Generalizations of the Aluthge Transform of Operators

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Abstract. Let \( A \) be an operator with the polar decomposition \( A = U|A| \). The Aluthge transform of the operator \( A \), denoted by \( \tilde{A} \), is defined as \( \tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}. \) In this paper, first we generalize the definition of Aluthge transform for non-negative continuous functions \( f, g \) such that \( f(x)g(x) = x \) (\( x \geq 0 \)). Then, by using this definition, we get some numerical radius inequalities. Among other inequalities, it is shown that if \( A \) is bounded linear operator on a complex Hilbert space \( \mathbb{H} \), then
\[
h(w(A)) \leq \frac{1}{4} \left\| h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right) \right\| + \frac{1}{2} h\left(f(|A|g(|A|))\right),
\]
where \( f, g \) are non-negative continuous functions such that \( f(x)g(x) = x \) (\( x \geq 0 \)), \( h \) is a non-negative and non-decreasing convex function on \([0, \infty)\) and \( \tilde{A}_{fg} = f(|A|)Ug(|A|) \).

1. Introduction and preliminaries

2. Introduction

Let \( \mathcal{B}(\mathbb{H}) \) denotes the \( C^\ast \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathbb{H} \) with an inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \). In the case when \( \dim \mathbb{H} = n \), we identify \( \mathcal{B}(\mathbb{H}) \) with the matrix algebra \( M_n \) of all \( n \times n \) matrices with entries in the complex field. For an operator \( A \in \mathcal{B}(\mathbb{H}) \), let \( A = U|A| \) (\( U \) is a partial isometry with \( \text{ker} U = \text{range} |A|^\perp \)) be the polar decomposition of \( A \). The Aluthge transform of the operator \( A \), denoted by \( \tilde{A} \), is defined as \( \tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}. \) In [7, 21], a more general notion called \( t \)-Aluthge transform has been introduced which has later been studied. This is defined for any \( 0 < t \leq 1 \) by \( \tilde{A}_t = |A|^{\frac{t}{2}}U|A|^{\frac{1-t}{2}}. \) Clearly, for \( t = \frac{1}{2} \) we obtain the usual Aluthge transform. For the case \( t = 1 \), the operator \( \tilde{A}_1 = |A|U \) is called the Duggal transform of \( A \in \mathcal{B}(\mathbb{H}) \). For \( A \in \mathcal{B}(\mathbb{H}) \), we generalize the Aluthge transform of the operator \( A \) to the form
\[
\tilde{A}_{fg} = f(|A|)Ug(|A|),
\]
in which \( f, g \) are non-negative continuous functions such that \( f(x)g(x) = x \) (\( x \geq 0 \)). The numerical radius of \( A \in \mathcal{B}(\mathbb{H}) \) is defined by
\[
w(A) := \sup\{\|\langle Ax, x \rangle\| : x \in \mathbb{H}, \|x\| = 1\}.
\]
It is well known that \( w(\cdot) \) defines a norm on \( B(H) \), which is equivalent to the usual operator norm \( \| \cdot \| \). In fact, for any \( A \in B(H) \), \( \frac{1}{2} \|A\| \leq w(A) \leq \|A\| \); see [8]. Let \( r(\cdot) \) denote the spectral radius. It is well known that for every operator \( A \in B(H) \), we have \( r(A) \leq w(A) \). An important inequality for \( w(A) \) is the power inequality stating that \( w(A^n) \leq w(A)^n \) (\( n = 1, 2, \cdots \)). For further information about the numerical radius we refer the reader to [10–12] and references therein. The quantity \( w(A) \) is useful in studying perturbation, convergence and approximation problems as well as integrative methods, etc. For more information see [3, 6, 9, 13–15, 17].

Let \( A, B, C, D \in B(H) \). The operator matrices \[
\begin{bmatrix}
A & 0 & 0 & D \\
0 & B & C & 0
\end{bmatrix}
\]
are called the diagonal and off-diagonal parts of the operator matrix
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]
respectively.

