Hyers-Ulam Stability of Substitution Vector-Valued Integral Operator

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Abstract. For a substitution vector-valued integral operator $T_{\phi}^u$, we determine necessary and sufficient conditions to have Hyers-Ulam stability using conditional expectation operators. Then, we present an example to illustrate our result.

1. Introduction and Preliminaries

It seems that S. M. Ulam [15] first raised the stability problem of functional equations. The problem can be stated as follows. Let $G_1$ be a group and $(G_2,d)$ a metric group. Given $\epsilon > 0$, does there exists $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for each $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \epsilon$, for each $x, y \in G_1$? The first (partial) answer to it was published in 1941 by Hyers[5]. It reads as follows. Let $E$ and $Y$ be Banach spaces and $\epsilon > 0$. Then, for each $g : E \to Y$ with

$$\sup_{x,y \in E} \|g(x + y) - g(x) - g(y)\| \leq \epsilon,$$

there is a unique solution $f : E \to Y$ of the Cauchy equation $f(x + y) = f(x) + f(y)$ such that $\sup_{x \in E} \|g(x) - f(x)\| \leq \epsilon$. This result is called the Hyers-Ulam stability of the additive Cauchy equation.

For the last 50 years, that issue has been a very popular subject of investigations and we refer the reader to [1, 2, 6–8, 10] for further information, some discussions, and examples of recent results.

T. Miura, S. Miyajima and S. -E. Takahasi [9] introduced the notion of the Hyers-Ulam stability of a mapping between two normed linear spaces as follows:

Definition 1.1 ([9]). Let $X, Y$ be normed linear spaces and $T$ be a (not necessarily linear) mapping from $X$ into $Y$. We say that $T$ has the Hyers-Ulam stability if there exists a constant $M > 0$ with the following property:

For any $g \in T(X)$, $\epsilon > 0$ and $f \in T(X)$ satisfying $\|Tf - g\| \leq \epsilon$, we can find $f_0 \in T(X)$ such that $Tf_0 = g$ and $\|f - f_0\| \leq Me$.

We call $M$ a HUS constant for $T$, and denote the infimum of all HUS constants for $T$ by $M_T$. We refer the reader for the Hyers-Ulam stability of substitution operators on function spaces to [4, 9, 13, 14] and the
From now on, by an operator we will a non-zero linear operator. Let \( \mathcal{B} \) be a Banach space and let \( T \) be an operator from \( \mathcal{B} \) into itself. The linearity of \( T \) implies that \( T \) has the H-U stability if and only if there exists a constant \( M \) with the following property:

For any \( \epsilon > 0 \) and \( f \in \mathcal{B} \) with \( \| Tf \| \leq \epsilon \) there exists \( f_0 \in \mathcal{B} \) such that \( Tf_0 = 0 \) and \( \| f - f_0 \| \leq Me. \)

For a bounded operator \( T : \mathcal{B} \to \mathcal{B} \), we denote the null space of \( T \) by \( N(T) \), the range of \( T \) by \( \mathcal{R}(T) \) and the induced one-to-one operator \( T \) from the quotient space \( \mathcal{B}/N(T) \to \mathcal{B} \) defined by \( T(f + N(T)) = Tf \), for all \( f \in \mathcal{B} \). Clearly \( \mathcal{R}(T) = \mathcal{R}(\tilde{T}) \).

Takagi, Miura and Takahashi [13] investigated the relation of the Hyers-Ulam stability of \( T \) and the inverse operator \( \tilde{T}^{-1} \) from \( \mathcal{R}(T) \) into \( \mathcal{B}/N(T) \) in the following sense.

**Theorem A** ([13], Theorem 2). For a bounded linear operator \( T \) on a Banach space, the following statements are equivalent:

1. \( T \) has the Hyers-Ulam stability.
2. \( T \) has closed range.
3. \( T^{-1} \) is bounded.

Moreover, in this case \( M_T = \| \tilde{T}^{-1} \| \).

The aim of this paper is to carry some of the results obtained for the linear operators on function spaces in \([4, 9, 13, 14]\) to a substitution vector-valued integral operator on \( L^1(X) \) space.

First of all, we introduce notations, definitions and preliminary facts that are used throughout the paper.

