On Warped Product Semi-Slant Submanifolds of Nearly Trans-Sasakian Manifolds


Abstract. In this paper, we study warped product semi-slant submanifold of type $M = N_T \times_f N_\theta$ with slant fiber, isometrically immersed in a nearly Trans-Sasakian manifold by finding necessary and sufficient conditions in terms of Weingarten map. A characterization theorem is proved as main result.

1. Introduction

The study on warped product submanifolds got momentum after B.-Y. Chen’s papers on CR-warped product [13, 14]. A contact CR-warped product submanifold is the Riemannian product of invariant and anti-invariant submanifold. It was proved in [21] that there does not exist any contact CR-warped product of type $M = N_L \times_f N_\theta$, of nearly Trans-Sasakian manifolds in both cases when structure field tangent to either base manifold or fiber. Also, it was also found in the same paper, the non-trivial contact CR-warped product of the form $M = N_T \times_f N_L$, in a nearly Trans-Sasakian manifold such that $N_T$ invariant tangent to Reeb vector field. Similarly, the Riemannian product of invariant and slant submanifolds with non constant warping function is called warped product semi-slant submanifold. The non-existence of the warped product semi-slant submanifold $M = N_0 \times_f N_T$, isometrically immersed in a nearly Trans-Sasakian manifold with structure vector field is tangent $N_0$ and $N_T$ has discussed in [22]. On the other hand, the existence case of the non-trivial warped product semi-slant submanifold of type $M = N_T \times_f N_\theta$, in a nearly Trans-Sasakian manifold has been proved in [22] with $N_T$ is an invariant submanifold which is tangent to the Reeb vector field and constructed a geometric inequality for the extrinsic invariant in terms of warping function. Therefore, it is natural to see that the warped product semi-slant submanifold is a generalized version of contact CR-warped product submanifold in case of a nearly Trans-Sasakian manifold. Similar notions have been studied in the series of articles [1–8, 17, 21, 22, 24–29]. In this paper, we prove a characterization theorem involving the shape operator under which a semi-slant submanifold of a nearly Trans-Sasakian manifold reduces to a warped product.
2. Preliminaries

An almost contact manifold is an odd-dimensional manifold $\tilde{M}$ which carries a field $\varphi$ of endomorphisms of tangent space, vector field $\xi$, called characteristic or Reeb vector field and a 1-form $\eta$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

(2.1)

where $I : TM \to TM$ is the identity mapping. Now from definition it follows that $\varphi \circ \xi = 0$ and $\eta \circ \varphi = 0$, then the $(1, 1)$ tensor field $\varphi$ has constant rank $2n$ (cf. [9]). An almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ is said to be normal when the tensor field $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis of $\varphi$.

An almost contact metric structure $(\varphi, \xi, \eta)$ is said to be a normal in the form of almost complex structure if almost complex structure $J$ on a product manifold $\tilde{M} \times \mathbb{R}$ given by

$$f(U, f \frac{d}{dt}) = \left(\varphi U - f \xi, \eta(U) \frac{d}{dt}\right),$$

where $f$ is a smooth function on $\tilde{M} \times \mathbb{R}$, has no torsion, i.e., $J$ is integrable. Every almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ admits a Riemannian metric $g$ which is satisfying

$$g(\varphi U, \varphi V) = g(U, V) - \eta(X)\eta(Y), \quad \eta(U) = g(U, \xi),$$

(2.2)

for all $U, V \in \Gamma(TM)$. This metric $g$ is called compatible metric and the manifold $\tilde{M}$ endowed with the structure $(\varphi, \eta, \xi, g)$ is called an almost contact metric manifold. As an immediate consequence of (2.1), we have $g(\varphi U, V) = -\eta(U)\eta(V)$. Hence, the second fundamental 2-form $\Phi$ is defined by $\Phi(U, V) = g(\varphi U, V)$. Any almost contact metric such that both $\eta$ and $\Phi$ are closed is called almost cosymplectic manifold and those for which $d\eta = \Phi$ are called contact metric manifolds. Finally, a normal almost cosymplectic manifold is called cosymplectic manifold and a normal contact manifold is called Sasakian manifold. In term of the covariant derivative of $\varphi$ the cosymplectic and the Sasakian manifolds conditions can be expressed respectively by

$$(\nabla_U \varphi)V = 0, \quad \text{and} \quad (\nabla_U \varphi)V = g(U, V)\xi - \eta(V)U,$$

for all $U, V \in \Gamma(TM)$ (see [9]). It should be noted that both in cosymplectic and Sasakian manifolds $\xi$ is killing vector field. On the other hand, the Sasakian and the cosymplectic manifolds represent the two external cases of the larger class of quasi-Sasakian manifolds. An almost contact metric structure $(\varphi, \eta, \xi)$ is said to be nearly Trans-Sasakian manifold (cf. [19]) i.e., if

$$(\nabla_U \varphi)V + (\nabla_V \varphi)U = a(2g(U, V)\xi - \eta(U)V - \eta(V)U) - \beta(\eta(V)\varphi U + \eta(U)\varphi V),$$

(2.3)

for any $U, V$ tangent to $\tilde{M}$, where $\nabla$ is the Riemannian connection metric $g$ on $\tilde{M}$. If we replace $U = \xi, V = \xi$ in (2.3), we find that $(\nabla_\xi \varphi)\xi = 0$ which is implies that $\varphi \nabla_\xi \xi = 0$. Now applying $\varphi$ and using (2.1), we get $\nabla_\xi \xi = 0$. Since from Gauss formula finally, we get $\nabla_\xi \xi = 0$ and $h(\xi, \xi) = 0$. For more classification (see [? ? ?]).