In [16], it has been shown that if \( A \) is an operator in \( B(H) \), then
\[
w(A) \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{1/2} \right).
\]
(1)

Several refinements and generalizations of inequality (1) have been given; see [1, 4, 5, 21–24]. Yamazaki [22] showed that for \( A \in B(H) \) and \( t \in [0, 1] \) we have
\[
w(A) \leq \frac{1}{2} \left( \|A\| + w(\tilde{A}_t) \right).
\]
(2)

Davidson and Power [7] proved that if \( A \) and \( B \) are positive operators in \( B(H) \), then
\[
\|A + B\| \leq \max \{|\|A\|, |\|B\|\} + |\|AB\||^{1/2}.
\]
(3)

Inequality (3) has been generalized in [2, 20] and improved in [18, 19]. In [20], the author extended this inequality to the form
\[
\|A + B^*\| \leq \max \{|\|A\|, |\|B\|\} + \frac{1}{2} \left( |\|A^tB^{t-1}\|| + |\|A^{t-1}B^t\||\right),
\]
(4)
in which \( A, B \in B(H) \) and \( t \in [0, 1] \).

In this paper, by applying the generalized Aluthge transform of operators, we establish some inequalities involving the numerical radius. In particular, we extend inequalities (2) and (4) for two non-negative continuous functions. We also show some upper bounds for the numerical radius of 2 \( \times \) 2 operator matrices.

3. main results

To prove our numerical radius inequalities, we need several known lemmas.

**Lemma 3.1.** [1, Theorem 2.2] Let \( X, Y, S, T \in B(H) \). Then
\[
r(XY + ST) \leq \frac{1}{2} \left( w(XY) + w(TS) \right) + \frac{1}{2} \sqrt{(w(XY) - w(TS))^2 + 4\|YS\|\|TX\|}.
\]

**Lemma 3.2.** [16, 22] Let \( A \in B(H) \). Then
\[
\begin{align*}
(a) \ w(A) & = \max_{\theta \in \mathbb{R}} \left\| \text{Re} \left( e^{i\theta} A \right) \right\|, \\
(b) \ w \left( \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) & = \frac{1}{2} \|A\|.
\end{align*}
\]
Proof. Let \( \langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} \left\| x + \hat{\theta} y \right\|^2 i^k \).

Now, we are ready to present our first result. The following theorem shows a generalization of inequality (2).

**Theorem 3.3.** Let \( A \in \mathbb{B}(\mathbb{H}) \) and \( f, g \) be two non-negative continuous functions on \([0, \infty)\) such that \( f(x)g(x) = x \) \((x \geq 0)\). Then, for all non-negative and non-decreasing convex function \( h \) on \([0, \infty)\), we have

\[
 h(w(A)) \leq \frac{1}{4} \left\| h \left( g^2 (|A|) \right) + h \left( f^2 (|A|) \right) \right\| + \frac{1}{2} h \left( \tilde{A}_{f,g} \right).
\]

**Proof.** Let \( x \) be any unit vector. Then

\[
\text{Re} \left\langle e^i \theta Ax, x \right\rangle = \text{Re} \left\langle e^i \theta U |A| x, x \right\rangle = \text{Re} \left\langle e^i \theta U g (|A|) f (|A|) x, x \right\rangle = \text{Re} \left\langle e^i \theta f (|A|) x, g (|A|) U^* x \right\rangle = \frac{1}{4} \left\| \left( e^i \theta f (|A|) + g (|A|) U^* \right) x \right\|^2 - \frac{1}{4} \left\| \left( e^i \theta f (|A|) - g (|A|) U^* \right) x \right\|^2 \quad \text{(by polarization identity)}
\]

\[
\leq \frac{1}{4} \left\| \left( e^i \theta f (|A|) + g (|A|) U^* \right) x \right\|^2 - \frac{1}{4} \left\| \left( e^i \theta f (|A|) - g (|A|) U^* \right) x \right\|^2.
\]

Now, taking the supremum over all unit vectors \( x \in \mathbb{H} \) and applying Lemma 3.2 in the above inequality produces

\[
w(A) \leq \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) \right\| + \frac{1}{2} w \left( \tilde{A}_{f,g} \right).
\]
Therefore,

\begin{align*}
  h \left( w(A) \right) & \leq h \left( \frac{1}{4} \left\| g^2(\|A\|) + f^2(\|A\|) \right\| + \frac{1}{2} w(\tilde{A}_{f,s}) \right) \\
  &= h \left( \frac{1}{2} \left\| g^2(\|A\|) + f^2(\|A\|) \right\| + \frac{1}{2} w(\tilde{A}_{f,s}) \right) \\
  \leq & \frac{1}{2} h \left( \left\| g^2(\|A\|) + f^2(\|A\|) \right\| + \frac{1}{2} h(\tilde{w}(\tilde{A}_{f,s})) \right) \\
  &= \frac{1}{2} \left\| h \left( \frac{g^2(\|A\|) + f^2(\|A\|)}{2} \right) \right\| + \frac{1}{2} h(\tilde{w}(\tilde{A}_{f,s})) \\
  \leq & \frac{1}{4} \left\| h \left( g^2(\|A\|) + f^2(\|A\|) \right) \right\| + \frac{1}{2} h(\tilde{w}(\tilde{A}_{f,s})) \\
  \text{(by the convexity of } h) \\
\end{align*}

