Leaping Cauchy numbers

Takao Komatsu

School of Mathematics and Statistics, Wuhan University, Wuhan 430072 China

Abstract. We introduce leaping Cauchy numbers, that are generalizations of the analogous numbers to Euler numbers, as Cauchy numbers corresponds to Bernoulli numbers, in particular, in terms of determinant expressions. We also give several properties including sums of products.

The theory of determinants has been established and studied by many Mathematicians through the history. For example, T. Muir, who was remembered as an authority on determinants, studied and collected nearly all of the known facts about determinants up to the early 1930s. The basic facts, including permutations and combinations, general principles of simple determinants, compound determinants, co-factors, adjugates, rectangular arrays and matrices, and linear dependence can be found in [12]. His book the history of determinants [11], whose Volume 1 covered the origins to Leibniz in 1693 until 1840, was published in 1890. The remaining volumes were Volume 2 1840-1860 (1911), Volume 3 1860-1880 (1920), Volume 4 1880-1900 (1923), Volume 5 1900-1920 (1929). Muir was working on Volume 6 1920-1940 at the time of his death. Although many theories about determinants had been established, they have been unknown, ignored, or forgotten. Unfortunately, many Mathematicians has declared new results, though they had already been established. One can safely says that we have rediscovered the original results. However, it would be important to remember the first or the original results. Nowadays, it is really difficult to get the original papers. Nevertheless, we should or cite them through the definite sources including [2], [4] and [15] by way of [11].

In this paper by remembering the classical results, that have been often ignored, we shall introduce one kind of generalized numbers of the original Cauchy numbers, which are related to the Bernoulli numbers of the second kind, in particular, in terms of determinants.

The classical Cauchy numbers $c_n$ ([3, p.293]) may be defined by the generating function:

$$\frac{x}{\log(1 + x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$  

(1)

The numbers $b_n = c_n/n!$ are sometimes called Bernoulli numbers of the second kind (e.g. [5, 13]).

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Email address: komatsu@whu.edu.cn (Takao Komatsu)
It is known that Cauchy numbers have a determinant expression ([4, p.50]):

\[
c_n = n! \begin{vmatrix}
\frac{1}{1!} & 1 & 0 \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2} & 1 \\
\end{vmatrix}
\]  \tag{2}

On the other hand, Bernoulli numbers \(B_n\), determined by

\[
x e^x - 1 = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},
\]

also have a determinant expression ([4, p.53]):

\[
B_n = (-1)^n n! \begin{vmatrix}
\frac{1}{1!} & 1 & 0 \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2} & 1 \\
\end{vmatrix}
\]  \tag{3}

If we choose only the factorials of even numbers, this is reduced to a determinant expression of Euler numbers (cf. [4, p.52]):

\[
E_{2n} = (-1)^n (2n)! \begin{vmatrix}
\frac{1}{1!} & 1 & 0 \\
\vdots & \vdots & \ddots & 0 \\
\frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\end{vmatrix}
\]  \tag{4}

Here, classical Euler numbers \(E_n\) are defined by

\[
\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.
\]

One of the natural questions is what the number \(U_n\) is from Cauchy number \(c_n\), as Euler number \(E_n\) from Bernoulli number \(B_n\). Namely, what kind of number \(U_n\) is, expressing the following determinant expression?

\[
U_{2n} = (2n)! \begin{vmatrix}
\frac{1}{1} & 1 & 0 \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{1}{2n-2} & \frac{1}{2n-4} & \cdots & \frac{1}{3} & \frac{1}{2} \\
\end{vmatrix}
\]  \tag{5}

In this paper, we shall answer this natural question by introducing the leaping Cauchy numbers as one natural extension of the classical Cauchy numbers, in particular, in terms of deteterminanal expressions.
1. Definitions and preliminary properties

For a nonnegative integer \( m \), let \( F_m(x) \) be the partial summation of the logarithm function \( \log(1 + x) \), defined by

\[
F_m(x) = \sum_{n=1}^{m} (-1)^{n-1} \frac{x^n}{n}
\]

and \( F_{\text{odd},m}(x) \) be its odd version, defined by

\[
F_{\text{odd},m}(x) = \sum_{n=1}^{m} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}.
\]