**Note 2.1.** If $\alpha = 0$ and $\beta = 0$ in (2.3), then nearly Trans-Sasakian becomes nearly cosymplectic manifold, if $\alpha = 1$ and $\beta = 0$ in (2.3), thus its called nearly Sasakian manifold. Let $\alpha = 0$ and $\beta = 1$ in (2.3), then nearly Trans-Sasakian turn into nearly Kenmotsu manifold. Similarly nearly $\alpha$-Sasakian manifold and nearly $\beta$–Kenmotsu manifold can be defined from the nearly Trans-Sasakian manifold by substituting $\beta = 0$ and $\alpha = 0$ in (2.3), respectively.

Now let $M$ be a submanifold of $\tilde{M}$, then we will denote by $\nabla$ is the induced Riemannian connection on $M$ and $g$ is the Riemannian metric on $M$ as well as the metric induced on $M$. Let $TM$ and $T^*M$ be the Lie algebra of vector fields tangent to $M$ and normal to $M$, respectively and $\nabla^\perp$ the induced connection on $T^*M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on $M$ and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of $TM$ over $M$. Then the Gauss and Weingarten formulas are given by

$$\nabla_U V = \nabla_U T + h(U, V).$$

(2.4)

$$\nabla_U N = -A_N U + \nabla_U T,$$

(2.5)
for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$) respectively for the immersion of $M$ into $\tilde{M}$. They are related as:

$$g(h(U, V), N) = g(A_N U, V).$$

(2.6)

Now, for any $U \in \Gamma(TM)$, we defined as:

$$\varphi U = TU + FU,$$

(2.7)

where $TU$ and $FU$ are the tangential and normal components of $\varphi U$, respectively. If $M$ is invariant and anti-invariant, then $FU$ and $TU$ are identically zero, respectively. Similarly for any $N \in \Gamma(T^\perp M)$, we have

$$\varphi N = tN + fN,$$

(2.8)

where $tN$ (resp. $fN$) is the tangential (resp. normal) components of $\varphi N$. From (2.2) and (2.7), it is easy to observe that for each $U, V \in \Gamma(TM)$, $g(TU, V) = -g(U, TV)$. Further, the covariant derivative of the endomorphism $\varphi$ is defined as

$$(\nabla_U \varphi)V = \nabla_U \varphi V - \varphi \nabla_U V, \ \forall U, V \in \Gamma(TM).$$

(2.9)

Proposition 2.1. On a nearly Trans-Sasakian manifold, the following condition is satisfied

$$g(\nabla_V \xi, U) + g(\nabla_U \xi, V) = 2\beta g(\varphi X, \varphi Y),$$

for any vector fields $U, V$ tangent to $\tilde{M}$, where $\tilde{M}$ is a nearly Trans-Sasakian manifold

Proof. Setting $U = \xi$ in (2.3), then we find

$$(\nabla_\xi \varphi)V + (\nabla_V \varphi)\xi = \alpha |2g(\xi, V)\xi - V - \eta(V)\xi\} - \beta \varphi V.$$

Taking the inner product with $\varphi U$ in the above equation we get

$$g((\nabla_\xi \varphi)V, \varphi U) = -g((\nabla_V \varphi)\xi, \varphi U) - \alpha g(V, \varphi U) - \beta g(\varphi V, \varphi U).$$

(2.10)

Interchanging $U$ and $V$ in the above equation, we derive

$$g((\nabla_\xi \varphi)U, \varphi V) = -g((\nabla_V \varphi)\xi, \varphi U) - \alpha g(U, \varphi V) - \beta g(\varphi V, \varphi U).$$

(2.11)

Adding equation (2.10) and (2.11), we find

$$g((\nabla_\xi \varphi)V, \varphi U) + g((\nabla_\xi \varphi)U, \varphi V) = -g((\nabla_V \varphi)\xi, \varphi U) - g((\nabla_V \varphi)\xi, \varphi V) - 2\beta g(\varphi V, \varphi U).$$

As left hand side of the above equation should be zero from the fact that $\nabla_\xi \varphi = 0$, for almost contact metric manifold, hence

$$-g((\nabla_V \varphi)\xi, \varphi U) - g((\nabla_V \varphi)\xi, \varphi V) = 2\beta g(\varphi V, \varphi U).$$

The proof follows from the above equations and this complete the proof of the Proposition. \qed