\[ \frac{1}{2} \left\| h \left( \frac{g^2(\|A\|) + f^2(\|A\|)}{2} \right) \right\| + \frac{1}{2} h(\tilde{w}(\tilde{A}_{f,s})) \]

\[ \text{(by the convexity of } h). \]

\[ \square \]

Theorem 3.3 includes some special cases as follows.

**Corollary 3.4.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then, for all non-negative and non-decreasing convex function \( h \) on \([0, \infty)\) and all \( t \in [0, 1] \), we have

\[ h \left( w(A) \right) \leq \frac{1}{4} \left\| h \left( |A|^{2t} \right) + h \left( |A|^{2(1-t)} \right) \right\| + \frac{1}{2} h \left( w \left( \tilde{A}_t \right) \right). \]

\[ (5) \]

**Corollary 3.5.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then, for all \( t \in [0, 1] \) and \( s \geq 1 \), we have

\[ w^s(A) \leq \frac{1}{4} \left\| |A|^{2s} + |A|^{2(1-t)s} \right\| + \frac{1}{2} w^s \left( \tilde{A}_t \right). \]

In particular,

\[ w^s(A) \leq \frac{1}{2} \left( \| |A| \|^s + w^s \left( \tilde{A} \right) \right). \]

**Proof.** The first inequality follows from inequality (5) for the function \( h(x) = x^t \) \((s \geq 1)\). For the particular case, it is enough to put \( t = \frac{1}{2} \). \[ \square \]

Theorem 3.3 gives the next result for the off-diagonal operator matrix \( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \).

**Theorem 3.6.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \), \( f, g \) be two non-negative continuous functions on \([0, \infty)\) such that \( f(x)g(x) = x \) \((x \geq 0)\) and \( s \geq 1 \). Then

\[ w^s \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{1}{4} \max \left( \left\| g^{2s}(\|A\|) + f^{2s}(\|A\|) \right\|, \left\| g^{2s}(\|B\|) + f^{2s}(\|B\|) \right\| \right) \]

\[ + \frac{1}{4} \left( \| f(\|B\|)g(\|A\|) \|^s + \| f(\|A\|)g(\|B\|) \|^s \right). \]
Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of $A$ and $B$, respectively, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$.

It follows from the polar decomposition of $T$ that

$$
\hat{T}_{f,g} = f(\|T\|) \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} g(\|T\|)
$$

$$
= \begin{bmatrix} f(|B|) & 0 \\ 0 & f(|A|) \end{bmatrix} \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} g(|B|) & 0 \\ 0 & g(|A|) \end{bmatrix}
$$

$$
= \begin{bmatrix} f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) \end{bmatrix}.
$$

Using $|A|^* = AA^* = U|A|^2U^*$ and $|B|^* = BB^* = V|B|^2V^*$ we have $g(|A|) = U^*g(|A^*|)U$ and $g(|B|) = V^*g(|B^*|)V$ for every non-negative continuous function $g$ on $[0, \infty)$. Therefore,

$$
\omega(\hat{T}_{f,g}) = \omega\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix} \right)
$$

$$
\leq \omega\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix} \right) + \omega\left( \begin{bmatrix} 0 & 0 \\ f(|A|)Vg(|B|) & 0 \end{bmatrix} \right)
$$

$$
= \omega\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix} \right) + \omega\left( \begin{bmatrix} W^* & 0 \\ 0 & f(|A|)Vg(|B|) \end{bmatrix} \right)
$$

$$
= \frac{1}{2}\|f(|B|)Ug(|A|)|| + \frac{1}{2}\|f(|A|)Vg(|B|)||
$$

(by Lemma 3.2(b))

$$
= \frac{1}{2}\|f(|B|)UU^*g(|A^*|)U\| + \frac{1}{2}\|f(|A|)VV^*g(|B^*|)V\|
$$

$$
\leq \frac{1}{2}\|f(|B|)g(|A^*|)|| + \frac{1}{2}\|f(|A|)g(|B^*|)||
$$

(6)