Let \( (X, \Sigma, \mu) \) be a \( \sigma \)-finite measure space and \( \varphi : X \to X \) be a non-singular measurable transformation; i.e. \( \mu \circ \varphi^{-1} \ll \mu \). Here the non-singularity of \( \varphi \) guarantees that the operator \( f \to f \circ \varphi \) is well defined as a mapping on \( L^p(\Sigma) \) where \( L^p(\Sigma) \) denotes the linear space of all equivalence classes of \( \Sigma \)-measurable functions on \( X \). Let \( h_0 = d\mu \circ \varphi^{-1}/d\mu \) be the Radon-Nikodym derivative. We also assume that \( h_0 \) is almost everywhere finite-valued, or equivalently \( \varphi^{-1}(\Sigma) \subseteq \Sigma \) is a sub-\(\sigma\)-finite algebra see [12]. As usual, \( \Sigma \) is said to be \( \varphi \)-invariant if \( \varphi(\Sigma) \subseteq \Sigma, \) where \( \varphi(\Sigma) = \{\varphi(A), A \in \Sigma\}. \) The measure \( \mu \) is said to be normal if \( \mu(A) = 0 \) implies that \( \mu(\varphi(A)) = 0 \). The support of a measurable function \( f \) is defined by \( \sigma(f) = \{x \in X : f(x) \neq 0\}. \) All comparisons between two functions or two sets are to be interpreted as holding up to a \( \mu \)-null set. For a sub-\(\sigma\)-finite algebra \( \mathcal{A} \subseteq \Sigma \), the conditional expectation operator associated with \( \mathcal{A} \) is the mapping \( f \to E^\mathcal{A} f \), defined for all non-negative \( f \) as well as for all \( f \in L^p(\Sigma), 1 \leq p \leq \infty \), where \( E^\mathcal{A} f \), by Radon-Nikodym Theorem, is the unique \( \mathcal{A} \)-measurable function satisfying

\[
\int_A f d\mu = \int_A E^\mathcal{A} f d\mu, \quad \forall A \in \mathcal{A}.
\]

We recall that \( E^\mathcal{A} : L^2(\Sigma) \to L^2(\mathcal{A}) \) is an orthogonal projection. For more details on the properties of \( E^\mathcal{A} \) see [11]. Throughout this paper, we assume that \( \mathcal{A} = \varphi^{-1}(\Sigma) \) and \( E^{\varphi^{-1}(\Sigma)} = E \).

For a given complex Hilbert space \( \mathcal{H} \), let \( u : X \to \mathcal{H} \) be a mapping. We say that \( u \) is weakly measurable if for each \( h \in \mathcal{H} \) the mapping \( x \mapsto \langle u(x), h \rangle \) of \( X \) to \( \mathbb{C} \) is measurable. We will denote this map by \( \langle u, h \rangle \). Let \( U(\Sigma) \) be the class of all measurable mappings \( f : X \to C \) such that \( \| f \|_p = \int_X |f(x)|^p d\mu < \infty \) for \( p \geq 1 \).

Let \( \varphi : X \to X \) be a non-singular measurable transformation and let \( u : X \to \mathcal{H} \) be a weakly measurable function. Then the pair \((u, \varphi)\) induces a substitution vector-valued integral operator \( T^\varphi_u : U(\Sigma) \to \mathcal{H} \) defined
by

$$\langle T_u^\gamma f, h \rangle = \int_X \langle u, h \rangle f \circ \varphi d\mu, \quad h \in \mathcal{H}, \ f \in L^p(\Sigma).$$

It is easy to see that $T_u^\gamma$ is well defined and linear. Moreover for each $f \in L^p(\Sigma),$

$$\sup_{h \in \mathcal{H}_1} |\langle T_u^\gamma f, h \rangle| \leq \sup_{h \in \mathcal{H}_1} \|T_u^\gamma f\| = \|T_u^\gamma f\| = |\langle T_u^\gamma f, \frac{T_u^\gamma f}{\|T_u^\gamma f\|} \rangle| \leq \sup_{h \in \mathcal{H}_1} \|T_u^\gamma f, h\|,$$

where $\mathcal{H}_1$ is the closed unit ball of $\mathcal{H}$. Hence $\|T_u^\gamma f\| = \sup_{h \in \mathcal{H}_1} \|T_u^\gamma f, h\|$, for each $f \in L^p(\Sigma)$. Some fundamental properties of this operator on $L^2(\Sigma)$ space are studied by the author et al in [3].

**Definition 2.1.** Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that $(u, \varphi, \mathcal{H})$ has absolute property, if for each $f \in L^p(X)$, there exists $h_f \in H_1$ such that $\sup_{h \in H_1} \int_X |\langle u, h \rangle f \circ \varphi| d\mu = \int_X |\langle u, h_f \rangle f \circ \varphi| d\mu$, and $\langle u, h_f \rangle = e^{i - \arg f = \varphi + 0} \langle u, h \rangle$, for a constant $\varphi_f$.