We denote the tangential and normal parts of $(\nabla_U \varphi)V$ by $P_U V$ and $Q_U V$ such that

$$(\nabla_U \varphi)V = P_U V + Q_U V$$
Then in a nearly Trans-Sasakian manifold, we have
\begin{align}
\mathcal{P}_U V + \mathcal{P}_V U &= \alpha(2\langle U, V \rangle \xi - \eta(U)V - \eta(V)U) - \beta[\eta(V)TU + \eta(U)TV], \\
\mathcal{Q}_U V + \mathcal{Q}_V U &= -\beta[\eta(V)FU + \eta(U)FV],
\end{align}
for any \( U, V \) are tangent to \( M \). It is straightforward to verify the following properties of \( \mathcal{P} \) and \( \mathcal{Q} \),
\begin{align}
(i) \quad \mathcal{P}_{U+V} W &= \mathcal{P}_U W + \mathcal{P}_V W, \\
(ii) \quad \mathcal{Q}_{U+V} W &= \mathcal{Q}_U W + \mathcal{Q}_V W, \\
(iii) \quad \mathcal{P}_U(W + Z) &= \mathcal{P}_U W + \mathcal{P}_U Z, \\
(iv) \quad \mathcal{Q}_U(W + Z) &= \mathcal{Q}_U W + \mathcal{Q}_U Z, \\
(v) \quad g(\mathcal{P}_U V, W) &= -g(V, \mathcal{P}_U W), \\
(vi) \quad g(\mathcal{Q}_U V, N) &= -g(V, \mathcal{Q}_U N), \\
(vii) \quad \mathcal{P}_U \phi V + \mathcal{Q}_U \phi V &= -\phi(\mathcal{P}_U V + \mathcal{Q}_U V).
\end{align}

Next we will give the definition of slant submanifold as follows:

**Definition 2.1.** [11] For each non zero vector \( U \) tangent to \( M \) at \( p \), such that \( U \) is not proportional to \( \xi \), we denote by \( 0 \leq \theta(U) \leq \pi/2 \), the angle between \( \phi U \) and \( T_p M \) is called the Wirtinger angle. If the angle \( \theta(U) \) is constant for all \( U \in T_p M \), \( <\xi> \), and \( p \in M \). Then \( M \) is said to be a slant submanifold and the angle \( \theta \) is called slant angle of \( M \). Obviously if \( \theta = 0 \), \( M \) is invariant and if \( \theta = \pi/2 \), \( M \) is anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

In a contact metric manifold, J. L. Cabrerizo (cf. [11]) obtained the following theorem.

**Theorem 2.1.** Let \( M \) be a submanifold of an almost contact metric manifold \( \bar{M} \) such that \( \xi \in T M \). Then \( M \) is slant if and only if there exists a constant \( \lambda \in [0, 1] \) such that
\begin{equation}
T^2 = \lambda(-I + \eta \otimes \xi).
\end{equation}
Furthermore, in such a case, if \( \theta \) is slant angle, then it satisfies that \( \lambda = \cos^2 \theta \).

Hence, for a slant submanifold \( M \) of an almost contact metric manifold \( \bar{M} \), the following relations are consequences of the above theorem.
\begin{align}
g(TX, TY) &= \cos^2 \theta g(X, Y) - \eta(X)\eta(Y), \\
g(FX, FY) &= \sin^2 \theta g(X, Y) - \eta(X)\eta(Y)
\end{align}
for any \( X, Y \in \Gamma(TM) \). Another characterization of consequence of the Theorem 2.1 is easily derived as follows:

**Theorem 2.2.** Let \( M \) be a slant submanifold of an almost contact metric manifold \( \bar{M} \) such that \( \xi \in T M \). Then
\begin{align}
(a) \quad tFX &= \sin^2 \theta(-X + \eta(X)\xi), \quad \text{and} \quad (b) \quad fFX &= -FTX,
\end{align}
for any \( X \in \Gamma(TM) \).

3. Warped Product Semi-Slant Submanifolds

A natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant distribution was described by N. Papaghiuc (cf. [23]). These submanifolds are known as semi-slant submanifolds. The semi-slant submanifolds of almost contact metric manifolds were defined and studied by Cabrerizo. They defined these submanifolds as:
Definition 3.1. [12] A Riemannian submanifold \( M \) of an almost contact manifold \( \tilde{M} \) is said to be a semi-slant submanifold if there exist two orthogonal distributions \( D \) and \( D^0 \) such that

(i) \( TM = D^0 \oplus D \oplus \langle \xi \rangle \), where \( \langle \xi \rangle \) is 1-dimensional distribution spanned by \( \xi \).

(ii) \( D \) is invariant distribution under \( \varphi \) i.e., \( \varphi D \subseteq D \).

(iii) \( D^0 \) is slant distribution with slant angle \( \theta \neq 0, \frac{\pi}{2} \).

If we denote the dimension of \( D_i \) by \( d_i \) for \( i = 1, 2 \), then it is clear that contact CR-submanifolds and slant submanifolds are semi-slant submanifolds with \( \theta = \pi/2 \) and \( d_1 = 0 \), respectively. It is called proper semi-slant if slant angle is different from 0 and \( \pi/2 \). Moreover, if \( \mu \) is an invariant subspace under \( \varphi \) of normal bundle \( T^i M \), then in case of semi-slant submanifold, the normal bundle \( T^i M \) can be decomposed as \( T^i M = FD^0 \oplus \mu \). A semi-slant submanifold is said to be a mixed totally geodesic if \( h(X, Z) = 0 \), for any \( X \in \Gamma(D^0) \) and \( Z \in \Gamma(D) \).

