Equitable List Vertex Colourability and Arboricity of Grids

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Abstract. A graph $G$ is equitably $k$-list arborable if for any $k$-uniform list assignment $L$, there is an equitable $L$-colouring of $G$ whose each colour class induces an acyclic graph. The smallest number $k$ admitting such a colouring is named equitable list vertex arboricity and is denoted by $\rho^*_L(G)$. Zhang in 2016 posed the conjecture that if $k \geq \lceil(\Delta(G) + 1)/2 \rceil$ then $G$ is equitably $k$-list arborable. We give some new tools that are helpful in determining values of $k$ for which a general graph is equitably $k$-list arborable. We use them to prove the Zhang’s conjecture for $d$-dimensional grids where $d \in \{2, 3, 4\}$ and give new bounds on $\rho^*_L(G)$ for general graphs and for $d$-dimensional grids with $d \geq 5$.

1. Introduction

All graphs considered in this paper are simple and undirected. For a graph $G$, we use $V(G)$, $E(G)$, and $\Delta(G)$ to denote vertex set, edge set, and the maximum degree of $G$, respectively. By $	ext{G}[V']$ we mean the subgraph of $G$ induced by a vertex subset $V'$. To simplify the notation we write $G - V'$ instead of $G[V(G) \setminus V']$. Analogously, we write $G - E'$ to denote the graph obtained from $G$ by the deletion of an edge subset $E'$. By $G_1 \cup G_2$ we mean the union of disjoint graphs $G_1$, $G_2$, i.e. the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

The symbol $\mathbb{N}$ stands for the set of positive integers, and moreover $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $a, b \in \mathbb{N}_0$. If $a < b$ then $[a, b]$ denotes the set $\{a, a + 1, \ldots, b - 1, b\}$, if $a = b$ then $[a, b] = \{a\}$, and if $a > b$ then $[a, b] = \emptyset$. We adopt the convention $[1, b] = [b]$, moreover $[b]_{\text{odd}}$ and $[b]_{\text{even}}$ denote the sets of odd integers and even integers in $[b]$, respectively.

A colouring of a graph $G$ is a mapping $c : V(G) \to \mathbb{N}$. A coloured graph is then a pair $(G, c)$, where $G$ is a graph and $c$ is its colouring. A colouring of a graph $G$ is proper if each colour class induces an edgeless graph. A $k$-colouring of a graph $G$ is a mapping $c : V(G) \to [k]$. A graph $G$ is properly $k$-colourable if there is a proper $k$-colouring of $G$. A graph $G$ is $k$-arborable if there is a $k$-colouring of $G$ such that each colour class induces an acyclic graph.

Let $L$ be a list assignment (for a graph $G$), i.e. a mapping that assigns to each vertex $v \in V(G)$ a set $L(v)$ of allowable colours. An $L$-colouring of $G$ is a colouring of $G$ such that for every $v \in V(G)$ the colour on $v$ belongs to $L(v)$. A list assignment $L$ is $k$-uniform if $|L(v)| = k$ for all $v \in V(G)$. A graph $G$ is $k$-choosable if for each $k$-uniform list assignment $L$, we can find a proper $L$-colouring of $G$. A graph $G$ is $k$-list arborable if,
Conjecture 1.3 ([13]). If \( k \in \mathbb{N} \) and \( k \geq \Delta(G) + 1 \) then every graph \( G \) is equitably \( k \)-choosable.

It has to be mentioned herein that equitable \( k \)-colouring is not monotone with respect to \( k \). It means that there are graphs that are equitably \( k \)-colourable and not equitably \( t \)-colourable for some \( t < k \). To the best of our knowledge there are no results of this type on equitable \( k \)-choosability nor equitable \( k \)-list arborability.

On the other hand, Zhang [13] formulated in 2016 the following conjectures.

Conjecture 1.2 ([13]). For every graph \( G \) it holds \( \rho^+(G) \leq \lceil (\Delta(G) + 1)/2 \rceil \).

Conjecture 1.3 ([13]). If \( k \in \mathbb{N} \) and \( k \geq \lceil (\Delta(G) + 1)/2 \rceil \) then every graph \( G \) is equitably \( k \)-list arborable.

Zhang [13] confirmed above two conjectures for complete graphs, 2-degenerate graphs, 3-degenerate claw-free graphs with maximum degree at least 4, and planar graphs with maximum degree at least 8. Our results confirm above conjectures for some Cartesian products of paths, i.e. for some grids.

Given two graphs \( G_1 \) and \( G_2 \), the Cartesian product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \square G_2 \), is defined to be a graph whose vertex set is \( V(G_1) \times V(G_2) \) and edge set consists of all the edges joining vertices \( (x_1, y_1) \) and \( (x_2, y_2) \) when either \( x_1 = x_2 \) and \( y_1, y_2 \in E(G_2) \) or \( y_1 = y_2 \) and \( x_1, x_2 \in E(G_1) \). Note that the Cartesian product is commutative and associative. Hence the graph \( G_1 \square \cdots \square G_d \) is unambiguously defined for any \( d \in \mathbb{N} \). Let \( P_n \) denote a path on \( n \) vertices. Notice that when \( G = G_1 \square \cdots \square G_d \) and each of the factors \( G_i \) of \( G \) is \( P_1 \) then \( G \) is a \( d \)-dimensional hypercube. Similarly, when each of the factors \( G_i \) is a path on at least two vertices then \( G \) is a \( d \)-dimensional grid (cf. Fig. 1). By grids we mean the class of all \( d \)-dimensional grids taken over all \( d \in \mathbb{N} \).

Nakprasit and Nakprasit [10] proved that the problem of equitable vertex arboricity is \( \mathsf{NP} \)-hard. Thus the problem of equitable list vertex arboricity cannot be easier. We are interested in determining polynomially solvable cases. We will use the following known lemmas. By \( N_G(x) \) we denote neighborhood of a vertex \( x \) in \( G \), i.e. the set of adjacent vertices to \( x \).
Lemma 1.4 ([7, 11]). Let $k \in \mathbb{N}$ and $S = \{x_1, \ldots, x_k\}$, where $x_1, \ldots, x_k$ are distinct vertices of $G$. If $G - S$ is equitably $k$-choosable and
\[ |N_G(x_i) \setminus S| \leq i - 1 \]
holds for every $i \in [k]$ then $G$ is equitably $k$-choosable.

Lemma 1.5 ([13]). Let $k \in \mathbb{N}$ and $S = \{x_1, \ldots, x_k\}$, where $x_1, \ldots, x_k$ are distinct vertices of $G$. If $G - S$ is equitably $k$-list arborable and
\[ |N_G(x_i) \setminus S| \leq 2i - 1 \]
holds for every $i \in [k]$ then $G$ is equitably $k$-list arborable.

In this paper we investigate the problem of equitable list vertex arboricity of graphs. The remainder of the paper is organized as follows. In Section 2 we generalize Lemmas 1.4 and 1.5 in such a way that their new versions guarantee the continuity of the equitable choosability and equitable list vertex arboricity of graphs. We give a new tool using the equitable choosability of a subgraph $H$ covering graph $G$ (Lemma 2.7). These tools (Lemmas 2.5, 2.6, and 2.7) lead to new bounds on $\rho^*_l(G)$, for any graph $G$. Since the new tool uses the notation of equitable choosability we dedicate Section 3 to this notation for some graphs related to grids. Finally, we apply all the lemmas to confirm the correctness of Zhang’s conjectures for $d$-dimensional grids, $d \in \{2, 3, 4\}$, and to give new bounds on $\rho^*_l(G)$ for $d$-dimensional grids with $d \geq 5$ (Section 4). We conclude the paper with posing some new conjectures concerning equitable list vertex arboricity of graphs.

2. Some auxiliary tools and general bounds on $\rho^*_l(G)$

In the literature a lot of proofs of results on equitable choosability are done by induction on the number of vertices of a graph and by usage of Lemma 1.4. It means, to show that $G$ is equitably $k$-choosable, the set $S \subseteq V(G)$ that fulfills the inequality (1) is determined and next the induction hypothesis is applied to the graph $G - S$. Repeated application of this approach defines a partition $S_1 \cup \cdots \cup S_{\eta+1}$ of $V(G)$ such that the following both conditions hold:

- $|S_1| \leq k$ and $|S_j| = k$ for $j \in [2, \eta + 1]$;
- for each $j \in [2, \eta + 1]$ there is an ordering of vertices of $S_j$, say $x^j_1, \ldots, x^j_k$, that fulfills the inequality $|N_G(x^j_i) \setminus (S_1 \cup \cdots \cup S_{j-1})| \leq i - 1$ for every $i \in [k]$.
In this section we prove that if \( G \) has such a partition then \( G \) is not only equitably \( k \)-choosable but also is equitably \( t \)-choosable for every \( t \in \mathbb{N} \) satisfying \( t \geq k \). Next, we observe that the similar result for a graph to be equitably \( k \)-list arborable can be formulated.

Let \( k \in \mathbb{N} \). A \( k \)-partition of a graph \( G \) is a partition of the vertex set of \( G \) into \([|V(G)|/k]\) sets. The \( k \)-partition is special if all sets of the \( k \)-partition, except at most one, have \( k \) elements. Let \( G \) be a graph and \( c \) be its vertex colouring (not necessarily proper). A set \( S \subseteq V(G) \) is rainbow in the coloured graph \((G, c)\) if all vertices in \( S \) are coloured differently. A \( k \)-partition of the coloured graph \((G, c)\) is rainbow if every set of the \( k \)-partition is rainbow. It is easy to see the following fact.

**Observation 2.1.** Let \( k \in \mathbb{N} \) and \((G, c)\) be a coloured graph. If there is a rainbow \( k \)-partition of \((G, c)\) then each colour appears on at most \([|V(G)|/k]\) vertices of \( G \).

**Lemma 2.2.** Let \( k \in \mathbb{N} \). A graph \( G \) is equitably \( k \)-choosable if and only if for every \( k \)-uniform list assignment \( L \) there is a proper \( L \)-colouring \( c \) of \( G \) such that \((G, c)\) has a rainbow \( k \)-partition.

**Proof.** Obviously, if for every \( k \)-uniform list assignment \( L \) there is a proper \( L \)-colouring \( c \) of \( G \) such that \((G, c)\) has a rainbow \( k \)-partition then each colour class has the cardinality at most \([|V(G)|/k]\), by Observation 2.1. It means that this \( L \)-colouring is equitable, and hence \( G \) is equitably \( k \)-choosable.

To prove the opposite implication, suppose that \( G \) is equitably \( k \)-choosable and \( L \) is a \( k \)-uniform list assignment for \( G \). It follows that there is a proper \( L \)-colouring \( c \) of \( G \) such that each colour class has at most \([|V(G)|/k]\) elements. Let \(|V(G)| = \eta k + r\), where \( \eta \in \mathbb{N}_0 \) and \( r \in [k] \). Thus \( \eta + 1 = \lceil |V(G)|/k \rceil \) and so each colour class contains at most \( \eta + 1 \) vertices. Assume, on the contrary, that there is no rainbow \( k \)-partition of \((G, c)\). Among all partitions of \((G, c)\) into rainbow sets, let \( V_1 \cup \cdots \cup V_t \) be one with the smallest \( t \). Since there is no rainbow \( k \)-partition, we have \( t > \eta + 1 \). Without loss of generality, we may assume that \( V_1 \cup \cdots \cup V_{t-1} \cup V_{t+1} \cup \cdots \cup V_t \) is the rainbow partition with \(|V_1| \leq \cdots \leq |V_i|\) and with the minimum cardinality of \( V_i \). Let \(|V_i| = s \) and \( x \in V_1 \). Since we have at most \( \eta + 1 \) vertices coloured with \( c(x) \) and \( t > \eta + 1 \), there is a set \( V_i \) such that \( V_i \cup \{x\} \) is rainbow. If \( s = 1 \) then \( V_2 \cup V_3 \cup \cdots \cup V_{t-1} \cup (V_i \cup \{x\}) \cup V_{t+1} \cup \cdots \cup V_t \) is the partition with less number of rainbow sets, a contradiction. If \( s > 1 \) then we get the rainbow partition \( V_1 \setminus \{x\} \cup V_2 \cup \cdots \cup V_{t-1} \cup (V_i \cup \{x\}) \cup V_{t+1} \cup \cdots \cup V_t \) that contradicts with the minimum cardinality of \( V_1 \).