Define the leaping Cauchy numbers \( U_n^{(m)} \) by

\[
\sum_{n=0}^{\infty} U_n^{(m)} \frac{x^n}{n!} = \left( 1 - \frac{(-1)^\frac{m}{2} \left( \log(1 + x^2) - F_{m/2}(x^2) \right)}{2x^m} \right)^{-1}
\]

if \( m \) is even, and

\[
\sum_{n=0}^{\infty} U_n^{(m)} \frac{x^n}{n!} = \left( 1 + \frac{(-1)^\frac{m-1}{2} \left( \arctan x - F_{\text{odd},m/2}(x) \right)}{x^m} \right)^{-1}
\]

if \( m \) is odd.

In particular, we have

\[
\sum_{n=0}^{\infty} U_n^{(0)} \frac{x^n}{n!} = \left( 1 - \frac{\log(1 + x^2)}{2} \right)^{-1}
\]

and

\[
\sum_{n=0}^{\infty} U_n^{(1)} \frac{x^n}{n!} = \frac{x}{\arctan x}.
\]

In [8, Theorem 1], different types of Cauchy numbers \( c_{n,\leq m} \) and \( c_{n,\geq m} \) are defined in terms of the partial function \( F_m(x) \):

\[
\sum_{n=0}^{\infty} c_{n,\leq m} \frac{x^n}{n!} = \frac{e^{F_m(x)} - 1}{F_m(x)}
\]

and

\[
\sum_{n=0}^{\infty} c_{n,\geq m} \frac{x^n}{n!} = \frac{e^{\log(1 + x) - F_{m-1}(x)} - 1}{\log(1 + x) - F_{m-1}(x)}.
\]

so that \( c_n = c_{n,\leq m} = c_{n,\geq m} \).

There is a recurrence relation among the leaping Cauchy numbers.

**Lemma 1.1.** For integers \( n \geq 1 \) and \( m \geq 0 \),

\[
U_n^{(m)} = \sum_{k=0}^{n-1} (-1)^{n-k-1}(2n)! \frac{U_k^{(m)}}{(2n - 2k + m)(2k)!} U_{2k}^{(m)}
\]

with \( U_0^{(m)} = 1 \).
Proof. By the definition, we see that for any nonnegative integer \( m \),
\[
\sum_{n=0}^{\infty} U_n^{(m)} \frac{x^n}{n!} = \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n + m} \right)^{-1}.
\] (6)

By this definition, \( U_n^{(m)} = 0 \) if \( n \) is odd. Therefore,
\[
1 = \left( \sum_{n=0}^{\infty} U_n^{(m)} \frac{x^{2n}}{(2n)!} \right) \left( 1 + \sum_{l=1}^{\infty} \frac{(-1)^l x^{2l}}{2l + m} \right) = \sum_{n=0}^{\infty} U_{2n}^{(m)} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{(-1)^{n-k} U_{2k}^{(m)}}{(2n - 2k + m)(2k)!} x^{2n}.
\]
Comparing the coefficients on both sides, we have \( U_{2n}^{(m)} = 1 \) and for \( n \geq 1 \)
\[
\frac{U_{2n}^{(m)}}{(2n)!} + \sum_{k=0}^{n-1} \frac{(-1)^{n-k} U_{2k}^{(m)}}{(2n - 2k + m)(2k)!} = 0.
\]

The leaping Cauchy numbers have an explicit expression.

