = a\bar{a}^4 + a\bar{a}^4 = (a - a\bar{a})^4 + a\bar{a}^4. Since (a - a\bar{a})^4 + a\bar{a}^4 = 0, (a - a\bar{a})^4 = a\bar{a}^4 = 0, and (a - a\bar{a})^4 + a\bar{a}^4 = (a - a\bar{a})^4 + a\bar{a}^4 = (a - a\bar{a})^4 + a\bar{a}^4 = (a - a\bar{a})^4 + a\bar{a}^4 = 0. Hence (a - a\bar{a})^4 = a\bar{a}^4 = a\bar{a}^4.

We don’t know whether a is EP under the conditions of Theorem 5.2. However, we have the following theorem.

Theorem 5.3. Let R be a NEC ring and a \in R^4. If Ra is a minimal left ideal of R, then a is EP.

Proof. Since R is NEC and a \in R^4, by Lemma 5.1, a \in R^5. If a\bar{a}^4 = a\bar{a}^4, then aR = \bar{a}R = a\bar{a}^4 = a\bar{a}^4 + a\bar{a}^4 = a\bar{a}^4 + a\bar{a}^4 = a\bar{a}^4. Hence one obtains that (1 - a\bar{a}^4)Ra = (1 - a\bar{a}^4)Ra = 0, a = a\bar{a}^4, it follows that a is an EP element. If a\bar{a}^4 \neq a\bar{a}^4, then, by Theorem 5.2(4), we have a\bar{a}^4 \neq a\bar{a}^4, so (1 - a\bar{a}^4)a = 0. Noting that a\bar{a}^4 = a\bar{a}^4. Then (1 - a\bar{a}^4)a \neq 0. Since Ra is a minimal left ideal of R, Ra = R(1 - a\bar{a}^4)a\bar{a}^4. Write a = c(1 - a\bar{a}^4)a\bar{a}^4 for some c \in R. Then

Ra = Ra = R(1 - a\bar{a}^4)a\bar{a}^4 = Ra(1 - a\bar{a}^4)a\bar{a}^4. By Theorem 5.2(4), Ra = R(a\bar{a}^4 + 1)Ra \subseteq Ra = Ra. Hence Ra = Ra, which implies that a is EP. □

Let a \in R^2 \cap R^4 and write \chi_a = \{a, a^4, a^4, (a^4)^4, (a^4)^4\}. Then we have the following theorem.
Theorem 5.4. Let \( a \in \mathbb{R}^2 \cap R^2 \). Then \( a \) is an EP element if and only if the equation
\[
d^+axa = ax
\]
has at least a solution in \( \chi_a \).

**Proof** The necessity is clear.

Conversely, we assume that the equation (4.1) has at least a solution in \( \chi_a \).

(1) If \( x = a \) is a solution, then \( a^*a^2 = a^2 \), this implies \( a^*a = aa^* \). Hence \( a \) is EP.

(2) If \( x = a^2 \) is a solution, then \( a^1aa^2a = aa^2 \), that is, \( a^2 = aa^2 \). Hence \( a \) is EP.

(3) If \( x = a^3 \) is a solution, then \( a^3aa^3a = aa^3 \), that is \( a^3 = aa^3 \). Hence \( a \) is EP.

(4) If \( x = a^4 \) is a solution, then \( a^4aa^4a = aa^4 \). Noting that \( a^4 = aa^4 \). Then \( a^4 = aa^4 \).

(5) If \( x = (a^2)^\cdot \) is a solution, then \( a^2a(a^2)^\cdot = a(a^2)^\cdot \), it follows that \( (a^2a)^2a = a(a^2)^\cdot \). Noting that \( a^2 = aa^2 \).

Then \( (a^2)^\cdot a = a(a^2)^\cdot \). Applying the involution to the last equation, we have \( a^2a^2 = a^2a^2 \), this gives that \( Ra^2 = Ra^2 = Ra^2 = Ra^2 \). Hence \( (a^2)^\cdot a = a(a^2)^\cdot \). Applying the involution to the last equation, we have \( a^2a^2 = a^2a^2 \).

Multiplying by \( a \) from the left sided, one has \( (a^2)^\cdot a = a^2a^2a^\cdot a, \) this gives that \( a^2a^2 = a^2a^2a^\cdot a \). Hence \( Ra^2 = a^2R = a^2a^2 = a^2(a^\cdot a)Ra^2 \subseteq a^2R \), which implies that \( (a^2)^\cdot a = a(a^2)^\cdot \).

Hence \( a \) is EP. \( \square \)

**References**