**Lemma 2.3.** Let \( k \in \mathbb{N} \). A graph \( G \) is equitably \( k \)-list arborable if and only if for every \( k \)-uniform list assignment \( L \) there is an \( L \)-colouring \( c \) in which every colour class induces an acyclic graph and such that \((G, c)\) has a rainbow \( k \)-partition.

**Proof.** We repeat all the steps of the proof of Lemma 2.2, but in each case when we refer to the colouring \( c \) of a graph \( G \) we assume or state that each colour class in \( c \) is acyclic instead of the assumption that \( c \) is proper. Additionally, we substitute the notion of equitable \( k \)-choosability by the notion of equitable \( k \)-list arborability.

**Lemma 2.4.** Let \( k \in \mathbb{N} \) and \((G, c)\) be a coloured graph. If there is a rainbow special \( k \)-partition of \((G, c)\) then there is also a rainbow special \( x \)-partition of \((G, c)\) for every integer \( x \) such that \( x \leq k \).

**Proof.** Let \(|V(G)| = \eta k + r_1\), where \( \eta \in \mathbb{N}_0 \) and \( r_1 \in [k] \). Let \( S_1 \cup S_2 \cup \cdots \cup S_{\eta+1} \) be a rainbow special \( k \)-partition of \((G, c)\) such that \(|S_1| = r_1\) and \(|S_i| = k\) for \( i \in \{2, \eta + 1\} \). We show that there is a rainbow special \( x \)-partition, for every \( x \leq k \).

Arrange vertices of \( G \) in the list in such a way that:

- vertices from \( S_i \) are placed before vertices from \( S_j \) for \( i < j \);
- vertices from \( S_1 \) are placed in any order at the top of the list;
- each vertex from \( S_i \) for \( i > 1 \), is placed in the list in such a way that its colour is different from the colours of \( k - 1 \) previous vertices in the list or its colour is different from the colours of all previous vertices in the list, if the number of previous vertices is smaller than \( k - 1 \).
Since sets $S_i$ are rainbow, for every $i$, then the above described arrangement of vertices is possible. Assume that $(r_1, r_2, \ldots, r_{|V(G)|})$ is the list of vertices created in such a way. Let $|V(G)| = \beta x + r_2$, where $\beta \in \mathbb{N}_0$, $r_2 \in [x]$.

Sets $R_i = [v_{i−1} + 1, \ldots, v_i]$, for $1 \leq i \leq \beta$ and $R_{\beta+1} = [v_{\beta+1}, \ldots, v_{|V(G)|}]$ form an $x$-partition. It is easy to see that this partition is rainbow and special. □

**Lemma 2.5.** Let $k \in \mathbb{N}$. If a graph $G$ has a special $k$-partition $S_1 \cup \cdots \cup S_{k+1}$ such that $|S_1| \leq k$ and $|S_j| = k$ for $j \in [2, \eta + 1]$, moreover, if for every $j \in [2, \eta + 1]$ there is an ordering $x_1^j, \ldots, x_r^j$ of vertices of the set $S_j$ that for every $i \in [k]$ the inequality

$$|N_G(x_i^j) \cap (S_1 \cup \cdots \cup S_{j−1})| \leq i − 1$$

is fulfilled then $G$ is equitably $t$-choosable for every integer $t$ satisfying $t \geq k$.

**Proof.** Let $k, t$ be fixed and $L$ be a $t$-uniform list assignment for $G$. We show that there is a proper $L$-colouring $c$ of $G$ such that the coloured graph $(G, c)$ has a rainbow special $t$-partition. Since $L$ is chosen freely, it will follow that $G$ is equitably $t$-choosable, by Lemma 2.2. Let

- $|V(G)| = \eta k + r_1$, where $\eta, r_1$ are non-negative integers, $r_1 \in [k]$, and
- $|V(G)| = \beta t + r_2$, where $\beta, r_2$ are non-negative integers, $r_2 \in [t]$, and
- $\gamma = \gamma k + r$ where $\gamma, r$ are non-negative integers, $r \in [k]$.

Thus $|V(G)| = \beta \gamma k + \beta r + r_1$. We split $V(G)$ into two subsets $V_1$ and $V_2$, where $V_1 = S_1 \cup \cdots \cup S_{\eta+1−\gamma}$ and $V_2 = S_{\eta+1−(\gamma−1)} \cup \cdots \cup S_{\eta+1}$. Observe that $|V_1| = \beta t + r_2$ and $|V_2| = \beta \gamma k$. First, we properly colour the vertices in $V_1$, next we spread the colouring on $V_2$. We colour vertices in each set $S_i$ of $V_1$ in such a way that we obtain a rainbow set. It is easy to see that we can colour vertices from $S_1$ such that we obtain a rainbow set, since each vertex has assigned a list of length $t$ and $|S_1| = r_1 \leq t$. Next, we colour vertices $x_1^2, \ldots, x_r^2$ in $S_2$. We assign to $x_1^2$ a colour from its list that is not used in $S_1$. Since $|N_G(x_1^2) \cap S_1| \leq k − 1$ and $|L(x_1^2)| = t \geq k$, this may be done. Next we assign to $x_2^2, \ldots, x_r^2$ (in the sequence) a colour from its list that is different from the ones assigned to the vertices with higher subscript and not used in $S_1$. All these steps may be completed since $|N(x_i^2) \cap S_1| \leq i − 1$ and $|L(x_i)| = t \geq k$. Similarly, we colour the vertices of each set $S_j$ ($j \in [\eta + 1 − \beta \gamma]$). Consider the coloured subgraph $(G_1, c)$, where $G_1 = G[V_1]$. Since each set $S_j$ ($j \in [\eta + 1 − \beta \gamma]$) is rainbow, we obtain a rainbow $k$-partition of $(G_1, c)$. If $\eta r_2 \leq r_1$, we take $r_2$ vertices of $S_1$ and denote this set by $R$. Otherwise, we additionally choose $r_2 − r_1$ vertices from $S_2$ that have colours different than colours of vertices in $S_1$ and then these vertices together with $S_1$ form $R$. Observe that also $(G_1 − R, c)$ has a rainbow $k$-partition. Furthermore, $|S_1 \cup \cdots \cup S_{\eta+1−\gamma} \setminus R| = |V(G_1 − R)| = \beta \gamma r$. By Lemma 2.4, $G_1 − R$ has a rainbow $t$-partition. Let $T_1, \ldots, T_\beta$ be a rainbow $t$-partition of $(G_1 − R, c)$.

Now we colour the vertices in $V_2$. Recall that $|V_2| = \beta \gamma k$. Let us divide $V_2$ into $\beta$ subsets, each containing $\gamma k$-sets $S_{\eta+1−(\beta−i)\gamma}$ in the following way:

- $H_1 = S_{\eta+1−(\beta−1)\gamma} \cup S_{\eta+1−(\beta−2)\gamma} \cup \ldots \cup S_{\eta+1−(\beta−1)\gamma}$
- $H_2 = S_{\eta+1−(\beta−1)\gamma} \cup S_{\eta+1−(\beta−2)\gamma} \cup \ldots \cup S_{\eta+1−(\beta−1)\gamma}$
- $H_3 = S_{\eta+1−(\beta−1)\gamma} \cup S_{\eta+1−(\beta−2)\gamma} \cup \ldots \cup S_{\eta+1−(\beta−1)\gamma}$
- $\cdots$
- $H_\beta = S_{\eta+1−(\beta−1)\gamma} \cup S_{\eta+1−(\beta−2)\gamma} \cup \ldots \cup S_{\eta+1−(\beta−1)\gamma}$

We will properly colour vertices in $H_1, \ldots, H_\beta$ from their lists, step by step, in such a way that each set $T_i \cup H_i$ for $i \in [\beta]$ is rainbow.

First, consider a colouring of vertices of $H_i$. To simplify the notation let $A = \alpha + 1 − ((\beta − i + 1)\gamma − 1)$. Thus $H_i = S_A \cup S_{A+1} \cup \ldots \cup S_{A+1−\gamma−1}$. Recall that vertices $x_1^A, \ldots, x_r^A$ in $S_A$ fulfill the inequality (3). We delete colours that are used on vertices in $T_i$ from lists of vertices in $S_A$. Now the lists of vertices in $S_A$ are shorter than $t$, however each vertex still has at least $\gamma k$ colours on the list. Assign to $x_i^A$ a colour from its list that
is not used on vertices from $S_1 \cup \cdots \cup S_{A-1}$. Since $|N_G(x_i^k) \cap (S_1 \cup \cdots \cup S_{A-1})| \leq k - 1$ and $|L(x_i)| = \gamma k \geq k$, this may be done. Then assign to $x_i^{k-1}, \ldots, x_i^1$ (in a sequence) a colour from its list that is different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_{A-1}$. All these steps may be done since $|N_G(x_i^k) \cap (S_1 \cup \cdots \cup S_{A-1})| \leq i - 1$ and $|L(x_i^1)| = \gamma k \geq k$. Now, we colour vertices in $S_{A+1}$, where $S_{A+1} = \{x_1^{A+1}, \ldots, x_{\gamma k}^{A+1}\}$. We delete colours that are used on vertices in $T_i$ and $S_A$ from lists of vertices in $S_{A+1}$. Observe that after deleting colours from lists, each vertex in $S_{A+1}$ has at least $(\gamma - 1)k$ colours on the list. Similarly as above, first we colour the vertex $x_i^{A+1}$ with a colour from its list that is not used in $S_1 \cup \cdots \cup S_A$ and then we colour, one by one, vertices $x_i^{A+1}, \ldots, x_i^{1+1}$ with colours from their lists that are different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_A$. We can do this since $|N_G(x_i^1) \cap (S_1 \cup \cdots \cup S_A)| \leq i - 1$ and $|L(x_i^1)| = (\gamma - 1)k \geq k$. Observe that in the same way we can colour vertices from sets $S_{A+2}, \ldots, S_{A+\gamma-1}$. Indeed, let $S_{A+i} = \{x_i^{1+i}, \ldots, x_i^{\gamma k+i}\}$. We delete from lists of vertices in $S_{A+i}$ colours that are used on vertices in $T_i, S_A, \ldots, S_{A+\gamma-1}$ and then we assign the colour different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_{A+i}$.

Thus finally, we have obtained a proper colouring $c$ that admits a rainbow $t$-partition of $(G, c)$ which completes the proof. \qed

To stress the difference between Lemma 1.4 and Lemma 2.5 let us consider the equitable choice number of a special family of graphs, which we denote by $K$. Assume $K_3 \in K$ and if $G \in K$, then the graph obtained from $G$ by adding three new vertices, say $x, y, z$, and six edges such that $x, y, z$ are pairwise adjacent and additionally the vertex $x$ is adjacent to any two vertices of $G$ and the vertex $y$ is adjacent to any one vertex of $G$ is also in $K$. The definition of $K$ very naturally indicates the special 3-partition satisfying the assumptions of Lemma 2.5 of every graph $G \in K$. Thus Lemma 2.5 implies that if $G \in K$, then $G$ is equitably $t$-choosable for every integer $t$ satisfying $t \geq 3$. However, such a result cannot be easily proved by Lemma 1.1. Obviously, Lemma 1.1 implies that $G$ is equitably 3-choosable, but to prove that $G$ is equitably 4-choosable for every integer $t \geq 4$ we have to involve more arguments.

The next result generalizes Lemma 1.5. We give only a sketch of its proof because it imitates the proof of Lemma 2.5.

**Lemma 2.6.** Let $k \in \mathbb{N}$. If a graph $G$ has a special $k$-partition $S_1 \cup \cdots \cup S_{\eta+1}$ such that $|S_1| \leq k$ and $|S_i| = k$ for $j \in [2, \eta + 1]$, moreover, if for every $j \in [2, \eta + 1]$ there is an ordering $x_i^1, \ldots, x_i^j$ of vertices of the set $S_j$ that for every $i \in [k]$ the inequality

$$|N_G(x_i^j) \cap (S_1 \cup \cdots \cup S_{j-1})| \leq 2i - 1,$$

is fulfilled then $G$ is equitably $t$-list arborable for any integer $t$ satisfying $t \geq k$.