**Theorem 1.2.** For integers \( n \geq 1 \) and \( m \geq 0 \),
\[
U_n^{(m)} = (2n)! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\eta_1+\cdots+\eta_k=m} 1 \frac{1}{(2\eta_1 + m) \cdots (2\eta_k + m)}.
\] (7)

Proof. We shall prove that
\[
U_n^{(m)} = (2n)! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\eta_1+\cdots+\eta_k=m} 1 \frac{1}{(2\eta_1 + m) \cdots (2\eta_k + m)}.
\] (7)

When \( n = 1 \), by Lemma 1.1, we have
\[
U_2^{(m)} = \frac{2m+2}{m+2} U_0^{(m)} = \frac{2m+2}{m+2}.
\]
This matches the result of (7) for \( n = 1 \). Assume that (7) is valid up to \( n - 1 \). Then by Lemma 1.1, we have
\[
\frac{U_n^{(m)}}{(2n)!} = \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{2n + m} U_0^{(m)} + \sum_{l=1}^{n-1} \frac{(-1)^{n-l-1}}{2n - 2l + m} \sum_{k=1}^{l} (-1)^{k} \sum_{\eta_1+\cdots+\eta_k=m} 1 \frac{1}{(2\eta_1 + m) \cdots (2\eta_k + m)}
\]
\[
= \frac{(-1)^{n-1}}{2n + m} U_0^{(m)} + \sum_{l=1}^{n-1} (-1)^{k-1} \sum_{l=1}^{n-1} (-1)^{n-l} \sum_{\eta_1+\cdots+\eta_k=m} 1 \frac{1}{(2\eta_1 + m) \cdots (2\eta_k + m)}
\]
\[
\frac{1}{m+2} \begin{array}{cccc}
\frac{1}{m+4} & \frac{1}{m+2} & 0 \\
\vdots & \vdots & \ddots & 0 \\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+2} & \frac{1}{m+2} \\
\end{array}
\] (8)

Remark 2.2. When \( m = 0 \) in Theorem 2, the result is reduced to (5), where \( U_{2n} = U_{2n}^{(0)} \). It is known ([7, 10]) that Euler numbers of the second kind \( \hat{E}_n \), defined by

\[
\frac{x}{\sinh x} = \sum_{n=0}^{\infty} \hat{E}_n \frac{x^n}{n!},
\]

have the determinant expression

\[
\frac{1}{m+2} \begin{array}{cccc}
\frac{1}{m+4} & 1 & 0 \\
\vdots & \ddots & \ddots & 0 \\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+2} & \frac{1}{m+2} \\
\end{array}
\] 

When \( m = 1 \) in Theorem 2, we have its analogous expression:

\[
\frac{1}{m+2} \begin{array}{cccc}
\frac{1}{m+4} & 1 & 0 \\
\vdots & \ddots & \ddots & 0 \\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+2} & \frac{1}{m+2} \\
\end{array}
\]
Proof. For simplicity, put $\tilde{U}^{(m)}_n = U^{(m)}_n / n!$ and prove that

$$
\tilde{U}^{(m)}_{2n} = \begin{bmatrix}
\frac{1}{m+2} & 1 & 0 \\
\frac{1}{m+4} & \frac{1}{m+2} & \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+2} & 1 \\
\end{bmatrix}.
$$

(9)

By Theorem 1.2, we see that

$$
\tilde{U}^{(m)}_2 = \frac{1}{m+2}.
$$

Assume that (9) is valid up to $n - 1$. By Lemma 1.1,

$$
\tilde{U}^{(m)}_{2n} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{2n-2k+m} \tilde{U}^{(m)}_{2k}
$$

with $\tilde{U}^{(m)}_0 = 1$. Expanding at the first row of the right-hand side of (9), we have

$$
\frac{U^{(m)}_{2n-2}}{m+2} - \begin{bmatrix}
\frac{1}{m+4} & 1 & 0 \\
\frac{1}{m+6} & \frac{1}{m+2} & \\
\vdots & \vdots & \ddots & 1 & 0 \\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+2} & 1 \\
\end{bmatrix} = \frac{U^{(m)}_{2n-2}}{m+4} - \frac{U^{(m)}_{2n-4}}{m+4} + \cdots + (-1)^{n-2} \begin{bmatrix}
\frac{1}{m+2n-2} & 1 & \\
\end{bmatrix}
$$

$$
= \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{2n-2k+m} U^{(m)}_{2k} = U^{(m)}_{2n}.
$$

\[ \square \]