**Proposition 1.3** ([3]). Assume that $(u, \varphi, \mathcal{H})$ has the absolute property. Then

$$\sup_{h \in \mathcal{H}_1} \int_X |\langle u, h \rangle f \circ \varphi| d\mu = \sup_{h \in \mathcal{H}_1} \int_X |\langle u, h \rangle f \circ \varphi| d\mu.$$

Throughout of this paper we assume that $(u, \varphi, \mathcal{H})$ has the absolute property.

**2. The main results**

In this section, we determine the Hyers-Ulam stability the substitution vector-valued integral operator $T_u^\gamma : L^1(\Sigma) \to \mathcal{H}$, with the norm of the inverse of the one-to-one operator induced by this operator.

First, we present an auxiliary lemma which plays a key role in the sequel.

**Lemma 2.1.** Let $\Sigma$ be $\varphi$-invariant. If $T_u^\gamma : L^1(\Sigma) \to \mathcal{H}$ is a bounded substitution vector-valued integral operator. Then we have

$$\|f + \mathcal{N}(T_u^\gamma)\| = \int_{\varphi(u \cap \{0\})} |f|.$$

**Proof.** Put $\mathcal{D} = \varphi(u \cap \{0\})$ and $\mathcal{D}^c = X \setminus \mathcal{D}$. we can write

$$L^1(X, \Sigma, \mu) = L^1(\mathcal{D}, \Sigma_1, \mu) \oplus L^1(\mathcal{D}^c, \Sigma_2, \mu),$$

where $\Sigma_1 = \Sigma \cap \mathcal{D}$ and $\Sigma_2 = \Sigma \cap \mathcal{D}^c$. Moreover we have

$$N(T_u^\gamma) = \{f \in L^1(\Sigma) : |f| = 0 \text{ on } \mathcal{D} \} = L^1(\Sigma_2).$$

If $T_u^\gamma$ is one-to-one, then $\mu(\mathcal{D}^c) = 0$ and hence there is nothing to prove. Choose $g \in N(T_u^\gamma)$ arbitrary. Thus for each $f \in L^1(\Sigma)$ we obtain

$$\int_{\mathcal{D}} |f| d\mu = \int_{\mathcal{D}} |f + g| d\mu \leq \int_X |f + g| d\mu = \|f + g\|.$$

Therefore, we deduce that $\int_{\mathcal{D}} |f| d\mu \leq \|f + N(T_u^\gamma)\|$. Now, put $p = -\chi_{\mathcal{D}^c} f$. It is easy to see that $p \in N(T_u^\gamma)$. Hence, we get that for all $f \in L^1(\Sigma),$

$$\|f + N(T_u^\gamma)\| \leq \|f + p\| = \|f(1 - \chi_{\mathcal{D}^c})\| = \|f\chi_{\mathcal{D}^c}\| = \int_{\mathcal{D}} |f| d\mu.$$

Therefore the lemma is proved. \(\Box\)
In the following theorem we give necessary and sufficient conditions for $T^\mu_u : L^1(\Sigma) \to \mathcal{H}$ to have the Hyers-Ulam stability.

**Theorem 2.2.** Let $T^\mu_u$ be a bounded operator from $L^1(\Sigma)$ into $\mathcal{H}$. Also let $\Sigma$ be $\varphi$-invariant. If $\mu$ is normal, then the following assertions are equivalent:

(i) $T^\mu_u$ has the Hyers-Ulam stability.

(ii) $T^\mu_u$ has closed range.

(iii) There exists $r > 0$ such that $\sup_{h \in \mathcal{H}_1} h_0E((u,h)) \circ \varphi^{-1} \geq r$ for $\mu$-almost all $x \in \cup_{h \in \mathcal{H}_1} \sigma(J_h)$, where $J_h := h_0E((u,h)) \circ \varphi^{-1}.$

(iv) $\varphi(\cup_{h \in \mathcal{H}_1} h_0E((u,h))) \subseteq \{x \in X; \sup_{h \in \mathcal{H}_1} h_0E((u,h)) \circ \varphi^{-1}(x) \geq r\}$, for some $r > 0$.

(v) There exists $M > 0$ such that $\|f + N(T^\mu_u f)\| \leq M\|T^\mu_u f\|$, for each $f \in L^1(\Sigma)$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is direct consequent of Theorem A.