**Proof.** For fixed $k, t$ and a $t$-uniform list assignment $L$ for $G$, we construct an $L$-colouring $c$ of $G$ such that the coloured graph $(G, c)$ has a rainbow special $t$-partition and each colour class in $c$ induces an acyclic graph. We do it in the same manner as in the proof of Lemma 2.5, but if we put a colour on the vertex $x_i^j$, $i \in [k]$, $j \in [2, \eta + 1]$ then we use Lemma 1.5 (instead of Lemma 1.4) to guarantee that each colour class in $c$ induces an acyclic graph (instead of to guarantee that the constructed colouring is proper). \qed

Next, we give new tool that help us in proving further results concerning exact values as well as bounds on equitable list vertex arboricity of graphs.

A spanning graph $H$ of a graph $G$ is any subgraph of $G$ such that $V(H) = V(G)$. We say that a graph $H$ covers all cycles of $G$ if it is spanning and for any cycle $C$ contained in $G$ there are $x, y \in V(C)$ such that $xy \in E(H)$.

**Lemma 2.7.** Let $k \in \mathbb{N}$. If $H$ is a graph that covers all cycles of $G$ and $H$ is equitably $k$-choosable then $G$ is equitably $k$-list arborable.
Theorem 2.8 ([6]). Let \( r \in \mathbb{N} \) and \( G \) be a graph such that \( \Delta(G) \leq r \).

(i) If \( r \leq 7 \) and \( k \geq r + 1 \) then \( G \) is equitably \( k \)-choosable.

(ii) If \( k \geq r + \left\lfloor \frac{1 + \frac{1}{r}}{2} \right\rfloor \) if \( r \leq 30 \) then \( G \) is equitably \( k \)-list arborable.

(iii) If \( |V(G)| \geq r^3 \) and \( k \geq r + 2 \) then \( G \) is equitably \( k \)-choosable.

(iv) If \( \omega(G) \leq r \) and \( |V(G)| \geq 3(3r+1)^8 \) then \( G \) is equitably \((r+1)\)-choosable (\( \omega(G) \) is the clique number of \( G \)).

Theorem 2.8 and Lemma 2.7 imply the general upper bound on equitable list vertex arboricity.

Theorem 2.9. Let \( r \in \mathbb{N} \) and \( G \) be a graph with at least one edge and \( \Delta(G) - 1 \leq r \).

(i) If \( r \leq 7 \) and \( k \geq r + 1 \) then \( G \) is equitably \( k \)-list arborable.

(ii) If \( k \geq r + \left\lfloor \frac{1 + \frac{1}{r}}{2} \right\rfloor \) if \( r \leq 30 \) then \( G \) is equitably \( k \)-list arborable.

(iii) If \( |V(G)| \geq r^3 \) and \( k \geq r + 2 \) then \( G \) is equitably \( k \)-list arborable.

(iv) If \( \omega(G) \leq r \) and \( |V(G)| \geq 3(3r+1)^8 \) then \( G \) is equitably \((r+1)\)-list arborable.

Proof. Let \( F \) be a spanning forest of \( G \) such that the numbers of connected components of \( F \) and \( G \) are the same. Thus \( G - F \) covers all cycles of \( G \). By Lemma 2.7, if \( G - F \) is equitably \( k \)-choosable then \( G \) is equitably \( k \)-list arborable. Since \( \Delta(G - F) \\leq \Delta(G) - 1 \), the theorem follows directly from Theorem 2.8.

If we restrict our consideration to particular graph classes or to graphs with particular properties, we get even better bounds on equitable list arboricity that, in addition, confirm Zhang’s conjecture.

Theorem 2.10 ([7]). Let \( k \in \mathbb{N} \) and let \( F \) be a forest. If \( k \geq \Delta(F)/2 + 1 \) then \( F \) is equitably \( k \)-choosable.

We can apply Theorem 2.10 to show an upper bound on equitable list vertex arboricity of graphs with (edge) arboricity equal to 2. The (edge) arboricity of a graph \( G \) is the minimum number of forests into which its edges can be partitioned.

Theorem 2.11. Let \( k \in \mathbb{N} \) and let \( G \) be a graph with arboricity 2. If \( k \geq \lceil (\Delta(G) + 1)/2 \rceil \) then \( G \) is equitably \( k \)-list arborable.

Proof. Let \( F_1 = (V(G), E_1) \) and \( F_2 = (V(G), E_2) \) be two forests into which \( E(G) \) was partitioned. Of course, \( E(G) = E_1 \cup E_2 \). It is clear that \( F_1 \) covers all cycles of \( G \). If \( \Delta(F_1) < \Delta(G) \) then by Theorem 2.10 and Lemma 2.7 \( G \) is equitably \( k \)-list arborable for \( k \geq \Delta(F_1)/2 + 1 \). It means that \( G \) is equitably \( k \)-list arborable for \( k \geq \lceil (\Delta(G) + 1)/2 \rceil \). Suppose that \( \Delta(F_1) = \Delta(G) \). Let \( D \) be the set of vertices of maximum degree in \( F_1 \). Observe that every vertex in \( D \) is adjacent only with edges from \( E_1 \). Let \( E'_1 \subseteq E_1 \) be the minimal set of edges such that \( D \subseteq \bigcup_{e \in E'_1} e \). Since \( E'_1 \) is minimal, the subgraph induced by \( E'_1 \) is a star-forest. Furthermore, in the subgraph induced by \( E_2 \cup E'_1 \) every edge in \( E'_1 \) is a pendant edge. Thus the subgraph induced by \( E_2 \cup E'_1 \) is acyclic and so \( F_1 - E'_1 \) covers all cycles of \( G \). Since \( \Delta(F_1 - E'_1) < \Delta(G) \), by Theorem 2.10 and Lemma 2.7, \( G \) is equitably \( k \)-list arborable for \( k \geq (\Delta(F_1 - E'_1))/2 + 1 \). It means that \( G \) is equitably \( k \)-list arborable for \( k \geq (\Delta(G) + 1)/2 \).

A graph \( G \) is \( d \)-degenerate if every subgraph of \( G \) has a vertex of degree at most \( d \). Since every 2-degenerate graph has arboricity 2, Theorem 2.11 confirms the result for 2-degenerate graphs obtained by Zhang [13].

Corollary 2.12 ([13]). Let \( k \in \mathbb{N} \) and let \( G \) be a 2-degenerate graph. If \( k \geq \lceil (\Delta(G) + 1)/2 \rceil \) then \( G \) is equitably \( k \)-list arborable.
3. Equitable choosability of grids

Since our new tool (Lemma 2.7) uses the notion of equitable choosability we dedicate this section to this notion for some graphs related to grids. Nethertheless, before we consider it, we give some sufficient conditions for graphs to be equitably 2-choosable.

**Lemma 3.1.** If $G$ has a matching of size $|V(G)|/2$ and $G$ is 2-choosable then $G$ is equitably 2-choosable.

**Proof.** Observe that the assumption that $G$ has a matching of size $|V(G)|/2$ implies that $\alpha(G) \leq |V(G)|/2$ ($\alpha(G)$ denotes the cardinality of the largest independent vertex set of $G$). Thus each colour class has at most $|V(G)|/2$ vertices in any proper colouring of $G$. Let $L$ be a 2-uniform list assignment for $G$. Since $G$ is 2-choosable, there is a proper $L$-colouring $c$ of $G$. Furthermore, every colour class in $c$ has at most $|V(G)|/2$ vertices, and so $c$ is equitable proper $L$-colouring of $G$. □

The graphs that are 2-choosable were characterized by Erdős, Rubin and Taylor in [4]. The core of $G$ is a graph obtained from $G$ by recursive removing all vertices of degree one. Thus the core of $G$ has no vertices of degree one. A graph is called a $\Theta_{2,2,r}$-graph if it consists of two vertices $x$ and $y$ and three internally disjoint paths of lengths 2, 2 and $p$, joining $x$ and $y$.

**Theorem 3.2 ([4]).** A connected graph $G$ is 2-choosable if and only if the core of $G$ is either $K_1$, or an even cycle, or a $\Theta_{2,2,r}$-graph, where $r \in \mathbb{N}$.

**Lemma 3.3.** Let $k \in \mathbb{N}$ with $k \geq 2$. If $G$ is a bipartite graph with $\Delta(G) \leq 2$ then $G$ is equitably $k$-choosable.

**Proof.** Observe first that each component of $G$ is either an even cycle or a path. If $G$ has more than one component that is a path, let $G'$ be a graph obtained from $G$ by adding edges so that $G'$ has one component that is a path and all other components are even cycles. In the case when $G$ has at most one component that is a path, we assume $G' = G$. We will show that $G'$ is equitably $k$-choosable for any $k \geq 2$. By Theorem 3.2, being applied to each connected component of $G'$, $G'$ is 2-choosable (it is clear that if each component is 2-choosable then the whole graph is also 2-choosable). Since $G'$ has a matching of size $|V(G')|/2$ then $G'$ is equitably 2-choosable by Lemma 3.1. Furthermore, Theorem 2.8(i) follows that $G'$ is equitably $k$-choosable for every $k \geq 3$ (since $\Delta(G') \leq 2$). Hence the arguments that $G'$ is equitably $k$-choosable for any $k \geq 2$ and that $G$ is a spanning subgraph of $G'$ imply that $G$ is equitably $k$-choosable for any $k \geq 2$. □

Now, we define $\mathcal{G}_1$ to be a family of all grids $P_n \Box P_2$ and all graphs resulting from grids $P_n \Box P_2$ by removing one vertex of minimum degree, taken over all $n_1 \in \mathbb{N}$. The following results will be used in the next section to determine equitable list vertex arboricity of grids.

**Lemma 3.4.** Let $k \in \mathbb{N}$ with $k \geq 3$. If every component of a graph $G$ is in $\mathcal{G}_1$ then $G$ is equitably $k$-choosable.

**Proof.** We show that there is a special 3-partition of $G$ that fulfills the assumptions of Lemma 2.5, i.e. there are disjoint sets $S_1, \ldots, S_{n+1}$ such that the following conditions hold:

- $V(G) = S_1 \cup \cdots \cup S_{n+1}$;
- $|S_j| \leq 3$ and $|S_j| = 3$ for $j \in [2, \eta + 1]$;
- there is an ordering of vertices of each set $S_j$, say $x^j_1, x^j_2, x^j_3$, fulfilling the inequality $|N_G(x^j_i) \cap (S_1 \cup \cdots \cup S_{j-1})| \leq i - 1$ for $i \in [3]$;

and hence, by Lemma 2.5, $G$ is equitably $k$-choosable for any $k \geq 3$. We prove the existence of the partition by induction on the number of vertices of $G$. It is easy to see that it is true for a graph with at most 3 vertices. Thus suppose that if every component of a graph is in $\mathcal{G}_1$ and the graph has less than $n$ vertices, $n \geq 4$, then it has a special 3-partition that fulfills the assumptions of Lemma 2.5. Let $G$ be an $n$-vertex graph having
Lemma 2.5. Suppose that the assertion is true for graphs with less than \( n \) vertices that satisfies assumptions of the lemma. We show that there is a set \( \mathcal{G}_1 \) of degree 2. Let \( G \) be the neighbor of \( x \), and we see that the vertices of \( S \) satisfy \( |N_G(x_i) \setminus S| \leq i - 1 \) for \( i \in [3] \) and each component of \( G - S \) is in \( \mathcal{G}_1 \). Thus, by induction, the lemma follows.

Let \( x_1 \) be a vertex of the minimum degree in \( G \), thus \( \deg_G(x_1) \leq 2 \). Suppose first that \( \deg_G(x_1) = 2 \). In this case each component has at least four vertices. Let \( x_2, x_3 \) be the neighbors of \( x_1 \) such that \( \deg_G(x_2) = 2 \) and \( \deg_G(x_3) \leq 3 \). Let \( S = \{x_1, x_2, x_3\} \), then \( |N_G(x_i) \setminus S| = 0 \) for \( i \in [3] \). Thus, by induction hypothesis, \( G \) has a special 3-partition that fulfills the assumptions of Lemma 2.5, and so we are done.