3. Table of $U^{(m)}_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
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<td>$U^{(0)}_n$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>30</td>
<td>-840</td>
<td>60480</td>
<td>6153840</td>
</tr>
<tr>
<td>$U^{(1)}_n$</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>704</td>
<td>-54784</td>
<td>269376</td>
<td>109763256</td>
</tr>
<tr>
<td>$U^{(2)}_n$</td>
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<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{7}$</td>
<td>-48</td>
<td>48</td>
<td>4136</td>
</tr>
<tr>
<td>$U^{(3)}_n$</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{9}$</td>
<td>-1613872</td>
<td>1368573552</td>
</tr>
<tr>
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<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{11}$</td>
<td>-1368573552</td>
</tr>
<tr>
<td>$U^{(5)}_n$</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{11}$</td>
<td>$\frac{1}{13}$</td>
</tr>
<tr>
<td>$U^{(6)}_n$</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{11}$</td>
<td>$\frac{1}{13}$</td>
</tr>
</tbody>
</table>

The following properties are easily seen.

Theorem 3.1. For $m \geq 0$

$$
U^{(m)}_2 = \frac{2}{m+2}.
$$
The sequence of numerators of \( U_{2n}^{(1)} / (2n)! \) \((n \geq 1)\) is seen in [16, A216272], and the sequence of denominators of \( U_{2n}^{(1)} / (2n)! \) \((n \geq 2)\) is seen in [16, A195466]. We have an explicit expression:

\[
\frac{U_{2n}^{(1)}}{(2n)!} = \frac{(-1)^{n+1}}{2n-1} \sum_{l=0}^{2n-1} \frac{2^{l+1}}{l+1} \left\{ \begin{array}{c} l+1 \\ k \end{array} \right\} \frac{1}{(l+k)!} \quad (n \geq 1),
\]

where \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) denotes the Stirling numbers of the second kind given by

\[
\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} (k) \binom{k}{j}.
\]

and \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) denotes the (unsigned) Stirling numbers of the first kind arising as coefficients of the rising factorial

\[
x(x + 1) \cdots (x + n - 1) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k.
\]

4. Applications by the Trudi’s formula

We shall use the Trudi’s formula to obtain different explicit expressions and inversion relations for the numbers \( B_n^{(m)} \).

**Lemma 4.1.** For a positive integer \( n \), we have

\[
\begin{vmatrix}
  a_1 & a_0 & 0 & \cdots \\
a_2 & a_1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots \\
a_{n-1} & \cdots & a_1 & a_0 \\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix} = \sum_{t_1+2t_2+\cdots+mt_m=n} \left( t_1 + \cdots + t_m \right) \left(-a_0\right)^{n-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m},
\]

where \( \left( t_1 + \cdots + t_m \right) = \frac{t_1 + \cdots + t_m}{t_1 + \cdots + t_m} \) are the multinomial coefficients.

This relation is known as Trudi’s formula [11, Vol.3, p.214], [15] and the case \( a_0 = 1 \) of this formula is known as Brioschi’s formula [2], [11, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g. [9]), which is based upon the relation:

\[
\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n - k) = 0 \quad (n \geq 1).
\]

**Lemma 4.2.** If \( \{\alpha_n\}_{n \geq 0} \) is a sequence defined by \( \alpha_0 = 1 \) and

\[
\alpha_n = \begin{vmatrix}
  D(1) & 1 & & \\
  D(2) & \ddots & \ddots & \\
  \vdots & \ddots & \ddots & 1 \\
  D(n) & \cdots & D(2) & D(1)
\end{vmatrix}, \quad \text{then } D(n) = \begin{vmatrix}
  \alpha_1 & 1 & & \\
  \alpha_2 & \ddots & \ddots & \\
  \vdots & \ddots & \ddots & 1 \\
  \alpha_n & \cdots & \alpha_2 & \alpha_1
\end{vmatrix}.
\]


From Trudi’s formula, it is possible to give the combinatorial expression

\[ \alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1 + \cdots + t_n}{t_1, \ldots, t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}. \]

By applying these lemmata to Theorem 8, we obtain an explicit expression for leaping Cauchy numbers.