where $W = \begin{bmatrix} 0 & 1 \\ I & 0 \end{bmatrix}$ is unitary. Applying Theorem 3.3 and inequality (6), we have

$$
\omega^r(T) \leq \frac{1}{4}\left\|g^{2\ast}(\|T\|) + f^{2\ast}(\|T\|)\right\| + \frac{1}{2}\left(\omega^r(\hat{T}_{f,g})\right)
$$

$$
\leq \frac{1}{4}\max\left(\left\|g^{2\ast}(|A|) + f^{2\ast}(|A|)\right\|, \left\|g^{2\ast}(|B|) + f^{2\ast}(|B|)\right\|\right)
$$

$$
+ \frac{1}{2}\left(\frac{1}{2}\left\|f(|B|)g(|A^*|)|| + \|f(|A|)g(|B^*|)\||\right\|\right)
$$

$$
\leq \frac{1}{4}\max\left(\left\|g^{2\ast}(|A|) + f^{2\ast}(|A|)\right\|, \left\|g^{2\ast}(|B|) + f^{2\ast}(|B|)\right\|\right)
$$

$$
+ \frac{1}{4}\left\|f(|B|)g(|A^*|)|| + \frac{1}{4}\left\|f(|A|)g(|B^*|)\||\right\|\right\|\right)
$$

(by the convexity $h(x) = x^r$).
Applying a commutativity property of the spectral radius, we get
\[ w^2(AB) \leq \frac{1}{4} \max \left( \|A^{2s} + |A|^{2(1-t)s}\|, \|B^{2s} + |B|^{2(1-t)s}\| \right) \]
\[ + \frac{1}{4} \left( \|A^\dagger |B^\dagger|^{1-t}\| + \|B^\dagger |A^\dagger|^{1-t}\| \right). \]

**Proof.** Applying the power inequality of the numerical radius \((w(A^n) \leq w^n(A))\), we have
\[ w^2(AB) \leq \max \left( w^2(AB), w^2(BA) \right) \]
\[ = w^2 \left( \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix} \right) \]
\[ = w^2 \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \]
\[ \leq \frac{1}{4} \max \left( \|A^{2s} + |A|^{2(1-t)s}\|, \|B^{2s} + |B|^{2(1-t)s}\| \right) \]
\[ + \frac{1}{4} \left( \|A^\dagger |B^\dagger|^{1-t}\| + \|B^\dagger |A^\dagger|^{1-t}\| \right) \]
(by Theorem 3.6).

**Corollary 3.8.** Let \(A, B \in B(H)\) be positive operators. Then, for all \(t \in [0, 1]\) and \(s \geq 1\), we have
\[ \left\| A^{\frac{i}{2}} B^{\frac{i}{2}} \right\| \leq \frac{1}{4} \max \left( \|A^{is} + A^{1-t}B^{1-t}\|, \|B^{is} + B^{1-t}A^{1-t}\| \right) \]
\[ + \frac{1}{4} \left( \|A^\dagger B^\dagger|^{1-t}\| + \|B^\dagger A^\dagger|^{1-t}\| \right). \]

**Proof.** Since the spectral radius of any operator is dominated by its numerical radius, then \(r^2(AB) \leq w^2(AB)\).
Applying a commutativity property of the spectral radius, we get
\[ r^2(AB) = r^2 \left( A^{\frac{i}{2}} A^{\frac{i}{2}} B^{\frac{i}{2}} B^{\frac{i}{2}} \right) \]
\[ = r^2 \left( A^{\frac{i}{2}} B^{\frac{i}{2}} A^{\frac{i}{2}} B^{\frac{i}{2}} \right) \]
\[ = r^2 \left( A^{\frac{i}{2}} B^{\frac{i}{2}} \left( A^{\frac{i}{2}} B^{\frac{i}{2}} \right)^\dagger \right) \]
\[ = \left\| A^{\frac{i}{2}} B^{\frac{i}{2}} \left( A^{\frac{i}{2}} B^{\frac{i}{2}} \right)^\dagger \right\|^2 \]
\[ = \left\| A^{\frac{i}{2}} B^{\frac{i}{2}} \right\|. \]
(7)

Now, the result follows from Corollary 3.7.

An important special case of Theorem 3.6, which generalizes inequality (4) can be stated as follows.