Suppose now that \( \deg_G(x_1) = 1 \). Let \( x_2 \) be the neighbor of \( x_1 \). If \( \deg_G(x_2) = 3 \) then let \( x_3 \) be the neighbor of \( x_2 \) of degree 2. Let \( S = \{x_1, x_2, x_3\} \). Hence every component of \( G - S \) is in \( \mathcal{G}_1 \), and so we are done.

Let \( S = \{x_1, x_2, x_3\} \), then \( |N_G(x_i) \setminus S| \leq i - 1 \) for \( i \in [3] \). If \( \deg_G(x_2) > 2 \) then let \( x_3 \) be the neighbor of \( x_2 \), other than \( x_1 \). Observe that in this case the vertices \( x_1, x_2, x_3 \) form a component of \( G \). Again \( S = \{x_1, x_2, x_3\} \) satisfies \( |N_G(x_i) \setminus S| = 0 \) for \( i \in [3] \), and so, by induction hypothesis, \( G \) has a special 3-partition that fulfills the assumptions of Lemma 2.5. If \( \deg_G(x_2) = 1 \) then as \( x_3 \) in \( S \) we put a vertex of the minimum degree in \( G - \{x_1, x_2\} \).

Finally suppose that \( \deg_G(x_1) = 0 \). In this case let \( x_2, x_3 \) be two adjacent vertices of degree at most two. If there are no such vertices then \( G \) is an edgeless graph and we can choose \( x_2, x_3 \) arbitrarily. Similarly as above we can see that every component of \( G - S \) is in \( \mathcal{G}_1 \), and so \( S \) satisfies \( |N_G(x_i) \setminus S| \leq i - 1 \) for \( i \in [3] \). It implies that \( G \) has a special 3-partition that fulfills the assumptions of Lemma 2.5, and so \( G \) is equitably \( k \)-choosable for \( k \geq 3 \).

It should be mentioned here that for each component of graph \( G \) in \( \mathcal{G}_1 \), we have \( \Delta(G) \leq 3 \). Thus, by Theorem 2.8, such a graph is equitably \( k \)-choosable for \( k \geq 4 \). Hence Lemma 3.4 extends this result to \( k \geq 3 \).

Let \( n_1, n_2 \in \mathbb{N} \), \( n_2 \geq 2 \), and \( \ell \in [0, n_1 - 1] \). The symbol \( (P_{n_1} \boxplus P_{n_2}, \ell) \) denotes a graph obtained from \( P_{n_1} \boxplus P_{n_2} \) by the deletion of a set \( V' \) (cf. Fig. 2), where

\[
V' = \{(n_1 - p, n_2) : p \in [0, \ell - 1] \} \cup \{(n_1 - p, n_2 - 1) : p \in [0, \ell - 1] \}.
\]

Observe that \( (P_{n_1} \boxplus P_{n_2}, 0) \) is a grid \( P_{n_1} \boxplus P_{n_2} \).

Let \( \mathcal{G}_2 = \{(P_{n_1} \boxplus P_{n_2}, \ell) : n_1 \geq 1, n_2 \geq 2, \ell \in [0, n_1 - 1] \} \).

Lemma 3.5. Let \( n \in \mathbb{N} \) with \( n \geq 4 \). If each component of a graph \( G \) is in \( \mathcal{G}_2 \) then \( G \) is equitably \( k \)-choosable.

Proof. We show that there is a special 4-partition of \( G \) that fulfills the assumptions of Lemma 2.5. We prove it by induction on the number of vertices. Observe that every graph in \( \mathcal{G}_2 \) has at least two vertices and it is easy to see that if \( G \) has at most 4 vertices then \( G \) has a special 4-partition that fulfills the assumptions of Lemma 2.5. Suppose that the assertion is true for graphs with less than \( n \) vertices, \( n \geq 5 \). Let \( G \) be a graph with \( n \) vertices that satisfies assumptions of the lemma. We show that there is a set \( S_i \), say \( \{x_1, x_2, x_3, x_4\} \), such that \( |N_G(x_i) \setminus S| \leq i - 1 \) for \( i \in [4] \) and each component of \( G - S \) is in \( \mathcal{G}_2 \).

We choose the set \( S \) as follows. First suppose that there is a component \( (P_{n_1} \boxplus P_{n_2}, \ell) \) of \( G \) such that \( n_1 - \ell \geq 2 \) and \( n_2 \geq 2 \). Let us consider the set \( S = \{x_1, x_2, x_3, x_4\} \) with \( x_1 = (n_1 - \ell, n_2), x_2 = (n_1 - 1, n_2), x_3 = (n_1 - \ell, n_2 - 1) \), and \( x_4 = (n_1 - \ell - 1, n_2 - 1) \).

Figure 2: A graph a) \((P_3 \boxplus P_3, 3)\) and b) \((P_3 \boxplus P_5, 2)\) being isomorphic to \( P_3 \boxplus P_2 \).
Thus $|N_G((n_1 - \ell, n_2)) \setminus S| = 0$, $|N_G((n_1 - \ell - 1, n_2)) \setminus S| \leq 1$, $|N_G((n_1 - \ell, n_2 - 1)) \setminus S| \leq 1$ and $|N_G((n_1 - \ell - 1, n_2 - 2)) \setminus S| \leq 2$. Furthermore, every component of $G - S$ is in $G_2$ and hence, by the induction hypothesis, $G$ has 4-partition of $G$ that fulfills the assumptions of Lemma 2.5. If there is a component ($P_n \odot P_{n_2}, \ell$) of $G$ such that $n_1 - \ell = 1$ and $n_2 = 2$, then we put $x_1 = (1, n_2)$, $x_2 = (1, n_2 - 1)$, $x_3 = (1, n_2 - 2)$, and $x_4 = (1, n_2 - 3)$. Every component of $G - S$ is in $G_2$ and $|N_G((1, n_2)) \setminus S| = 0$, $|N_G((1, n_2 - 1)) \setminus S| = 0$, $|N_G((1, n_2 - 2)) \setminus S| \leq 1$ and $|N_G((1, n_2 - 3)) \setminus S| \leq 2$, so by the induction hypothesis, the assumptions of Lemma 2.5 are satisfied. Otherwise, every component of $G$ is a path. If there is a component with at least four vertices then four consecutive vertices of the path form the set $S$ that satisfies $|N_G(x_i) \setminus S| \leq i - 1$ for $i \in [4]$. If each component of $G$ has less than four vertices then, to obtain $S$, we take all vertices of one component and we next complete the set $S$ by vertices of some other component or even components, if the number of vertices chosen to set $S$ is still to small. It is easy to see that also in such a case the assumptions of Lemma 2.5 are fulfilled, which finishes the proof.

**Lemma 3.6.** Let $n_1, n_2, t \in \mathbb{N}$. If $G$ is a graph with $t$ components such that each one is isomorphic to $P_{n_1} \odot P_{n_2}$, then $G$ is equitably 3-choosable.

![Figure 3: Illustration for the proof of Lemma 3.6; $G' = P_2 \odot P_3$.](image)

**Proof.** If $n_1 \leq 2$ or $n_2 \leq 2$ then the proof follows from Lemma 3.4. Thus we may assume that $n_1 \geq 3$ and $n_2 \geq 3$. Let $G' = P_{n_1} \odot P_{n_2}$ for $p \in [t]$ be components of $G$ and $(t, j)^p : i \in [n_1], j \in [n_2]$ be the vertex set of the component $G^p$. Let $n_1 = 3q + r$ where $r \in [0, 2]$ and let $L$ be a 3-uniform list assignment for the graph $G$. We show that there is a proper $L$-colouring $c$ such that $(G, c)$ has a rainbow 3-partition. Let $H$ be a subgraph of $G$ induced by the set $\{(1, j)^p : j \in [n_2], p \in [t]\}$ if $r = 1$, and induced by the set $\{(1, j)^p, (2, j)^p : j \in [n_2], p \in [t]\}$ if $r = 2$. Moreover, let $S_{ij}^p = ((3i + 1 + r, j)^p, (3i + 2 + r, j)^p, (3i + 3 + r, j)^p)$ if $i \in [0, q - 1]$, $j \in [n_2], p \in [t]$ (cf. Fig. 3).

First, we colour the vertices of $H$. Let $c'$ be an equitable proper $L$-colouring of $H$ guaranteed by Lemma 3.4. Thus, by Lemma 2.2, there is a rainbow 3-partition of $(H, c')$. After this step all vertices of the first and the second column are coloured if $r = 2$, all vertices of the first column are coloured if $r = 1$, and graph is uncoloured if $r = 0$. Next, in each component, we colour uncoloured vertices of the first row, i.e., $(r + 1, 1)^p, (r + 2, 1)^p, \ldots, (n_1, 1)^p$ for $p \in [t]$. We properly colour these vertices in such a way that the sets $S_{ij}^p$, $i \in [0, q - 1]$ are rainbow. Now we divide the uncoloured vertices of each component into 3-element subsets $S_{ij}^p$ where $i \in [0, q - 1], j \in [2, n_2]$, and $p \in [t]$. In each component we define linear ordering $<^p$ on these sets in the following way: $S_{ij}^p < S_{ij}^p$ if $(j < s)$ or $(j = s$ and $i < r)$. According to this ordering, we properly colour vertices of each set $S_{ij}^p$ with the following rules:

- if it is only possible, we colour vertices in $S_{ij}^p$ in such a way that vertices of this set obtain different colours;
- if we cannot colour vertices in $S_{ij}^p$ in such a way that $S_{ij}^p$ is rainbow then we color vertices in $S_{ij}^p$ in such a way that two vertices have the same colour, let us say $c_1$, and there is no vertex coloured with $c_1$ in
We will show that such a colouring exists. Let \( c'' \) be a proper \( L \)-colouring of \( G - H \) such that these rules are maintained. Suppose that we are at the step when we have just coloured vertices in \( S_{ij} \), i.e. two vertices in \( S_{ij-1}^p \) are coloured with the same colour, let us say \( c_2 \), then there is no vertex coloured with \( c_2 \) in \( S_{ij}^p \).

Above described rules imply that either \( S_{ij}^p \) is rainbow or \( S_{ij}^p \cup S_{ij-1}^p \) can be divided into two 3-element rainbow sets in \((G - H, c'')\): \( S_{ij}^{pr} \cup S_{ij-1}^{pr} \) (cf. Fig. 4). We use this property to show that there is a rainbow special 3-partition of \((G - H, c'')\). We divide \( V(G - H) \) in the following way:

- the set of vertices of each component is divided step by step;
- in each component \( G_i' \), we start with the last set, with respect to \( <^p \), and go down due to this ordering;
- if \( S_{ij}^p \) is rainbow then it forms a set of the rainbow special 3-partition of \((G - H, c'')\); otherwise, we partite \( S_{ij}^p \cup S_{ij-1}^p \) into two 3-element rainbow sets \( S_{ij}^{pr} \cup S_{ij-1}^{pr} \) (cf. Fig. 4); we modify \( <^p \) by removing sets that are already included in the rainbow 3-partition.

Recall that the sets \( S_{ij}^p \) for \( i \in [0, q - 1] \) (sets of the first row) are rainbow, so the above partition results in a rainbow special 3-partition of \((G - H, c'')\). Thus together with the rainbow 3-partition of \((H, c')\) we obtain the rainbow 3-partition of \((G, c' \cup c'')\). Hence for every 3-uniform list assignment \( L \) there is a proper \( L \)-colouring \( c \) such that \((G, c)\) has a rainbow 3-partition and next, by Lemma 2.2, \( G \) is equitably 3-choosable.

Lemma 3.5 and Lemma 3.6 immediately imply the following result.

**Lemma 3.7.** Let \( n_1, n_2, k \in \mathbb{N} \) with \( k \geq 3 \). If each component of a graph \( G \) is isomorphic to \( P_{n_1} \square P_{n_2} \) then \( G \) is equitably \( k \)-choosable.
If each component of graph $G$ is in $P_n \Box P_n$, then $\Delta(G) \leq 4$. Thus, by Theorem 2.8, such a graph is equitably $k$-choosable for $k \geq 5$. Hence Lemma 3.7 extends this result to $k \geq 3$.