**Theorem 4.3.** For \( n \geq m \geq 1 \), we have

\[ U_{2n}^{(m)} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1 + \cdots + t_n}{t_1, \ldots, t_n} \times (-1)^{n-t_1-\cdots-t_n} \left( \frac{1}{m+2} \right)^{t_1} \cdots \left( \frac{1}{m+2n} \right)^{t_n}. \]

By applying the inversion relation in Lemma 4.2 to Theorem 8, we have the following.

**Theorem 4.4.** For \( n \geq 1 \), we have

\[ \begin{vmatrix} \frac{U^{(0)}_{2n}}{2} & 1 & 0 \\ \frac{U^{(0)}_{2n-2}}{4} & \frac{U^{(0)}_{2n-4}}{2} \\ \vdots & \vdots & \ddots & 1 \\ \frac{U^{(0)}_{2n-2}}{2(2n-2)!} & \frac{U^{(0)}_{2n-4}}{2(2n-4)!} & \cdots & \frac{U^{(0)}_{2n}}{2} \end{vmatrix}. \]

5. Convolution identities

It is known ([17, Theorem 2.1]) that Cauchy numbers \( c_n \) satisfy the identity

\[ \sum_{k=0}^{n} \binom{n}{k} c_{n-k} = -(n-1)c_n - n(n-2)c_{n-1} \quad (n \geq 0). \]

Leaping Cauchy numbers satisfy the following relations.

**Theorem 5.1.** For any nonnegative integer \( n \), we have

\[ \sum_{k=0}^{n} \binom{n}{k} U_{n-k}^{(0)} U_{n+k}^{(0)} = \frac{U_{n+2}^{(0)}}{n+1} + n U_{n}^{(0)}, \]

(10)

\[ \sum_{k=0}^{n} \binom{n}{k} U_{n-k}^{(1)} U_{n+k}^{(1)} = -(n-1)U_{n}^{(1)} - n(n-1)(n-3)U_{n-2}^{(1)} \]

(11)

and for \( m \geq 2 \)

\[ \sum_{k=0}^{n} \binom{n}{k} U_{n-k}^{(m)} U_{n+k}^{(m)} = \frac{m-n}{m-1} U_{n}^{(m)} + \frac{1}{m(m-1)} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^{j} \binom{m-1}{j} \frac{n!(n-2j-m)}{(n-2j)!} U_{n-2j}^{(m)}. \]

(12)

**Remark 5.2.** When \( n \) is odd, both sides of any of (10), (11) and (12) are equal to 0.
Proof. Put
\[
  u(x) = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n + m}\right)^{-1}.
\]
We have
\[
  u'(x) = u^2(x) \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2n + m}
  = -u^2(x) \left( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n + m} - \frac{m}{x} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n + m} \right)
  = -u^2(x) \left( \frac{x}{1 + x^2} - \frac{m}{x} \left( \frac{1}{u(x)} - 1 \right) \right)
  = u^2(x) \left( \frac{x}{1 + x^2} + \frac{m}{x} u(x) \right).
\]
If \(m = 0\), we have
\[
  u'(x) = u^2(x) \frac{x}{1 + x^2}
\]
or
\[
  u^2(x) = \frac{1}{x} u'(x) + xu'(x).
\]
Hence, we get
\[
  \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} u^{(0)}_k u^{(0)}_{n-k} x^n n!
  = \frac{1}{x} \sum_{n=0}^{\infty} u^{(0)}_{n+1} x^n n! + x \sum_{n=0}^{\infty} u^{(0)}_{n+1} x^n n!
  = \sum_{n=0}^{\infty} u^{(0)}_{n+2} x^n n! + \sum_{n=0}^{\infty} n u^{(0)}_{n+1} x^n n!.
\]
Comparing the coefficients on both sides, we obtain the identity (10).
If \(m = 1\), we have
\[
  u'(x) = -\frac{u^2(x)}{x(1 + x^2)} + \frac{u(x)}{x}
\]
or
\[
  u^2(x) = (1 + x^2)u(x) - x(1 + x^2)u'(x).
\]
Hence, we get
\[
  \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} u^{(1)}_k u^{(1)}_{n-k} x^n n!
  = \sum_{n=0}^{\infty} u^{(1)}_{n+1} x^n n! - x^2 \sum_{n=0}^{\infty} u^{(1)}_{n+1} x^n n! - x^3 \sum_{n=1}^{\infty} u^{(1)}_{n+2} x^n n!
  = \sum_{n=0}^{\infty} u^{(1)}_{n+1} x^n n! + \sum_{n=0}^{\infty} n! \frac{u^{(1)}_{n+2} x^n}{(n-2)!} - \sum_{n=0}^{\infty} n u^{(1)}_{n+1} x^n n! - \sum_{n=1}^{\infty} (n-3)! \frac{u^{(1)}_{n+2} x^n}{n!}
  = -\sum_{n=0}^{\infty} (n-1) u^{(1)}_{n+1} x^n n! - \sum_{n=0}^{\infty} n(n-1)(n-3) u^{(1)}_{n+2} x^n n!.
\]
Comparing the coefficients on both sides, we obtain the identity (11).
If \(m \geq 2\), we have
\[
  u^2(x) = \frac{m(1 + x^2)}{m + (m-1)x^2} u(x) - \frac{x(1 + x^2)}{m + (m-1)x^2} u'(x).
\]
The first term on the right-hand side is equal to