Let \( f \) be a positive differentiable function on \( N_i \) of two Riemannian manifolds \( N_1 \) and \( N_2 \) endowed with two Riemannian metrics \( g_1 \) and \( g_2 \), respectively. Then warped product \( M = N_1 \times_f N_2 \) is the manifold \( N_1 \times N_2 \) equipped with the Riemannian metric \( g = g_1 + f^2 g_2 \). The function \( f \) is called warping function of the warped product. If for any \( X, Y \in \Gamma(T N_1) \) and \( Z, W \in \Gamma(T N_2) \), then

\[
V_Z X = V_X Z = (X \ln f) Z, \tag{3.1}
\]

where \( V \) denote the Levi-Civita connection on \( M \). On the other hand, \( V \ln f \) is the gradient of \( \ln f \) is defined as \( g(V \ln f, X) = X \ln f \). A warped product manifold \( M = N_1 \times_f N_2 \) is said to be trivial if the warping function \( f \) is constant.

There are two types of warped products between proper slant and invariant submanifolds. Now we study warped product semi-slant submanifolds and its characterization of type of \( M = N_T \times_f N_0 \). For first case, we recall the following result which was obtained by Mustafa, et.al (cf. [22]) for warped product semi-slant submanifolds of nearly Trans-Sasakian manifolds:

Theorem 3.1. [22] There do not exist warped product semi-slant submanifolds \( M = N_0 \times_f N_T \) in a nearly Trans-Sasakian manifold \( M \), where \( N_0 \) and \( N_T \) are proper slant and invariant submanifolds of \( \tilde{M} \), respectively.

First, we give the following definition which based on the results S. Hiepk[20].

Definition 3.2. Assume that \( M \) be a semi-slant submanifold of a nearly-Trans Sasakian manifold \( \tilde{M} \), then we say that \( M \) is a locally warped product manifold semi-slant submanifold of \( \tilde{M} \) if \( D \) defines a totally geodesic foliation on \( M \) and \( D^0 \) defines a spherical foliation on \( M \), that is each leaf of \( D^0 \) is totally umbilical with parallel mean curvature vector field in \( M \).

Now we give following results for later use given in (cf. [22]) for warped product semi-slant submanifold of nearly-Trans Sasakian manifolds as:

Lemma 3.1. [22] Assume that \( M = N_T \times_f N_0 \) be a warped product semi-slant submanifold of a nearly Trans-Sasakian manifold \( \tilde{M} \). Then

\[
\xi \ln f = \beta, \tag{3.2}
\]

\[
g(h(X, Y), FZ) = 0, \tag{3.3}
\]

\[
g(h(Z, X), FZ) = - \left( \langle \varphi X \lambda \rangle + \alpha \eta(X) \right) ||Z||^2, \tag{3.4}
\]

\[
g(h(X, Z), FTZ) = \frac{1}{3} \cos^2 \theta \left( X \ln f - \beta \eta(X) \right) ||Z||^2, \tag{3.5}
\]

for any \( Z \in \Gamma(T N_0) \) and \( X, Y \in \Gamma(T N_T) \).
Following relations are the particular case, of above lemma. By interchanging \( X \) by \( \varphi X \) in (3.5), we find

\[
g(h(\varphi X, Z), FTZ) = \frac{1}{3} \cos^2 \theta \varphi (X \ln f)||Z||^2,
\]

(3.6)

Now we prove our main result

**Theorem 3.2.** Let a proper semi-slant submanifold \( M \) of nearly Trans-Sasakian manifold \( \tilde{M} \) such that the normal component \( Q_X \) lies in \( \varphi \)–invariant normal subbundle of \( M \). Then \( M \) is locally a non-trivial warped product submanifold of type \( M = N_T \times_f N_0 \) if and only if the following condition is satisfied

\[
A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3} \left( 2 + \sin^2 \theta \right) (X\lambda)Z + \beta \eta(X) \left( 1 - \frac{1}{3} \cos^2 \theta \right) ||Z||^2,
\]

for any \( \lambda \in \Gamma(TM) \), \( X \in \Gamma(D \oplus \xi) \) and \( Z \in \Gamma(\mathcal{D}^0) \). Moreover, for a differentiable function \( \lambda \) on \( M \) such that \( Z \lambda = 0 \), for any \( Z \in \Gamma(\mathcal{D}^0) \).

**Proof.** Assume that \( M = N_T \times_f N_0 \) be a non-trivial warped product semi-slant submanifold of a nearly Trans-Sasakian manifold \( \tilde{M} \) such that \( N_0 \) and \( N_T \) are proper slant and \( \varphi \)–invariant submanifolds of \( \tilde{M} \), respectively. Thus from (3.3) and (2.6), we get \( g(A_{FZ}X, Y) = 0 \), for any \( X, Y \in \Gamma(TN_T) \) and \( Z \in \Gamma(TN_0) \). Since, \( N_T \) is an invariant submanifold, then rearranging \( X \) by \( \varphi X \), we find that \( g(A_{FZ}\varphi X, Y) = 0 \), which indicates that the components of linear operator \( A_{FZ}\varphi X \) are not lying in \( TN_T \). Similarly, rearranging \( Y \) by \( T \) in (3.3) and from (2.6), it is easily see that \( g(A_{FTZ}X, Y) = 0 \), which again shows that \( A_{FTZ}X \) has no components in \( TN_T \). Hence, this means that \( A_{FZ}\varphi X - A_{FTZ}X \) lies in \( TN_0 \). Therefore, from the Lemmas 3.2-(3.5), we have

\[
g(h(X, Z), FTZ) = \frac{1}{3} \cos^2 \theta (X \ln f)||Z||^2 - \frac{1}{3} \cos^2 \theta \beta \eta(X)||Z||^2.
\]