**Remark 3.8.** Observe that Lemma 3.6 and Lemma 3.7 are still true if each component of $G$ is an arbitrary 2-dimensional grid (components are not necessarily of the same sizes). Furthermore, the bound in Lemma 3.7 is tight, since $P_2 \Box P_3$ is not 2-choosable.

**Lemma 3.9.** Let $n_1, n_2 \in \mathbb{N}$ and $t, s \in \mathbb{N}_0$. If $G$ is a graph with $t$ components such that each one is isomorphic to $P_{n_1} \Box P_{n_2} \Box P_2$ and with $s$ components being isomorphic to $P_{n_1} \Box P_{n_2}$, then $G$ is equitably 4-choosable.

**Proof.** If $n_1 = 1$ or $n_2 = 1$ then the proof follows from Lemma 3.7. Thus, without loss of generality, we may assume that $n_1, n_2 \geq 2$. Let $G' = P_{n_1} \Box P_{n_2} \Box P_2$, $F_u = P_{n_1} \Box P_{n_2}$ for $p \in [t]$, $u \in [s]$ be components of $G$ and $V(G') = \{(i, j, 0)^u : i \in [n_1], j \in [n_2], \ell \in [2]\}$, $V(F_u) = \{(i, j)^u : i \in [n_1], j \in [n_2]\}$. Let $n_1 = 2q + r$ where $r \in \{0, 1\}$. Let $L$ be a 4-uniform list assignment for a graph $G$. We show that there is a proper $L$-colouring $c$ such that $(G, c)$ has a rainbow 4-partition. If $r = 1$ then let $H$ be a subgraph induced in $G$ by the set $\{(i, j, 0)^u : j \in [n_2], p \in [t], \ell \in [2]\} \cup \{(i, j)^u : i \in [n_1], j \in [n_2], u \in [s]\}$. If $r = 0$ then let $H$ be a subgraph induced in $G$ by the set $\{(i, j)^u : i \in [n_1], j \in [n_2], u \in [s]\}$. By Lemma 3.7 there is an equitable proper $L$-colouring $c'$ of $H$, and so by Lemma 2.2 there is a rainbow 4-partition of $(H, c')$. Now we start with colouring vertices of $G - H$ (vertices of $G$, if $r = 0$ and $G$ has no component isomorphic to $P_{n_1} \Box P_{n_2}$).

We divide the set of uncoloured vertices of each component into 4-element subsets.

$S_{ij}^p = \{(2i + 1 + r, j, 1)^p, (2i + 1 + r, j, 2)^p, (2i + 2 + r, j, 1)^p, (2i + 2 + r, j, 2)^p\}$, where $i \in [0, q - 1], j \in [n_2], p \in [t]$ (cf. Fig 5). In each component we define a linear ordering $\prec$ on the family of these sets in the following way: $S_{ij}^p < S_{ij'}^p$ if $(j < s)$ or $(j = s$ and $i < r)$. According to this ordering we properly colour vertices of each set with the following rules:

- if it is only possible, we colour vertices in $S_{ij}^p$ in such a way that the vertices from this set get different colours;
- if we cannot colour vertices in $S_{ij}^p$ in such a way that $S_{ij}^p$ is rainbow then we colour vertices in this set in such a way that two vertices have the same colour, let us say colour $c$, other vertices are coloured differently and there is no vertex coloured with $c$ in $S_{ij-1}^p$.

We show that there exists a proper $L$-colouring of $G - H$ such that these rules are maintained. It is easy to see that we can colour vertices in sets $\{S_{ij}^p : i \in [0, q - 1]\}$ such that these sets are rainbow. Suppose that we are at the step when we colour vertices in $S_{ij'}^p$, $j \geq 2$, so vertices of every set that precedes $S_{ij}^p$ are
coloured, the vertices in $S_{ij}^p$ are uncoloured. Let $c''$ be a proper $L$-colouring of the coloured part of $G - H$ constructed up to now. To simplify the notation let $S_{ij}^p = \{(x, j, 1), (x, j, 2), (x + 1, j, 1), (x + 1, j, 2)\}$. Thus each vertex in $\{(x, j, 1), (x, j, 2)\}$ has at most two coloured neighbours that are not in $S_{ij}^p$ and each vertex in $\{(x + 1, j, 1), (x + 1, j, 2)\}$ has one coloured neighbour that is not in $S_{ij}^p$. Suppose that we cannot colour vertices in $S_{ij}^p$ such that $S_{ij}^p$ is rainbow. Since every vertex has four colours on its list, we can always colour three vertices in $S_{ij}^p$ with different colours, only the last vertex being coloured in $S_{ij}^p$ obtains the colour just used on $S_{ij}^p$. Let $c''((x, j, 1)) = c_1, c''((x, j, 2)) = c_2, c''((x + 1, j, 1)) = c_3, c''((x + 1, j, 2)) = c_1$. If there is no vertex coloured with $c_1$ in $S_{ij}^p$, then we are done. Suppose that there is a vertex coloured with $c_1$ in $S_{ij-1}^p$. Since we are forced to use the colour $c_1$ on $(x + 1, j, 2)$, we necessarily have $L((x + 1, j, 2)) = \{c_1, c_2, c_3, c''(x + 1, j, 1)\}$. If in $L((x + 1, j, 1))$ there is a colour $b$ such that $b \notin \{c_1, c_2, c_3, c''(x + 1, j, 1)\}$ then we can colour $(x + 1, j, 1)$ with $b$ and next we colour $(x + 1, j, 2)$ with $c_3$, to obtain a rainbow set $S_{ij}^p$ achieving a contradiction. Thus $L((x + 1, j, 1)) = \{c_1, c_2, c_3, c''(x + 1, j, 1)\}$. Since each vertex has four different colours on the list, we have $c_1 \neq c''((x + 1, j, 1))$ and $c_1 \neq c''((x + 1, j, 1))$. Furthermore, $(x, j, 1)$ has a neighbour coloured with $c_1$, thus $c''((x, j, 1)) \neq c_1$. However, by our assumption in $S_{ij-1}^p$ there is a vertex coloured with $c_1$, so $c''((x, j, 1)) = c_1$. Observe that also $c_2 \neq c''((x + 1, j, 1))$ and $c_3 \neq c''((x + 1, j, 1))$. Thus if $c_2 \neq c''((x, j - 1, 1))$ then we can colour $(x + 1, j, 1)$ with $c_2$ and $(x + 1, j, 2)$ with $c_3$ to obtain desired colouring. Assume that $c_2 = c''((x, j - 1, 1))$. Observe that there is no vertex coloured with $c_3$ in $S_{ij-1}^p$. If $c_3 \in L((x, j, 1))$ then we colour $(x, j, 1)$ with $c_3$ and next $(x + 1, j, 1)$ with $c_3$ to obtain a desired colouring. Otherwise, $(x, j, 1)$ has a colour $b$ different from $c_1, c_2, c_3$ and $c''((x - 1, j, 1))$ on its list. If we colour $(x, j, 1)$ with $b$, the $S_{ij}^p$ is rainbow, a contradiction.

**Claim 3.10.** If the set $S_{ij}^p$ is not rainbow and $S_{ij-1}^p$ is not rainbow, i.e., in $S_{ij-1}^p$ there are two vertices coloured with $b_1$, then in $S_{ij}^p$ there is no vertex coloured with $b_1$.

**Proof.** Without loss of generality we may assume $c''((x, j, 1)) = c_1, c''((x, j, 2)) = c_2, c''((x + 1, j, 1)) = c_3, c''((x + 1, j, 2)) = c_1$. Similarly as above we observe that $L((x + 1, j, 1)) = \{c_1, c_2, c_3, c''((x + 1, j - 1, 1))\}$ and $L((x + 1, j, 2)) = \{c_1, c_2, c_3, c''((x + 1, j - 1, 2))\}$. Since the colours on lists are different, $c''((x + 1, j - 1, 1)) \notin \{c_1, c_2, c_3\}$ and $c''((x + 1, j - 1, 2)) \notin \{c_1, c_2, c_3\}$ and hence neither $c''((x + 1, j - 1, 1))$ nor $c''((x + 1, j - 1, 2))$ is used on $S_{ij}^p$. The argument that $c''((x, j - 1, 1)) \neq c''((x, j - 1, 2))$ completes the proof. 

Previous arguments imply that either $S_{ij}^p$ is rainbow or $S_{ij}^p \cup S_{ij-1}^p$ can be divided into two 4-elements rainbow sets in $(G - H, c'')$, as it has been shown that each colour is used in $S_{ij}^p \cup S_{ij-1}^p$ at most twice.

We use the similar method as in the proof of Lemma 3.6 to show that there is a rainbow 4-partition of $(G - H, c'')$. We divide $V(G - H)$ in the following way (cf. Fig. 5):

- the set of vertices of each component is divided step by step;
- in each component $G^p$, we start with the last set due to $<^p$ and go down according this ordering;

if $S_{ij}^p$ is rainbow then it forms a set of the rainbow special 4-partition of $(G - H, c'')$, otherwise, we partite $S_{ij}^p \cup S_{ij-1}^p$ into two rainbow 4-element sets that form two sets of the rainbow 4-partition of $(G - H, c'')$, we modify $<^p$ by removing sets that have been already included into the rainbow 4-partition.

Recall that for $i \in [0, q - 1]$ the sets $S_{ij}^p$ are rainbow, so the above partition results in a rainbow special 4-partition of $(G - H, c'')$. Thus, together with the rainbow 4-partition of $(H, c')$, we obtain the rainbow 4-partition of $(G, c' \cup c'')$. Hence for every 4-uniform list assignment $L$ there is a proper $L$-colouring $c$ such that $(G, c)$ has a rainbow 4-partition, and so $G$ is equitably 4-choosable, by Lemma 2.2.

**Remark 3.11.** Lemma 3.9 is still true when components of $G$ are of different size.
Observe that the 4-partition given in the proof of Lemma 3.9 does not meet the assumptions of Lemma 2.5, thus from that proof we cannot conclude that such a graph is equitably k-choosable for k > 4. However, if each component of G is isomorphic to \(P_n \Box P_2 \Box P_2\) or \(P_n \Box P_n\) then \(\Delta(G) \leq 5\) and by Theorem 2.8 we have that G is equitably k-choosable for \(k \geq 6\).

4. Equitable list vertex arboricity of grids

In this section we apply tools described in the previous sections what causes in giving new results concerning equitable list arboricity of d-dimensional grids \(P_n \Box \cdots \Box P_n\).

First, observe that every 2-dimensional grid has a spanning linear forest, i.e. a union of disjoint paths), that covers all cycles. Since every linear forest is equitably k-choosable for any \(k \geq 2\) (cf. Lemma 3.3) then, using Lemma 2.7, we have the following

**Theorem 4.1.** Let \(k \in \mathbb{N}\). If \(k \geq 2\) then every 2-dimensional grid is equitably k-list arborable.

4.1. 3-dimensional grids

**Theorem 4.2.** Let \(k, n_2, n_3 \in \mathbb{N}\) with \(n_2 \geq 2, n_3 \geq 2\). If \(k \geq 2\) then \(P_2 \Box P_{n_2} \Box P_{n_3}\) is equitably k-list arborable.

**Proof.** We will prove that \(P_2 \Box P_{n_2} \Box P_{n_3}\) contains a subgraph \(H\) with maximum degree at most two that covers all cycles. Since \(P_2 \Box P_{n_2} \Box P_{n_3}\) is bipartite then \(H\) is also bipartite so, by Lemma 3.3, \(H\) is certainly equitably k-choosable for any \(k \geq 2\). Hence, by Lemma 2.7, the proof will follow.