**Corollary 3.9.** Let \(A, B \in B(H)\) and \(s \geq 1\). Then
\[ \|A + B^n\| \leq \frac{1}{2s^n} \max \left( \|A^{2s} + |A|^{2(1-t)s}\|, \|B^{2s} + |B|^{2(1-t)s}\| \right) \]
\[ + \frac{1}{2s^n} \left( \|A^\dagger |B^\dagger|^{1-t}\| + \|B^\dagger |A^\dagger|^{1-t}\| \right). \]
In particular, if $A$ and $B$ are normal, then
\[
\|A + B\| \leq \frac{1}{2^{1-s}} \max(\|A\|^s, \|B\|^s) + \frac{1}{2^{1-s}} \|AB\|^s.
\]

Proof. Applying Lemma 3.2 and Theorem 3.3, we have
\[
\|A + B^*\|^s = \|T + T^*\|^s
\leq 2^s \max_{\theta \in \mathbb{R}} \|\text{Re}(e^{i\theta} T)\|^s
= 2^s \omega^s(T)
\leq \frac{2^s}{4} \max \left( \|A\|^{2s} + \|A\|^{2(1-t)s}, \|B\|^{2s} + \|B\|^{2(1-t)s} \right)
+ \frac{2^s}{4} \left( \|A\|^{1-t} \|A\|^{1-t} \right)
\text{(by Theorem 3.6)},
\]
where $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, $f(x) = x^t$, and $g(x) = x^{1-t}$. Now, the desired result follows by replacing $B$ by $B^*$. For the particular case $t = \frac{1}{2}$. If $A$ and $B$ are normal, then $\|B^*\| = \|B\|$ and $\|A^*\| = \|A\|$. Applying equality (7) for the operators $\|A\|^\frac{1}{2}$ and $\|B\|^\frac{1}{2}$, we have
\[
\|\|A\| \|B\|\|^s \leq \|A\| \|B\|^{\frac{s}{2}}
\leq \|A\| \|B\|^{\frac{s}{2}}
= \|U'AB'V\|^\frac{s}{2}
= \|AB^*\|^\frac{s}{2},
\]
where $A = U|A|$ and $B = V|B|$ are the polar decompositions of the operators $A$ and $B$. This completes the proof of the corollary.

In the next result, we show another generalization of inequality (2).

**Theorem 3.10.** Let $A \in B(H)$ and $f, g, h$ be non-negative and non-decreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$) and $h$ is convex. Then
\[
h(w(A)) \leq \frac{1}{2} \left( h\left( w\left( A\tilde{f}_{s,t} \right) \right) + h(|A||I|) \right).
\]

Proof. Let $A = U|A|$ be the polar decomposition of $A$. Then for every $\theta \in \mathbb{R}$, we have
\[
\|\text{Re}(e^{i\theta} A)\| = r\left( \text{Re}\left( e^{i\theta} A \right) \right)
= \frac{1}{2} r\left( e^{i\theta} A + e^{-i\theta} A^* \right)
= \frac{1}{2} r\left( e^{i\theta} U|A| + e^{-i\theta} |A|U^* \right)
= \frac{1}{2} r\left( e^{i\theta} U g(|A|) f(|A|) + e^{-i\theta} f(|A|) g(|A|) U^* \right).
\text{(8)}
Now, if we put $X = e^{i\theta}Ug(\|A\|)$, $Y = f(\|A\|)$, $S = e^{-i\theta}f(\|A\|)$ and $T = g(\|A\|)U^*$ in Lemma 3.1, then we get

$$r\left(e^{i\theta}Ug(\|A\|)f(\|A\|) + e^{-i\theta}f(\|A\|)g(\|A\|)U^*)\right)$$

$$\leq \frac{1}{2}\left(w(f(\|A\|)Ug(\|A\|)) + w(g(\|A\|)U^*f(\|A\|))\right)$$

$$+ \frac{1}{2}\sqrt{4|e^{-i\theta}f(\|A\|)g(\|A\|)|^2|g(\|A\|)U^*e^{-i\theta}Uf(\|A\|)||}$$

(by Lemma 3.1)

$$\leq w(f(\|A\|)Ug(\|A\|)) + \sqrt{\|f(\|A\|)||g(\|A\|)||g(\|A\|)||}$$

$$= w(f(\|A\|)Ug(\|A\|)) + \sqrt{\|A\||\|A\||}$$

$$= w(\tilde{A}_{f,g}) + \|A\|.$$  \hspace{1cm} (9)