(3.8)

On the other hand, replacing \( X \) by \( \varphi X \) in the Eqs (3.3) and using (2.1)(i), we derive

\[
g(h(Z, \varphi X), FZ) = (X \ln f)||Z||^2 - \beta \eta(X)\xi ||Z||^2 + \alpha \eta(\varphi X)||Z||^2.
\]

Since, from (3.2) and the fact that \( \eta(\varphi X) = \eta(\varphi X, \xi) = 0 \), we obtain

\[
g(h(Z, \varphi X), FZ) = (X \ln f)||Z||^2 - \beta \eta(X)||Z||^2,
\]

(3.9)

for any \( X \in \Gamma(TN_T) \) and \( Z \in \Gamma(TN_0) \). We get the following relation by follows (3.8), (3.9) and (2.6) as:

\[
g(A_{FZ}\varphi X - A_{FTZ}X, Z) = \left( 1 - \frac{1}{3} \cos^2 \theta \right) (X \ln f)||Z||^2 + \beta \eta(X) \left( 1 - \frac{1}{3} \cos^2 \theta \right) ||Z||^2.
\]

(3.10)

Thus \( A_{FZ}\varphi X - A_{FTZ}X \) lies in \( TN_0 \) and from (3.10), we get the required result (3.7). Hence, the first part is proved completely.

**Conversely** Let \( M \) be a semi-slant submanifold of nearly Trans-Sasakian manifold with the condition (3.7) holds. For any \( X, Y \in \Gamma(D \oplus \xi) \) and \( Z \in \Gamma(\mathcal{D}^0) \), we obtain

\[
g(\nabla_X Y, Z) = g(\varphi \nabla_X Y, \varphi Z) + \eta(\nabla_X Y)\eta(Z) = g(\nabla_X \varphi Y, \varphi Z) - g(\nabla_X \varphi Y, \varphi Z).
\]

From the structure Eqs (2.3) and from the property of Riemannian connection, one obtains

\[
g(\nabla_X Y, Z) = -g(\varphi Y, \nabla_X \varphi Z) - g(P_X Y, T Z) + g(Q_X Y, F Z)
\]

Taking the help of Eqs (2.8) and the property of covariant derivative of \( \varphi \), we find

\[
g(\nabla_X Y, Z) = g(\varphi \nabla_X T Z, Y) - g(\nabla_X F Z, \varphi Y) + g(Y, P_X T Z) - g(Q_X Y, F Z).
\]
Using (2.4), (2.5) and (2.12), we derive
\[ g(V_X Y, Z) = g(V_X T^2 Z, Y) + g(V_X FTZ, Y) - g((V_X \phi)TZ, Y) \]
\[ + g(FZ, h(X, \phi Y)) + g(Y, P_X T Z) - g(Q_X Y, FZ). \]

Following the Theorem 2.1 and (2.5), it is easy to see that
\[ \sin^2 \theta g(V_X Y, Z) = -g(A_{FTZ} X - A_{TZ} \phi X, Y) - g(P_X T Z, Y) - g(Q_X Y, FZ) + g(P_X T Z, Y). \]

Therefore, from the hypothesis of the Theorem 3.2, we know that \( Q_X Y \) lies in \( \mu \) and from the Eqs (3.7), we get
\[ \sin^2 \theta g(V_X Y, Z) = \frac{1}{3} \left( 2 + \sin^2 \theta \right) (X \lambda) g(Z, Y) + \beta \eta(X) \left( \frac{1}{3} \cos^2 \theta - 1 \right) g(Z, Y), \]
which implies that
\[ \sin^2 \theta g(V_X Y, Z) = 0. \quad (3.11) \]

But \( M \) is a proper semi-slant submanifold, i.e., \( \sin^2 \theta \neq 0 \). From (3.11) we know that \( g(V_X Y, Z) = 0 \), this means that \( V_X Y \in \Gamma(\mathcal{D} \oplus \xi) \), for every \( X, Y \in \Gamma(\mathcal{D} \oplus \xi) \). Therefore, \( \mathcal{D} \oplus \xi \) is integrable and its leaves are totally geodesic in \( M \). Moreover, for any \( Z, W \in \Gamma(\mathcal{D}^\perp) \) and \( X \in \Gamma(\mathcal{D} \oplus \xi) \), we have
\[ g([Z, W], X) = g(\phi V_W Z, \phi X) + (\xi(\nabla_W Z) \eta(X) - g(\phi \nabla_Z W, \phi X) - \eta(\nabla_Z W) \eta(X) \]