We can see \(P_2 \Box P_{n_2} \Box P_{n_3}\) as two copies of \(P_{n_2} \Box P_{n_3}\) (we call them layers \(G_1\) and \(G_2\)) joined by some edges. Let \(V(G_1) = \{(i, y, z) : y \in [n_2], z \in [n_3]\}\) be the vertex set of the layer \(G_1\) and let \(V(G_2) = \{(2, y, z) : y \in [n_2], z \in [n_3]\}\) be the vertex set of the layer \(G_2\) (cf. Fig. 6a)). In each layer we choose a maximal matching in the following way. In each column we choose a maximal matching. We start with the first edge if the column is odd and with the second edge if the column is even. More formally, for \(i \in [2]\), \(M_i = \{(i, 2p + 1, r)(i, 2p + 2, r) : p \in [0, (n_2 - 2)/2], r \in [n_3], r \text{ is odd}\}\) and \(M_i'' = \{(i, 2p, r)(i, 2p + 1, r) : p \in [(n_2 - 1)/2], r \in [n_3], r \text{ is even}\}\) (cf. Fig. 6b)). Let \(M_i\) be a spanning subgraph of \(G_i\) such that \(V(M_i) = V(G_i)\) and \(E(M_i) = M_i' \cup M_i''\). We show...
that \( M_i \) covers all cycles in \( G_i \). Since both \( G_1, G_2 \) are isomorphic to \( P_{n} \bigtriangleup P_{n} \), we simplify notation and show that \( M = M' \cup M'' \) covers all cycles in \( P_{n} \bigtriangleup P_{n} \), where \( M' = \{(2p + 1, r)(2p + 2, r) : p \in [0, [(n_2 - 2)/2]], r \in [n_3], r \text{ is odd}\} \) and \( M'' = \{(2p, r)(2p + 1, r) : p \in [[(n_2 - 1)/2]], r \in [n_3], r \text{ is even}\} \). We prove it by induction on \( n_3 \). It is obviously true for \( n_3 = 2 \). Thus by induction hypothesis we may assume that such a spanning subgraph covers all cycles of \( P_{n_3} \). Suppose that \( P_{n_3} \bigtriangleup P_{n_3} \) contains an edge whose vertices have second coordinates \( n \), \( n \). Thus by our choice of \( n \), \( n \) is odd \( r \), \( r \). \( P \), \( P \).\( P \), \( P \), \( P \) which contradicts that \( M \) does not cover \( C \). Now we construct a spanning subgraph \( H \) of \( P_2 \bigtriangleup P_2 \bigtriangleup P_2 \) in the following way. Let us denote the set of edges in \( P_2 \bigtriangleup P_2 \bigtriangleup P_2 \) by \( E(G_1, G_2) \). We set \( E(H) = M_1 \cup M_2 \cup E(G_1, G_2) \). Thus \( H \) covers all cycles of \( P_2 \bigtriangleup P_2 \bigtriangleup P_2 \) and \( \Delta(H) = 2 \), and so \( P_2 \bigtriangleup P_2 \bigtriangleup P_2 \) is equitably \( k \)-list arborable for every \( k \geq 2 \).

**Theorem 4.3.** Let \( n_3, k \in \mathbb{N} \). If \( k \geq 2 \) then \( P_3 \bigtriangleup P_3 \bigtriangleup P_3 \) is equitably \( k \)-list arborable.

**Proof.** Similarly as in the proof of Theorem 4.2, we prove that \( P_3 \bigtriangleup P_3 \bigtriangleup P_3 \) contains a spanning subgraph \( HP_{3 \times 3 \times n_3} \) with maximum degree at most two that covers all cycles. Since \( P_3 \bigtriangleup P_3 \bigtriangleup P_3 \) is bipartite, \( HP_{3 \times 3 \times n_3} \) is also bipartite so, by Lemma 3.3, \( HP_{3 \times 3 \times n_3} \) is equitably \( k \)-choosable for any \( k \geq 2 \). Thus, by Lemma 2.7, the proof will follow.

![Figure 7: Illustration for the proof of Theorem 4.3](image)

Let \( G_1, G_2 \) and \( G_3 \) be layers of \( P_3 \bigtriangleup P_3 \bigtriangleup P_3 \), such that \( V(G_i) = \{(i, y, z) : y, z \in [3], z \in [n_3]\} \) for \( i \in [3] \). In each layer \( G_i \) we choose the spanning subgraph \( M_i \) in the following way (cf. Fig. 7a):

- \( E(M_1) = \{(1, i, j)(1, i, j + 1) : i \in [3], j \in [n_3 - 1]\}; \)
- \( E(M_2) = \{(2, 1, i)(2, 1, i + 1) : i \in [n_3 - 1]\}_{ODD} \cup \{(2, 1, i)(2, 2, i), (2, 2, i)(2, 3, i) : i \in [n_3]\}; \)
- \( E(M_3) = \{(3, 1, i)(3, 1, i + 1) : i \in [n_3 - 1]\} \cup \{(3, 2, i)(3, 2, i + 1) : i \in [n_3 - 1]\} \cup \{(3, 3, i)(3, 3, i + 1) : i \in [n_3 - 1]\}_{EVEN}. \)

Moreover,

- \( E_{2,3} = \{(2, 3, i)(3, 3, i) : i \in [n_3]\}. \)

The subgraph \( HP_{3 \times 3 \times n_3} \) is defined in the following way: \( V(HP_{3 \times 3 \times n_3}) = V(P_3 \bigtriangleup P_3 \bigtriangleup P_3) \) and \( E(HP_{3 \times 3 \times n_3}) = E(M_1) \cup E(M_2) \cup E(M_3) \cup E_{2,3}. \)
We show that $HP_{3 \times 3 \times n_3}$ covers all cycles of $P_3 \Box P_3 \Box P_{n_3}$. Let $L_i$ for $i \in [n_3]$ be layers that are isomorphic to $P_3 \Box P_3$, so $V(L_i) = \{ (j, \ell, i) : j \in [3], \ell \in [3] \}$. Observe that the subgraphs induced by $V(HP_{3 \times 3 \times n_3}) \cap V(L_i)$ are isomorphic (cf. Fig. 7b).

If a cycle in $P_3 \Box P_3 \Box P_{n_3}$ contains an edge from $HP_{3 \times 3 \times n_3}$ then obviously it is covered by $HP_{3 \times 3 \times n_3}$. Thus we focus only on cycles in $P_3 \Box P_3 \Box P_{n_3} - E(HP_{3 \times 3 \times n_3})$. We use the induction method to prove that every cycle in $P_3 \Box P_3 \Box P_{n_3} - E(HP_{3 \times 3 \times n_3})$ contains two vertices $u$ and $v$ such that $uv \in E(HP_{3 \times 3 \times n_3})$.

It is easy to see that $HP_{3 \times 3 \times 1}$ covers all cycles in $P_3 \Box P_3 \Box P_1$. Let $n_3 \geq 2$, assume that $HP_{3 \times 3 \times (n_3-1)}$ covers all cycles in $P_3 \Box P_3 \Box P_{n_3-1}$ and consider $HP_{3 \times 3 \times n_3}$ in $P_3 \Box P_3 \Box P_{n_3}$. Thus if there is an uncovered cycle in $P_3 \Box P_3 \Box P_{n_3} - E(HP_{3 \times 3 \times n_3})$ then it must contain vertices from layer $L_{n_3}$. First observe that the only cycle of $L_{n_3}$ that contains no edge from $HP_{3 \times 3 \times n_3}$ contains vertices $(2,1,n_3)$ and $(2,2,n_3)$. Since $(2,1,n_3)(2,2,n_3) \in E(HP_{3 \times 3 \times n_3})$, all cycles of $L_{n_3}$ are covered by $HP_{3 \times 3 \times n_3}$. Thus if there is an uncovered cycle $C$ in $P_3 \Box P_3 \Box P_{n_3} - E(HP_{3 \times 3 \times n_3})$ then it must contain vertices from layers $L_{n_3}$ and $L_{n_3-1}$. We consider two cases.

**Case 1.** $n_3$ is even. $C$ must go through two out of three following edges: $a = (2,3,n_3-1)(2,3,n_3)$, $b = (2,2,n_3-1)(2,2,n_3)$, $c = (3,3,n_3-1)(3,3,n_3)$. If $C$ contains edges $a$ and $b$ (edges $a$ and $c$, resp.) then it is covered by the edge $(2,2,n_3)(2,3,n_3)$ $(3,3,n_3)(3,3,n_3)$, resp.). If $C$ goes through the edges $b$ and $c$ then it must contain the vertex $(3,2,n_3)$. On the other hand, edges $(3,3,n_3-2)(3,3,n_3-1)$ and $(2,3,n_3-1)(3,3,n_3-1)$ belong to $HP_{3 \times 3 \times n_3}$. Hence $C$ must go through $(3,2,n_3-1)(3,3,n_3-1)$. This implies that the cycle is covered by the edge $(3,2,n_3-1)(3,2,n_3)$.

**Case 2.** $n_3$ is odd. $C$ must go through two out of three following edges: $a = (2,3,n_3-1)(2,3,n_3)$, $b = (2,2,n_3-1)(2,2,n_3)$, $c = (2,1,n_3-1)(2,1,n_3)$. If $C$ contains edges $a$ and $b$ (and $c$, resp.) then it is covered by the edge $(2,2,n_3)(2,3,n_3)$ $(2,1,n_3)(2,2,n_3)$, resp.). If the cycle contains the edges $a$ and $c$ then, to avoid vertex $(2,2,n_3)$, it consecutively goes through the edge $a$, vertices $(1,3,n_3)$, $(1,2,n_3)$, $(1,1,n_3)$, $(2,1,n_3)$ and edge $c$. Observe that $(2,3,n_3-1)$ is incident with exactly two edges $(1,3,n_3-1)(2,3,n_3-1)$ and $(2,3,n_3-2)(1,3,n_3-1)$ that are not in $E(HP_{3 \times 3 \times n_3})$. Due to $n_3$ even’ case the cycle $C$ cannot go through the second one. If it goes through the first one then $(1,3,n_3-1) \in V(C)$ and $C$ is covered by $(1,3,n_3-1)(1,3,n_3)$.

Thus $HP_{3 \times 3 \times n_3}$ covers all cycles of $P_3 \Box P_3 \Box P_{n_3}$. $\Delta(HP_{3 \times 3 \times n_3}) = 2$, and so $P_3 \Box P_3 \Box P_{n_3}$ is equitably $k$-list arborable for every $k \geq 2$.

**Theorem 4.4.** Let $n_1, n_2, n_3, k \in \mathbb{N}$. If $k \geq 3$ then $P_3 \Box P_{n_1} \Box P_{n_2} \Box P_{n_3}$ is equitably $k$-list arborable.

**Proof.** Let $G = P_{n_1} \Box P_{n_2} \Box P_{n_3}$ be a 3-dimensional grid. Let us define a set of edges $X_{ij} = \{ (\ell, i, j)(\ell + 1, i, j) : \ell \in \mathbb{N} \}$ for $i \in [n_2]$ and $j \in [n_3]$. First, observe that the graph $(V(G), X)$, where $X = \bigcup_{i \in [n_2], j \in [n_3]} X_{ij}$, is a linear forest. Thus $G - X$ covers all cycles of $G$. Furthermore, every component of $G - X$ is isomorphic to $P_{n_1} \Box P_{n_3}$. Thus, by Lemma 3.7, $G - X$ is equitably $k$-choosable for every $k \geq 3$. Finally, Lemma 2.7 implies that $G$ is equitably $k$-list arborable for every $k \geq 3$.