(ii) $\Rightarrow$ (iii) Assume $T^\mu_u$ has closed range. Then $T^\mu_u|_{\cup_{h \in \mathcal{H}_1} L^1(\Sigma)}$ is closed in $\mathcal{H}$ for each $h \in \mathcal{H}_1$. Since $T^\mu_u|_{\cup_{h \in \mathcal{H}_1} L^1(\Sigma)}$ is injective for each $h \in \mathcal{H}_1$ see[3,Theorem 2.12]. Hence we can deduce that there exists a constant $d > 0$ such that $\|T^\mu_u|_{\cup_{h \in \mathcal{H}_1} L^1(\Sigma)} f\| \geq d\|f\|$ for any $f \in L^1(\Sigma)$ and for each $h \in \mathcal{H}_1$. Now, by the contrary assume that (iii) is not hold. Then for each $r > 0$ and for each $h \in \mathcal{H}_1$ we have $h_0E((u,h)) \circ \varphi^{-1} < r$ on $\cup_{h \in \mathcal{H}_1} \sigma(J_h).$

Put $f = \chi_B$ with $\mu(B) < \infty$ and $B \subseteq \cup_{h \in \mathcal{H}_1} \sigma(J_h)$. Therefore we have

$$d\|\chi_B\| \leq \|T^\mu_u|_{\cup_{h \in \mathcal{H}_1} L^1(\Sigma)} \chi_B\| = \sup_{h \in \mathcal{H}_1} \int_X h_0E((u,h)) \circ \varphi^{-1} \chi_B d\mu < r\mu(B).$$

It is sufficient put $r = d$, but this is a contradiction. Hence we conclude that (iii) is hold.

(iii) $\Rightarrow$ (iv) We have $\sup_{h \in \mathcal{H}_1} h_0E((u,h)) \circ \varphi^{-1} \geq r$ on $\cup_{h \in \mathcal{H}_1} \sigma(J_h)$ for some $r > 0$. It is enough to prove that $\varphi(\cup_{h \in \mathcal{H}_1} \sigma(u,h)) \subseteq \cup_{h \in \mathcal{H}_1} \sigma(J_h)$. If $\varphi(\cup_{h \in \mathcal{H}_1} \sigma(u,h)) \notin \cup_{h \in \mathcal{H}_1} \sigma(J_h)$, then we can choose $C \subseteq \cup_{h \in \mathcal{H}_1} \sigma(u,h)$ such that $0 < \mu(\varphi(C)) < \infty$ with $\varphi(C) \cap (\cup_{h \in \mathcal{H}_1} \sigma(J_h)) = \emptyset$. For any $h \in \mathcal{H}_1$ we have

$$0 = \int_X \chi_{\varphi(C)} h_0E((u,h)) \circ \varphi^{-1} d\mu = \int_X \chi_{\varphi^{-1}(\varphi(C))}(u,h) d\mu.$$ 

On the other hand we have

$$\mu(C) = \mu(C \cap (\cup_{h \in \mathcal{H}_1} \sigma(u,h)))) = \mu((\cup_{h \in \mathcal{H}_1} (C \cap \sigma(u,h))))$$

$$\leq \mu((\cup_{h \in \mathcal{H}_1} (\varphi^{-1}(\varphi(C)) \cap \sigma(u,h)))) \leq \sum_{h \in \mathcal{H}_1} \mu(\varphi^{-1}(\varphi(C)) \cap \sigma(u,h)))$$

Since $\mu$ is normal, we get that $\mu(\varphi(C)) = 0$. But this is a contradiction.

(iv) $\Rightarrow$ (v) Put $A := \{x \in X; \sup_{h \in \mathcal{H}_1} h_0E((u,h)) \circ \varphi^{-1} \geq r\}$. Take $\epsilon$ arbitrary, then there exists $h_1 \in \mathcal{H}_1$ such...
Therefore we obtain $M$ such that $\sup_{h \in H} \|h_0 E((u, h_1)) \circ \varphi^{-1} \| \leq 1$. By Lemma 2.1, we obtain that

$$\|f + \mathcal{N}(T_u^0)\| = \int_{\varphi(\bigcup_{h \in H_1} \sigma((u, h)))} |f| \, d\mu \leq \int_A |f| \, d\mu = \frac{1}{r - \epsilon} \int_A (r - \epsilon) |f| \, d\mu$$

$$\leq \frac{1}{r - \epsilon} \int_B h_0 E((u, h_1)) \circ \varphi^{-1} |f| \, d\mu$$

$$\leq \frac{1}{r - \epsilon} \sup_{h \in H_1} \int_X h_0 E((u, h)) \circ \varphi^{-1} |f| \, d\mu$$

$$= \frac{1}{r - \epsilon} \|T_u^0 f\|,$$

for each $f \in L^1(X)$. Since $\epsilon$ was arbitrary, consequently there is a constant $M = \frac{1}{r}$. 