From the covariant derivative property, we get
\[ g([Z, W], X) = g(\phi \nabla_W Z, \phi X) - g((\nabla_W \phi) Z, \phi X) - g(\phi \nabla_Z \phi W, \phi X) + g((\nabla_W \phi) Z, \phi X) \]
\[ + \eta(\nabla_W Z) \eta(X) - \eta(\nabla_Z W) \eta(X) \]
\[ = g(\phi \nabla_W T Z, \phi X) - (A_{FW} \phi X, W) - g(P_Z W, \phi X) + g(\phi \nabla_Z T W, X) \]
\[ + g(\phi \nabla_F T W, X) + g(P_Z W, \phi X) + g(A_{FW} \phi X, Z) + \eta(\nabla_W Z) \eta(X) \]
\[ - \eta(\nabla_Z W) \eta(X) - g(A_{TZ} \phi X, W). \]

It is implies from the Eqs (2.5), i.e.,
\[ g([Z, W], X) = -g(\phi \nabla_W T Z, X) - g((A_{FW} \phi X, W) - g(P_Z W, \phi X) + g(\phi \nabla_Z T W, X) \]
\[ + g(\phi \nabla_F T W, X) + g(P_Z W, \phi X) + g(A_{FW} \phi X, Z) + \eta(\nabla_W Z) \eta(X) \]
\[ - \eta(\nabla_Z W) \eta(X) - g(A_{TZ} \phi X, W). \]

From Theorem 2.1 and Eq. (2.5) one derives
\[ g([Z, W], X) = g(P_Z T W, X) + \cos^2 \theta g(\nabla_W Z, X) + g(A_{FTZ} X, W) - (P_Z W, \phi X) - g(P_Z T W, X) \]
\[ - \cos^2 \theta g(\nabla_Z W, X) + g(A_{FW} \phi X, Z) + g(P_Z W, \phi X) - g(A_{FTW} X, Z) \]
\[ + \eta(\nabla_Z W) \eta(X) - \eta(\nabla_W Z) \eta(X) - g(A_{TZ} \phi X, W). \quad (3.12) \]

Using the properties of \( P - Q \) from (2.14), we find
\[ g(P_Z T W, X) - g(P_Z T W, X) = g(P_Z (\phi Z - F Z), X) - g(P_Z (\phi W - F W), X) \]
\[ - g(P_Z (\phi Z, X) - g(P_Z F Z, X) - g(P_Z \phi W, X) + g(P_Z F W, X) \]
\[ = - g(\phi P_Z W, X) + g(Q_Z W, X) + g(\phi P_Z W, X) - g(Q_Z W, X) + g(Q_Z W, F Z) \]
\[ - g(P_Z W, \phi X) + g(Q_Z W, F Z) - g(P_Z W, \phi X) - g(Q_Z W, F W). \]
Therefore, using the above relation in the Eqs (3.12), we get

\[ \sin^2 \theta g([Z, W], X) = g(A_{FTZ} X - A_{FZ} \phi X, W) + 2g(P_W Z, \phi X) + g(Q_W Z, FZ) - g(A_{FTW} X - A_{FW} \phi X, Z) - 2g(P_Z W, \phi X) - g(Q_Z X, FW) + \eta(\tilde{V}_W Z) \eta(X) - \eta(\tilde{V}_Z W) \eta(X). \]

Using properties (2.12)-(2.13) and (2.17), we arrive at

\[
\begin{align*}
\sin^2 \theta g([Z, W], X) &= g(A_{FTZ} X - A_{FZ} \phi X, W) + 2g(P_W Z + P_Z W, \phi X) - g(Q_X W, FZ) \\
& - \sin^2 \theta \beta \eta(X) g(Z, W) - g(A_{FTW} X - A_{FW} \phi X, Z) + g(Q_Z Z, FW) \\
& + \sin^2 \theta \beta \eta(X) g(Z, W) + \eta(\tilde{V}_W Z) \eta(X) - \eta(\tilde{V}_Z W) \eta(X).
\end{align*}
\]

By assumption of the Theorem 3.2 that \(Q_X Z\) lies in \(\mu\) and again using (2.12) we derive

\[
\begin{align*}
\sin^2 \theta g([Z, W], X) &= g(A_{FTZ} X - A_{FZ} \phi X, W) + \eta(\tilde{V}_W Z) \eta(X) \\
& - \eta(\tilde{V}_Z W) \eta(X) - g(A_{FTW} X - A_{FW} \phi X, Z).
\end{align*}
\]

Applying Eqs (3.7) in the Eqs (3.13), one obtains

\[
\begin{align*}
\sin^2 \theta g([Z, W], X) &= - \frac{1}{3} \left(2 + \sin^2 \theta \right)(X \lambda) g(Z, W) - \beta \eta(X) \left(\frac{1}{3} \cos^2 \theta - 1\right) g(Z, W) \\
& + \eta(\tilde{V}_W Z) \eta(X) - \eta(\tilde{V}_Z W) \eta(X) + \frac{1}{3} \left(2 + \sin^2 \theta \right)(X \lambda) g(Z, W) \\
& + \beta \eta(X) \left(\frac{1}{3} \cos^2 \theta - 1\right) g(Z, W),
\end{align*}
\]

which means that

\[
\sin^2 \theta g([Z, W], X) = \eta(\tilde{V}_W Z) \eta(X) - \eta(\tilde{V}_Z W) \eta(X).
\]