4.2. 4-dimensional grids

**Theorem 4.5.** Let $n_4, k \in \mathbb{N}$. If $k \geq 2$ then $P_2 \Box P_2 \Box P_2 \Box P_{n_4}$ is equitably $k$-list arborable.
Proof. Let $G = P_2 \square P_2 \square P_2 \square P_{n_4}$. We can see $G$ as $n_4$ 3-dimensional cubes $Q^1, \ldots, Q^{n_4}$ joined by some edges. Let $H$ be a spanning subgraph of $G$ that contains two cycles of length 4 of each cube $Q$: ‘front’ and ‘back’ cycles of $Q$ with $i$ odd, ‘top’ and ‘bottom’ cycles of $Q$ with $i$ even (cf. Fig. 8). More formally, let us define a spanning subgraph $H$ of $G$ in the following way $E(H) = E_1 \cup E_2$, where

$$E_1 = \{(1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i), (1, 1, 1, i)(2, 1, 1, i) : i \in [n_4]_{ODD}\}$$

$$E_2 = \{(1, 1, 1, j)(2, 1, 2, j), (1, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j), (2, 1, 1, j)(2, 1, 2, j) : j \in [n_4]_{EVEN}\}$$

We prove by induction on $n_4$ that $H$ covers all cycles of $P_2 \square P_2 \square P_2 \square P_{n_4}$. It is obviously true for $n_4 = 1$. Assume that it is true for $P_2 \square P_2 \square P_2 \square P_{n_4-1}$. Without loss of generality we may assume that $n_4$ is even. Suppose that there is a cycle $C$ in $G$ that has no two vertices adjacent by an edge in $H$. Since there is no such a cycle in $P_2 \square P_2 \square P_2 \square P_{n_4-1}$, it follows that $C$ contains an edge of the cube $Q^{n_4}$ induced by the vertices of the form $(i, j, \ell, n_4)$, $i \in [2], j \in [2], \ell \in [2]$ that is not in $H$. By symmetry we may assume that $C$ contains $(1, 1, 1, n_4)(2, 1, 1, n_4)$. Thus $C$ must also contain vertices $(1, 1, 1, n_4-1)$ and $(1, 2, 1, n_4-1)$, however $(1, 1, 1, n_4-1)(2, 1, 1, n_4-1) \in E(H)$, a contradiction. Since $H$ is equitably $k$-choosable for $k \geq 2$ by Lemma 3.3, $G$ is equitably $k$-list arborable for $k \geq 2$ by Lemma 2.7. \hfill $\square$

**Theorem 4.6.** Let $n_3, n_4, k \in \mathbb{N}$. If $k \geq 3$ then $P_2 \square P_2 \square P_n \square P_{n_4}$ is equitably $k$-list arborable.

![Illustration for the proof of Theorem 4.6](image-url)

**Proof.** Let $G = P_2 \square P_2 \square P_2 \square P_{n_4}$. We show that there is a spanning subgraph $H$ of $G$ that covers all cycles of $G$ such that each component of $H$ is isomorphic to $P_2 \square P_n$. Since $H$ is equitably $k$-choosable for $k \geq 3$, by Lemma 3.4, we apply Lemma 2.7 to show that $G$ is equitably $k$-list arborable for every $k \geq 3$. We can cf. $G$ as $n_4$ layers $G_1, \ldots, G_{n_4}$, each of which is isomorphic to a 3-dimensional grid $P_2 \square P_2 \square P_{n_4}$ joined by some edges. To obtain $H$ from every grid $G_i$ we take two disjoint $P_2 \square P_{n_4}$, if $i$ is odd we take ‘top’ and ‘bottom’ $P_2 \square P_{n_4}$, if $i$ is even we take ‘left’ and ‘right’ $P_2 \square P_{n_4}$ (cf. Fig. 9). Let $H = \bigcup_{i \in [n_4]_{ODD}}(H_{1i} \cup H_{2i}) \cup \bigcup_{j \in [n_4]_{EVEN}}(H_{1j} \cup H_{2j})$ be a spanning subgraph of $G$, where

- $H_{1i} = G[\{(1, 1, p, i), (2, 1, 1, 1) : p \in [n_3]\}]$ (‘bottom’);
- $H_{2i} = G[\{(1, 2, p, i), (2, 2, p, i) : p \in [n_3]\}]$ (‘top’);
- $H_{1j} = G[\{(1, 1, p, j), (1, 2, p, j) : p \in [n_3]\}]$ (‘left’);
- $H_{2j} = G[\{(2, 1, p, j), (2, 2, p, j) : p \in [n_3]\}]$ (‘right’).

We prove by induction on $n_4$ that $H$ covers all cycles of $G$. It is easy to see that if $n_4 = 1$, the subgraph $H$ covers all cycles of $G$. Now, suppose that $H$ covers all cycles of $P_2 \square P_2 \square P_n \square P_{n_4-1}$. Without loss of generality we may assume that $n_4$ is odd. If $G$ contains a cycle $C$ not covered by $H$ then there is an edge in $C$ whose end vertices have the last coordinate $n_4$ and that are not in $H$. Let $(1, 1, p, n_4)(1, 2, p, n_4)$ be
such an edge. Since all edges adjacent to the edge \((1, 1, p, n_k)(1, 2, p, n_k)\) except \((1, 1, p, n_k)(1, 1, p, n_k - 1)\) and \((1, 2, p, n_k)(1, 2, p, n_k - 1)\) are in \(H\) then the vertices \((1, 1, p, n_k - 1)\) and \((1, 2, p, n_k - 1)\) must be in \(C\). However, \((1, 1, p, n_k - 1)(1, 2, p, n_k - 1)\) is not incident with \((2, 2, s, t, n_k)\), which contradicts the assumption that \(H\) does not cover \(C\). Thus, by Lemma 3.4 and Lemma 2.7, the theorem holds. \(\Box\)

**Theorem 4.7.** Every 4-dimensional grid is equitably 4-list arborable.

**Proof.** Let \(G = P_{n_1} \square P_{n_2} \square P_{n_3} \square P_{n_4}\). Again, we determine a graph \(H\), whose every component is isomorphic to \(P \square P \square P\), or \(P \square P \square P\), that covers all cycles of \(G\). Next we apply Lemmas 3.9 and 2.7, so \(G\) is equitably 4-list arborable.

![Illustration for the proof of Theorem 4.7.](image)

We can see \(G\) as 3-dimensional grids \(G_i = P_{n_1} \square P_{n_2} \square P_{n_3}\), \(i \in [n_4]\) joined by some edges, i.e. \(G_i = G[(r, s, t, i) : r \in [n_1], s \in [n_2], t \in [n_3]], i \in [n_4]\). To obtain \(H\) we take all copies of \(G_i\) after removing the matching \(E_i\) defined as follows (cf. Fig. 10).

\[
E_i = \{(r, s, t, i)(r + 1, s, t, i) : r \in [n_1 - 1]_{\text{ODD}}, s \in [n_2], t \in [n_3]\} \quad \text{if } i \text{ is odd},
\]

\[
E_i = \{(r, s, t, i)(r + 1, s, t, i) : r \in [n_1 - 1]_{\text{EVEN}}, s \in [n_2], t \in [n_3]\} \quad \text{if } i \text{ is even}.
\]

Now, \(H = \bigcup_{i \in [n_4]}(G_i - E_i)\). We prove by induction on \(n_4\) that \(H\) covers all cycles of \(G\). Since \(E_1\) is a matching, \(G_1 - E_1\) obviously covers all cycles of \(G_1\). Let \(G' = P_{n_1} \square P_{n_2} \square P_{n_3} \square P_{n_4} - 1\) and \(H' = \bigcup_{i \in [n_4 - 1]}(G_i - E_i)\). Assume that \(H'\) covers all cycles of \(G'\). Without of loss generality we may assume that \(n_4\) is odd. On the contrary, suppose that \(G\) contains a cycle \(C\) not covered by \(H\). Thus \(C\) contains an edge \(e\) of \(E_{n_4}\), say \(e = (2r + 1, s, t, n_k)(2r + 2, s, t, n_k)\). So vertices \((2r + 1, s, t, n_k), (2r + 2, s, t, n_k)\) are in \(V(C)\). Since all edges of \(G_{n_4}\) incident with \((2r + 1, s, t, n_k)\) and \((2r + 2, s, t, n_k)\), except \(e\), are in \(H\), we must have that \((2r + 1, s, t, n_k - 1)\) is a neighbour of \((2r + 1, s, t, n_k)\) in \(C\) and \((2r + 2, s, t, n_k - 1)\) is a neighbour of \((2r + 2, s, t, n_k)\) in \(C\). Thus \((2r + 1, s, t, n_k - 1), (2r + 2, s, t, n_k - 1)\) is in \(V(C)\), however \((2r + 1, s, t, n_k - 1)(2r + 2, s, t, n_k - 1)\) is in \(E(H)\), which contradicts that \(C\) is not covered by \(H\). \(\Box\)

In the proof of the next theorem we use Lemma 2.6. We determine a special 5-partition of a graph to show that the graph is equitably \(k\)-list arborable for every \(k \geq 5\).

**Theorem 4.8.** Let \(k \in \mathbb{N}\). If \(k \geq 5\) then every 4-dimensional grid is equitably \(k\)-list arborable.

**Proof.** Let \(G = P_{n_1} \square P_{n_2} \square P_{n_3} \square P_{n_4}\) and \(V(G) = \{(i, j, k, l) : i \in [n_1], j \in [n_2], k \in [n_3], l \in [n_4]\}\). We determine a special 5-partition \(S_i \cup \cdots \cup S_{n_4 + 1}\) of \(G\), with \(|V(G)| = 5n^3 + r\), that fulfills the assumptions of Lemma 2.6. So, by Lemma 2.6, the theorem will follow. We depict sets \(S_i\) of size 5 step by step in decreasing order, starting with determining a set \(S_{n_4 + 1}\) and next, in the same manner, sets \(S_{n_4}, S_{n_3}, S_{n_2}\). The last set \(S_1\) is formed by vertices in \(V(G) \setminus (S_2 \cup \cdots \cup S_{n_4 + 1})\), so its size is less than or equal to 5. Since the assumptions of Lemma 2.6 are obviously fulfilled for each 4-dimensional grid \(G\) satisfying \(|V(G)| \leq 5\) we may assume that \(|V(G)| \geq 6\).

Let \(j \in [2, n + 1]\). To determine a set \(S_i\) consisting of elements \(x'_i, y'_i, \ldots, x'_j, y'_j\), we use the sets \(S_{j-1}, \ldots, S_{n_4}\) constructed in the previous steps. Let \(G_i = G - (S_{j-1} \cup \cdots \cup S_{n_4 + 1})\). Thus \(G_i\) is the graph induced in \(G\) by the union of sets \(S_1, \ldots, S_j\), whose forms are unknown at this moment. Observe that \(V(G_{i-1})\) is equal to
In every case we have Lemma 4.9.

4.3. d-dimensional grids, the general upper bound

In Section 2 we give a general upper bound on the equitable list vertex arboricity of all graphs. Now we improve this bound for d-dimensional grids.

Assume that \( d \geq 3 \) and \( n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\} \). Let us define the following family of graphs.

\[
\mathcal{H}(n_1, \ldots, n_{d-2}) = \{ G : \text{each component of } G \text{ is isomorphic to } P_{n_1} \sqcup \cdots \sqcup P_{n_{d-2}} \sqcup P_2 \text{ or } P_{n_1} \sqcup \cdots \sqcup P_{n_{d-2}} \}.
\]

**Lemma 4.9.** Let \( d \in \mathbb{N} \) with \( d \geq 3 \), \( n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\} \) and \( G = P_{n_1} \sqcup \cdots \sqcup P_{n_{d-2}} \). There is a graph \( H \in \mathcal{H}(n_1, \ldots, n_{d-2}) \) that covers all cycles of \( G \).

**Proof.** The idea of determining a graph \( H \) is the same as in the proof of Theorem 4.7. We can see \( G \) as \( n_d \) copies of a \((d-1)\)-dimensional grid \( P_{n_1} \sqcup \cdots \sqcup P_{n_d} \) joined by some edges. Let \( G_i = G[\{y_1, \ldots, y_{d-1}, i \} : y_i \in [n_i], j \in [d-1]] \}, i \in [n_d] \). To obtain \( H \), we delete from every \( G_i \) the matching \( E_i \) defined as follows.