\[
\frac{m}{m-1} \left( 1 - \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3} x^2 \right) u(x) = \frac{m}{m-1} u(x) - \frac{1}{m-1} \sum_{i=0}^{\infty} \left( -1 \right)^i \left( \frac{m-1}{m} \right)^i x^n u(x)
\]

\[
= \frac{m}{m-1} \sum_{n=0}^{\infty} \binom{n}{m} \frac{x^n}{n!} - \frac{1}{m-1} \sum_{i=0}^{\infty} \left( -1 \right)^i \left( \frac{m-1}{m} \right)^i \sum_{n=0}^{\infty} \frac{(n+2l)!}{n!} U_{n+1}^{(m)} \frac{x^{n+2l}}{(n+2l)!}
\]

\[
= \frac{m}{m-1} \sum_{n=0}^{\infty} \binom{n}{m} \frac{x^n}{n!} - \frac{1}{m-1} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^i \left( \frac{m-1}{m} \right)^i \frac{n!}{(n-2l)!} U_{n-2l}^{(m)} \frac{x^n}{n!}.
\]

The second term on the right-hand side is equal to

\[
- \frac{x}{m-1} \frac{1}{1 + \frac{1}{m} + \frac{1}{m^2} x^2} u'(x)
\]

\[
= - \frac{x}{m-1} \sum_{n=0}^{\infty} \binom{n}{m} \frac{x^n}{n!} + \frac{1}{m(m-1)} \sum_{l=0}^{\infty} (-1)^i \left( \frac{m-1}{m} \right)^i \sum_{n=0}^{\infty} \frac{(n+2l+1)!}{n!} U_{n+1}^{(m)} \frac{x^{n+2l+1}}{(n+2l+1)!}
\]

\[
= - \frac{1}{m-1} \sum_{n=0}^{\infty} \frac{n!}{(n-2l)!} U_{n-2l}^{(m)} \frac{x^n}{n!} - \frac{1}{m(m-1)} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (-1)^i \left( \frac{m-1}{m} \right)^i \frac{n!}{(n-2l)!} U_{n-2l}^{(m)} \frac{x^n}{n!}.
\]

Therefore, we obtain

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \binom{m}{n-k} \frac{x^n}{n!}
\]

\[
= \frac{1}{m-1} \sum_{n=0}^{\infty} (m-n) U_{n}^{(m)} \frac{x^n}{n!} + \frac{1}{m(m-1)} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-1)^i \left( \frac{m-1}{m} \right)^i \frac{n!}{(n-2l)!} (n-2l-m) U_{n-2l}^{(m)} \frac{x^n}{n!}.
\]

Comparing the coefficients on both sides, we obtain the identity (12). □

References

[4] J. W. L. Glaisher, Expressions for Laplace’s coefficients, Bernoullian and Eulerian numbers etc. as determinants, Messenger (2) 6 (1875), 49–63.