Now interchanging \(X\) by \(\phi X\) in the above equation and using the fact that \(\eta(\phi X) = 0\), we derive

\[
\sin^2 \theta g([Z, W], \phi X) = 0.
\]

Since, \(M\) is a proper semi-slant submanifold, then from previous Eq, we deduce that the slant distribution \(\mathcal{D}^\beta\) is integrable. Therefore, we can assume that \(N_0\) be a leaf of \(\mathcal{D}^\beta\) and \(h^\beta\) be a the second fundamental form (extrinsic invariant) of \(N_0\) into \(M\). Then from Gauss formula (2.4), we have

\[
g(h^\beta(Z, W), X) = g(\tilde{V}_Z W, X) = g(\tilde{V}_Z W, \phi X) + \eta(\tilde{V}_Z W) \eta(X) \\
= g(\tilde{V}_Z \phi W, \phi X) - g((\tilde{V}_Z \phi) W, \phi X) + \eta(\tilde{V}_Z W) \eta(X).
\]

From (2.8) and tangential components of \((\tilde{V}_Z \phi) W\), it is easily seen that

\[
g(h^\beta(Z, W), X) = g(\tilde{V}_Z T W, \phi X) + g(\tilde{V}_Z F W, \phi X) - g(P_Z W, \phi X) + \eta(\tilde{V}_Z W) \eta(X).
\]

Using the covariant differentiation property of \(\phi\) and (2.5), we obtain

\[
g(h^\beta(Z, W), X) = g((\tilde{V}_Z \phi) T W, X) - g(\tilde{V}_Z F^2 W, X) - g(\tilde{V}_Z F T W, X) \\
- g(A_{FW} Z, \phi X) + g(\phi P_Z W, X) + \eta(\tilde{V}_Z W) \eta(X).
\]

Then using the Theorem 2.1 and (2.5), we derive

\[
g(h^\beta(Z, W), X) = g((P_Z T W, X) + \cos^2 \theta g(\tilde{V}_Z W, X) + g(A_{FTZ}, X) - g(A_{FW} Z, \phi X) - g(P_Z T W, X) \\
- g(P_Z F W, X) + \eta(\tilde{V}_Z W) \eta(X),
\]
Applying Proposition 2.1 in last Eq. then easily get the following

\[ \sin^2 \theta g(h_\theta(Z, W), X) = g(A_{FW}X - A_{FW} \varphi X, Z) + g(QX X, FW) + \eta(\tilde{\theta}_Z W \eta(X)). \]

Using Eq. (2.13) in the second term of the above Eqs. Then from (3.7) and (2.17), we arrive at

\[ \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{1}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) - \beta \eta(X) \frac{1}{3} \cos^2 \theta g(Z, W) + \beta \eta(X) g(Z, W) - g(QX X, FW) - \beta \eta(X) \sin^2 \theta g(Z, W) + \eta(\tilde{\theta}_Z W \eta(X)). \]

As we have assumed that \(QX Z\) lies in \(\mu\), finally we get

\[ \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{1}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) + \frac{2}{3} \cos^2 \theta \beta \eta(X) g(Z, W) - \eta(X) g(\tilde{\theta}_Z \xi, W). \tag{3.14} \]

Interchanging \(Z\) by \(W\) in (3.14), we find

\[ \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{1}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) + \frac{2}{3} \cos^2 \theta \beta \eta(X) g(Z, W) - \eta(X) g(\tilde{\theta}_Z \xi, W). \tag{3.15} \]

From the symmetry of extrinsic invariant \(h_\theta\), then (3.15) and (3.14) implies that

\[ 2 \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{2}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) + \frac{4}{3} \cos^2 \theta \beta \eta(X) g(Z, W) - \eta(X) g(\tilde{\theta}_Z \xi, W). \tag{3.16} \]

Applying the Proposition 2.1 in last Eq. then easily get the following

\[ 2 \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{2}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) + \frac{4}{3} \cos^2 \theta \beta \eta(X) g(Z, W) - 2 \beta \eta(X) g(Z, W), \]

which implies that

\[ 2 \sin^2 \theta g(h_\theta(Z, W), X) = -\frac{2}{3} (2 + \sin^2 \theta) (X \lambda) g(Z, W) + \left(\frac{4}{3} \cos^2 \theta - 2\right) \beta \eta(X) g(Z, W) \tag{3.17} \]

Hence, replacing \(X\) by \(\varphi X\) in the above relation (3.17) and using fact that \(\eta(\varphi X) = g(\varphi X, \xi) = -g(X, \varphi \xi) = 0\), which gives

\[ g(h_\theta(Z, W), \varphi X) = -\frac{1}{3} (2 + \csc^2 \theta + \cot^2 \theta) (\varphi X \lambda) g(Z, W) \]

Finally, from the property of gradient of \(\ln f\), simplification gives

\[ g(h_\theta(Z, W), \varphi X) = -\frac{1}{3} (2 + \csc^2 \theta + \cot^2 \theta) g(Z, W) g(\nabla \lambda, \varphi X) \]

It follows that

\[ h_\theta(Z, W) = -g(Z, W) \frac{1}{3} (2 + \csc^2 \theta + \cot^2 \theta) \nabla \lambda. \tag{3.18} \]
Theorem 3.2.