**Case 1** \( i \) is odd

\[
E_i = \{(y_1, y_2, \ldots, y_{d-1}, i)(y_1 + 1, y_2, \ldots, y_{d-1}, i) : y_1 \in [n_1 - 1]_{\text{ODD}}\}.
\]

**Case 2** \( i \) is even

\[
E_i = \{(y_1, y_2, \ldots, y_{d-1}, i)(y_1 + 1, y_2, \ldots, y_{d-1}, i) : y_1 \in [n_1 - 1]_{\text{EVEN}}\}.
\]

In both cases we take \( y_j \in [n_j] \) for \( j \in [2, d - 1] \). Put \( H = \bigcup_{i \in [n_d]} (G_i - E_i) \). Note that \( H \in \mathcal{H}(n_1, \ldots, n_{d-2}) \). We prove by induction on \( n_d \) that \( H \) covers all cycles of \( G \). Since \( E_i \) is a matching of \( G_i \), obviously \( G_i - E_i \) covers all cycles of \( G_i \). Let \( G' = P_{n_1} \sqcup \cdots \sqcup P_{n_{d-1}} \) and \( H' = \bigcup_{i \in [n_d]} (G_i - E_i) \). By the induction hypothesis, \( H' \) covers all cycles of \( G' \). Without loss of generality we may assume that \( n_d \) is odd. On the contrary, suppose that \( G \) contains a cycle \( C \) not covered by \( H \). Thus \( C \) contains an edge \( e \) of \( E_{n_d} \), say \( e = (2r + 1, y_2, \ldots, y_{d-1}, n_d)(2r + 2, y_2, \ldots, y_{d-1}, n_d) \). So vertices \((2r + 1, y_2, \ldots, y_{d-1}, n_d), (2r + 2, y_2, \ldots, y_{d-1}, n_d) \) are in \( V(C) \). Since all edges of \( G_{n_d} \) incident with \((2r + 1, y_2, \ldots, y_{d-1}, n_d) \) and \((2r + 2, y_2, \ldots, y_{d-1}, n_d) \), except \( e \), are in \( H \), we must have that \((2r + 1, y_2, \ldots, y_{d-1}, n_d - 1) \) is a neighbour of \((2r + 1, y_2, \ldots, y_{d-1}, n_d) \) in \( C \). Thus \((2r + 1, y_2, \ldots, y_{d-1}, n_d - 1)) \) is a neighbour of \((2r + 2, y_2, \ldots, y_{d-1}, n_d - 1)) \) in \( C \). This contradicts that \( C \) is not covered by \( H \).
Observation 4.10. Let \( d \in \mathbb{N}, n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\} \) and \( H \in \mathcal{H}(n_1, \ldots, n_{d-2}) \). If \( d \geq 3 \) then \( \Delta(H) \leq 2d - 3 \).

Observation 4.10 together with Theorem 2.8(i)-(ii) and Lemma 2.7 imply the following result.

Theorem 4.11. Let \( d, k \in \mathbb{N} \).

(i) If \( k \geq 8 \) then every 5-dimensional grid is equitably k-list arborable.

(ii) If \( d \in \{6, 16\} \) and \( k \geq 2d - 2 + \frac{3d-4}{2} \) then every \( d \)-dimensional grid is equitably k-list arborable.

(iii) If \( d \geq 17 \) and \( k \geq 2d - 3 + \frac{3d-6}{6} \) then every \( d \)-dimensional grid is equitably k-list arborable.

5. Concluding remarks

Note that our results confirm Zhang’s conjectures for \( d \)-dimensional grids, when \( d \in \{2, 4\} \). For many cases they are even stronger than the conjectures. More precisely, we have obtained the following facts.

Corollary 5.1. Let \( k \in \mathbb{N} \) and \( d \in \{2, 3, 4\} \). If \( G \) is a \( d \)-dimensional grid and \( k \geq \lceil (\Delta(G) + 1)/2 \rceil \) then \( G \) is equitably k-list arborable.

Corollary 5.2. Let \( d, k \in \mathbb{N} \) with \( d \geq 2 \) and \( k \geq 2 \). If \( G \) is a \( d \)-dimensional grid with \( \Delta(G) \leq 5 \) then \( G \) is equitably k-list arborable.

Corollary 5.3. Let \( k \in \mathbb{N} \), \( d \in \{2, 3, 4\} \), and let \( G \) be a \( d \)-dimensional grid with \( \Delta(G) \geq 6 \) that is different from \( P_{n_1} \square P_{n_2} \square P_{n_3} \) and \( n_1, n_2, n_3 \in \mathbb{N} \setminus \{1, 2\} \). If \( k \geq \lceil (\Delta(G))/2 \rceil \) then \( G \) is equitably k-list arborable.

Since \( d \)-dimensional grids have many special properties, we expect that the results that are better than Zhang’s conjectures hold for almost all of them. Among others, \( d \)-dimensional grids are bipartite and \( d \)-degenerate. The equitably colouring of such classes of graphs is analyzed in many papers. For instance, it was proven in [8] that the inequality \( \chi'(G) \leq \Delta(G) \) holds for every connected bipartite graph \( G \). We improve this result for all \( d \)-dimensional grids. The following two theorems will help us to post some conjectures.

Theorem 5.4. Let \( d, k \in \mathbb{N} \) with \( d \geq 2 \), and let \( G \) be a \( d \)-dimensional grid. If \( k \geq 2 \) then there exists an equitable proper k-colouring of \( G \).

The concept of layers in \( d \)-dimensional grids, used until now, must be extended on the purpose of the proof of Theorem 5.4. Let \( G = P_{n_1} \square \cdots \square P_{n_d} \) and \( \{i_1, \ldots, i_d\} \) be any \( s \)-subset of indexes from \( [d] \). Moreover, let \((a_{i_1}, \ldots, a_{i_s})\) be a fixed \( s \)-tuple from \([n_{i_1}] \times \cdots \times [n_{i_s}]\). Then each graph induced in \( G \) by the set

\[
\{(y_{i_1}, \ldots, y_{i_s}) : y_{i_j} = a_{i_j}, \ldots, y_{i_s} = a_{i_s}\}
\]

is called an \( s \)-layer of \( G \). Note that the layers used until now are 1-layers.

Proof. Let \( k \) be fixed and \( G = P_{n_1} \square \cdots \square P_{n_d} \) with \( n_1, \ldots, n_d \in \mathbb{N} \setminus \{1\} \). We construct a proper \( k \)-colouring of \( G \) in which every colour class has the cardinality either \( |V(G)|/k \) or \( |V(G)|/k \). The construction is given in \( d \) stages. For \( i \in [d] \), in the \( i \)-th stage we describe a proper \( k \)-colouring \( c_i \) of an \( i \)-dimensional grid \( P_{n_i} \square \cdots \square P_{n_i} \), which is a \((d - i)\)-layer \( G_i \) of \( G \) induced in \( G \) by the set of vertices \( V_i \), where

\[
V_i = \{(y_{i_1}, \ldots, y_{i_i}, 1, \ldots, 1) : y_{i_1} \in [n_{i_1}], \ldots, y_{i_j} \in [n_{i_j}]\}.
\]

We construct a proper \( k \)-coloring \( c_{i+1} \) on \( V_{i+1} \) as an extension of a proper \( k \)-colouring \( c_i \) on \( V_i \). Finally, we obtain a proper \( k \)-colouring \( c_d \) of \( G \). For each \( i \in [d] \) we care for \( c_i \) to be equitable, which means that each colour class of \( c_i \) is of the cardinality either \( (n_1 \cdots n_i)/k \) or \( (n_1 \cdots n_i)/k \).
Let us start with the construction of $c_1$. In this case $G_1 = P_n$ and we put $c_1((y_1,1,...,1)) \equiv y_1 (\mod k)$. Thus, depending on $n_1$, each of $k$ colours arises either $[n_1/k]$ or $[n_1/k]$ times and moreover, $c_1$ is a proper $k$-colouring of $G_1$. Note that this time we use colors from $[0,k-1]$.

Suppose that, for some $i \in [d-1]$, the colouring $c_i$ is constructed. Of course $c_i$ satisfies all requirements mentioned before. Now we permute colours used in $c_i$ on vertices in $V_i$ (recall that $|V_i| = n_1 \cdots n_d$) in such a way that each of the colours $1,\ldots,p$ is used $[(n_1 \cdots n_i)/k]$ times and each of the remaining $k-p$ colours $p+1,\ldots,k$ is used $[(n_1 \cdots n_i)/k]$ times. Of course it could be $p = k$. Now let us define $c_{i+1}$ for each tuple $(y_1,\ldots,y_{i+1}) \in [n_1] \times \cdots \times [n_{i+1}]$. We put

$$c_{i+1}((y_1,\ldots,y_{i+1},1,\ldots,1)) = \begin{cases} (c_i((y_1,\ldots,y_i,1,\ldots,1)) + p(y_{i+1} - 1))(\mod k), & \text{if } p \neq k, \\ (c_i((y_1,\ldots,y_i,1,\ldots,1)) + y_{i+1} - 1)(\mod k), & \text{if } p = k. \end{cases}$$

Note that $c_{i+1}$ is proper. Indeed, the graph induced in $G_{i+1}$ by vertices with fixed coordinate $y_{i+1}$ is isomorphic to $G_i$ and is coloured according to $c_i$ (with permuted colours). Moreover, each edge $e$ of $G_{i+1}$ that is not an edge of any copy of $G_i$ (any of the $n_{i+1}$ layers of $G_{i+1}$ that are isomorphic to $G_i$), joins vertices from the consecutive layers of $G_{i+1}$. Hence $e$ has end vertices coloured with $j$ and $(j+p)(\mod k)$, when $p \neq k$ and $j$ and $(j+1)(\mod k)$, when $p = k$ (for some $j \in [k]$). In both cases these two colours are different. Thus $c_{i+1}$ is proper.

Next we have to observe that $c_{i+1}$ is equitable. Suppose that $p = k$. In this case each of $k$ colours arises in $c_i$ on the same number of vertices in $V_i$. Since in $G_{i+1}$ each of $n_{i+1}$ copies of $G_i$ is coloured in the same manner (with permuted colours) we can see that in the whole graph $G_{i+1}$ each colour arises the same number $(n_1 \cdots n_{i+1})/k$ of times. Consequently $c_{i+1}$ is equitable in this case. Now, suppose that $p \neq k$. Recall that the vertices of the first layers of $G_i$ are coloured in such a way that colours $1,\ldots,p$ arise one more than colours $p+1,\ldots,k$. In the second layer the colours $(p+1)(\mod k),\ldots,(p+p)(\mod k)$ arise one more than the remaining $k-p$ colours $(p+p+1)(\mod k),\ldots,(p+k-1)(\mod k)$ and so on. Thus we use colours cyclically, which guarantees that $c_{i+1}$ is equitable also in this case.

It is very easy to observe the following fact valid for all $d$-degenerate graphs.

**Theorem 5.5.** Let $d,k \in \mathbb{N}$. If $k \geq [(d+1)/2]$ then every $d$-degenerate graph is $k$-list arborable.

**Proof.** Let $k$ be fixed. We order vertices $v_1,\ldots,v_n$ of $G$ such that $\text{deg}_{G[[v_{i-n-1}]]}(v_i) \leq d$. Such an ordering always exists since $G$ is $d$-degenerate. Let $L$ be an arbitrary $k$-uniform list assignment for $G$. We construct an $L$-colouring of $G$ whose each colour class induces an acyclic subgraph of $G$. We do it, step by step, putting on a vertex $v_i$ a colour from its list that is not present more than once on previously coloured vertices $v_1,\ldots,v_{i-1}$. Since the size of each list is at least $[(d+1)/2]$, such a colour exists. Obviously, we obtained an $L$-colouring for $G$. Moreover, putting the colour on $v_i$ we do not produce any monochromatic cycle since $v_i$ has at most one neighbour in the colour of $v_i$. □

As we mentioned previously, a $d$-dimensional grid is $d$-degenerate graph and hence it is $k$-list arborable for every $k \geq [(d+1)/2]$, by Theorem 5.5. Furthermore, when $k \neq 1$, by Theorem 5.4, for a $d$-dimensional grid there is a $k$-colouring, in which each colour class is of the cardinality at most $|V(G)|/k$ and induces an acyclic graph (each edgeless graph is acyclic). These two facts and some other investigation yield the proposition of a general conjecture. If the conjecture is true then it improves our results for 3-dimensional and 4-dimensional grids.

**Conjecture 5.6.** Let $k,d \in \mathbb{N}$. If $k \geq [(d+1)/2]$ then every $d$-dimensional grid is equitably $k$-list arborable.

In general, we think that if a graph has a $k$-colouring in which each colour class is of the cardinality at most $|V(G)|/k$ and induces an acyclic graph, then it may not be equitably $k$-list arborable, even if it is $k$-list arborable. Thus we propose the following conjecture.
Conjecture 5.7. There is a graph $G$ and $k \in \mathbb{N}$ such that $G$ is $k$-list arborable and $G$ has a $k$-colouring in which each colour class is of the cardinality at most $\lceil |V(G)|/k \rceil$ and induces an acyclic graph, however $G$ is not equitably $k$-list arborable.

Note that the motivation of the paper came from Zhang’s conjectures, but along the way, we have obtained some new results on equitable $k$-choosability of grids.

References