Note that, since $w(X) = w(X^*)$ ($X \in \mathcal{B}(\mathcal{H})$), in the first inequality we have

$$w(YX) = w(TS) = w(f(\|A\|)Ug(\|A\|)) - w(g(\|A\|)U^*f(\|A\|)) = 0.$$

Using inequalities (8), (9) and Lemma 3.2 we get

$$\omega(A) = \max_{a \in \mathbb{R}} \|\text{Re}(e^{i\theta}A)\| \leq \frac{1}{2}(w(\tilde{A}_{f,g}) + \|A\|).$$

Hence

$$h(\omega(A)) \leq h\left(\frac{1}{2}\left(w(\tilde{A}_{f,g}) + \|A\|\right)\right)$$

(by the monotonicity of $h$)

$$\leq \frac{1}{2}h\left(w(\tilde{A}_{f,g})\right) + \frac{1}{2}h(\|A\|)$$

(by the convexity of $h$)

$$= \frac{1}{2}h\left(w(\tilde{A}_{f,g})\right) + \frac{1}{2}\|h(\|A\|)||,$$

as required. \hspace{1cm} $\square$

**Remark 3.11.** We can obtain Theorem 3.3 from Theorem 3.10, but we keep the proof for the readers. To see this, first note that by the hypotheses of Theorem 3.3 we have

$$h(\|A\|) = h(g(\|A\|)f(\|A\|))$$

$$\leq h\left(\frac{g^2(\|A\|) + f^2(\|A\|)}{2}\right)$$

(by the arithmetic-geometric inequality)

$$\leq \frac{1}{2}h\left(g^2(\|A\|)\right) + h\left(f^2(\|A\|)\right)$$

(by the convexity of $h$). \hspace{1cm} (10)

Hence, using Theorem 3.10 and inequality (10) we get

$$h(\omega(A)) \leq \frac{1}{2}h\left(w(\tilde{A}_{f,g})\right) + \|h(\|A\|)||$$

$$\leq \frac{1}{2}h\left(w(\tilde{A}_{f,g})\right) + \frac{1}{2}\|h(g^2(\|A\|)) + h(f^2(\|A\|))\||$$

$$= \frac{1}{2}h\left(w(\tilde{A}_{f,g})\right) + \frac{1}{4}\|h(g^2(\|A\|)) + h(f^2(\|A\|))\||.$$
Remark 3.12. For the special case \( f(x) = x^t \) and \( g(x) = x^{1-t} \) (\( t \in [0, 1] \)), we obtain the inequality (2)

\[
    w(A) \leq \frac{1}{2} \left( w(A^t) + \|A\| \right),
\]

where \( A \in B(H) \).

Let \( T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \). Using Theorem 3.10, we get the following result.

Corollary 3.13. Let \( A, B \in B(H) \) and \( f, g \) be two non-negative and non-decreasing continuous functions such that \( f(x)g(x) = x \) (\( x \geq 0 \)). Then

\[
    2\omega^t \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \max \{\|A|^t, \|B|^t\} \left( \frac{1}{2} \left( \|f(|B|)g(|A^t|)|f(|B|)g(|A^t|)| + \|f(|A|)g(|B^t|)|f(|A|)g(|B^t|)| \right) \right)
\]

where \( s \geq 1 \).

Proof. Using Theorem 3.10 and inequality (6), we have

\[
    2\omega^t \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| + \omega^t (T_{f,g})
\]

\[
    = \max \{\|A|^t, \|B|^t\} \left( \frac{1}{2} \left( \|f(|B|)g(|A^t|)|f(|B|)g(|A^t|)| + \|f(|A|)g(|B^t|)|f(|A|)g(|B^t|)| \right) \right)
\]

and the proof is complete.

Using similar arguments to the proof of Corollary 3.9, we get the following result.

Corollary 3.14. Let \( A, B \in B(H) \) and \( f, g \) be two non-negative and non-decreasing continuous functions on \([0, \infty)\) such that \( f(x)g(x) = x \) (\( x \geq 0 \)). Then

\[
    \|A + B\| \leq \max \{\|A\|, \|B\|\} + \frac{1}{2} \left( \|f(|B|)g(|A|)|f(|B|)g(|A|)| + \|f(|A^t|)g(|B^t|)|f(|A^t|)g(|B^t|)| \right).
\]

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References