Proof. 

Directly part follows from (3.6) and (3.4). Moreover, converse part can be easily proved as the for any \( Z \in \text{semi-slant submanifolds in almost contact manifolds with } \xi \)

\[
\text{the table shows that the necessary and sufficient condition for the existence of warped product semi-slant submanifolds in almost contact manifolds with } \xi \text{ tangent to the first factor which are directly generalizing from } (\alpha, \beta)\text{-nearly-Trans Sasakian manifold i.e.,}
\]

\[
A_{FZ}\psi X - A_{FZ}X = \frac{1}{3}(2 - \sin^2 \theta)(\psi X\lambda)Z + \alpha\eta(X)Z.,
\]

for any \( U \in \Gamma(\mathcal{T}M), X \in \Gamma(\mathcal{D} \oplus \xi) \) and \( Z \in \Gamma(\mathcal{D}^0) \). Moreover, for a differentiable function \( \lambda \) on \( M \) such that \( Z\lambda = 0 \), for any \( Z \in \Gamma(\mathcal{D}^0) \).

Proof. Directly part follows from (3.6) and (3.4). Moreover, converse part can be easily proved as the Theorem 3.2.

Case 3.1. If we substitute \( \alpha = 0 \), and, \( \beta = 0 \) in Eqs.(2.3), we immediately get the following result from the Theorem 3.2, i.e.,
A proper semi-slant submanifold $M$ of a nearly cosymplectic manifold $\tilde{M}$ such that the normal components of $(\nabla_X \varphi) U$ lies in $\varphi$–invariant normal subbundle of $M$ for any $X \in \Gamma(D)$ and $U \in \Gamma(TM)$. Then $M$ is locally a non-trivial warped product submanifold of the type $M = N_T \times_1 N_0$ such that $N_0$ is proper slant and $N_T$ is $\varphi$–invariant submanifolds if and only if the following condition is satisfied

$$A_{FTZ}X - A_{FZ}\varphi X = - \frac{1}{3} (2 + \sin^2 \theta)(X \lambda) Z,$$

(3.21)

for any $X \in \Gamma(D \oplus \xi)$ and $Z \in \Gamma(D^\theta)$. Moreover, for a differentiable function $\lambda$ on $M$ such that $Z \lambda = 0$, for any $Z \in \Gamma(D^\theta)$.

Case 3.2. Rearranging $\alpha = 1$ and $\beta = 0$ in Eqs. (2.3), then nearly-Trans Sasakian manifold turn into nearly Sasakian manifold. Thus, we find the following Theorem which is a direct consequence of the Theorem 3.3, that is,

Theorem 3.5. Assume that $M$ be a proper semi-slant submanifold of a nearly Sasakian manifold $\tilde{M}$ such that the normal components of $(\nabla_X \varphi) U$ lies in $\varphi$–invariant normal subbundle of $M$ for any $X \in \Gamma(D)$ and $U \in \Gamma(TM)$. Then $M$ is locally a non-trivial warped product submanifold of the type $M = N_T \times_1 N_0$ such that $N_0$ is proper slant and $N_T$ is $\varphi$–invariant submanifolds if and only if the following condition is satisfied

$$A_{FTZ}X - A_{FZ}\varphi X = \left(\frac{1}{3} \cos^2 \theta + 1\right)(\varphi X \lambda) Z + \eta(X)Z,$$

(3.22)

for any $X \in \Gamma(D \oplus \xi)$ and $Z \in \Gamma(D^\theta)$. Moreover, for a differentiable function $\lambda$ on $M$ such that $Z \lambda = 0$, for any $Z \in \Gamma(D^\theta)$.

Equivalently, we give others necessary and sufficient conditions in the following table for a semi-slant submanifold to be a warped product semi-slant in numerous ambient manifolds, i.e.,

<table>
<thead>
<tr>
<th>Manifolds Name and warped product of the form $M = N_T \times_1 N_0$</th>
<th>Necessary and sufficient conditions with for a differentiable function $\lambda$ on $M$ such that $Z \lambda = 0$.</th>
<th>Cases to substitute in Eqs. (2.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nearly Kenmotsu</td>
<td>$A_{FTZ}X - A_{FZ}\varphi X = \frac{1}{3} (2 + \sin^2 \theta)(X \lambda) Z - \eta(X)(\frac{1}{3} \cos^2 \theta - 1) Z.$</td>
<td>$\alpha = 0, \beta = 1.$</td>
</tr>
<tr>
<td>Nearly $\alpha$–Sasakian</td>
<td>$A_{FTZ}\varphi X - A_{FZ}X = \left(\frac{1}{3} \cos^2 \theta + 1\right)(\varphi X \lambda) Z + \alpha \eta(X)Z.$</td>
<td>$\beta = 0.$</td>
</tr>
<tr>
<td>Nearly $\beta$–Kenmotsu</td>
<td>$A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3} (2 + \sin^2 \theta)(X \lambda) Z - \beta \eta(X)(\frac{1}{3} \sin^2 \theta + 2) Z.$</td>
<td>$\alpha = 0.$</td>
</tr>
</tbody>
</table>

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References