(v) $\Rightarrow$ (i) It is trivial by using Theorem A and definition of the Hyers-Ulam stability. 

**Theorem 2.3.** Under the same assumptions as in Theorem 2.2, if $R = \sup \{r > 0 : \varphi(\bigcup_{h \in H_1} \sigma((u, h))) \subseteq \{x \in X; \sup_{h \in H_1} h_0 E((u, h)) \circ \varphi^{-1} \geq r\}\}$. Then $M_{r_0} = 1/R$.

**Proof.** By theorem 2.2, if $r$ is taken over all numbers satisfying

$$\varphi(\bigcup_{h \in H_1} \sigma((u, h))) \subseteq \{\sup_{h \in H_1} h_0 E((u, h)) \circ \varphi^{-1} \geq r\},$$

we obtain $M_{r_0} = \|T_u^0\| \leq 1/R$. For the opposite inequality, assume that $\|T_u^0\| \leq 1/r$ and for each $h \in H_1$, $\varphi(\bigcup_{h \in H_1} \sigma((u, h))) \subseteq \{h \geq r\}$ for some $r > 0$. Hence we can choose $A \subseteq \varphi(\bigcup_{h \in H_1} \sigma((u, h)))$, with $0 < \mu(A) < \infty$ such that $\sup_{h \in H_1} h_0 E((u, h)) \circ \varphi^{-1} |A < r$. Put $f_0 = \frac{\chi_A}{\mu(A)}$. Then we get that

$$\|T_u^0 f_0\| = \sup_{h \in H_1} \int_X h_0 E((u, h)) \circ \varphi^{-1} \frac{X_A}{\mu(A)} \, d\mu = \sup_{h \in H_1} \int_A h_0 E((u, h)) \circ \varphi^{-1} \frac{1}{\mu(A)} \, d\mu < r.$$

Therefore

$$1 = \|f_0\| = \int_{\varphi(\bigcup_{h \in H_1} \sigma((u, h)))} |f_0| \, d\mu = \|f_0 + \mathcal{N}(T_u^0)\| = \|T_u^0 f_0\| \leq \|T_u^0\| \|T_u^0 f_0\| < 1.$$

Which is a contradiction. Hence we deduce that if $\|T_u^0\| < 1/r$ then $\varphi(\bigcup_{h \in H_1} \sigma((u, h))) \subseteq \{\sup_{h \in H_1} h_0 E((u, h)) \circ \varphi^{-1} \geq r\}$. This follows that $1/R \leq \|T_u^0\|$.

**Example 2.4.** Let $X = (0, 1)$, $\Sigma$ be the Lebesgue subsets of $X$ and let $\mu$ be the Lebesgue measure on $X$. Also let $\varphi : X \to X$ be defined by

$$\varphi(x) = \begin{cases} 
2x & 0 < x < \frac{1}{2}, \\
2 - 2x & \frac{1}{2} \leq x < 1.
\end{cases}$$
Direct computation shows that \( h_0(x) = 1 \). Define \( u : X \to \mathcal{R} \) by \( u(x) = x + 1 \). Then for each \( h \in \mathcal{H}_1 \), we have
\[
\int_X h_0 E[(u(x), h)] \circ \varphi^{-1} d\mu = \int_{(0, \frac{1}{2})} E[(u(x), h)] \circ \varphi^{-1} d\mu \circ \varphi^{-1} \\
+ \int_{(\frac{1}{2}, 1)} E[(u(x), h)] \circ \varphi^{-1} d\mu \circ \varphi^{-1} \\
= \int_{\varphi^{-1}(0, \frac{1}{2})} E[(u(x), h)] d\mu + \int_{\varphi^{-1}(\frac{1}{2}, 1)} E[(u(x), h)] d\mu \\
= \int_{\varphi^{-1}(0, \frac{1}{2})} |(u(x), h)| d\mu + \int_{\varphi^{-1}(\frac{1}{2}, 1)} |(u(x), h)| d\mu \\
= \frac{1}{2} \int_{(0, 1)} |(u(\frac{x}{2}), h)| dx + \frac{1}{2} \int_{(0, 1)} |(u(1 - \frac{x}{2}), h)| dx.
\]
Hence for each \( h \in \mathcal{H}_1 \) we get that
\[
h_0 E[(u(x), h)] \circ \varphi^{-1} = \frac{1}{2} \left( |(u(\frac{x}{2}), h)| + |(u(1 - \frac{x}{2}), h)| \right).
\]
Therefore
\[
h_0 E[(u(x), h)] \circ \varphi^{-1} = \frac{1}{2} \left( |\frac{xh}{2} + h| + |2h - \frac{xh}{2}| \right) \geq \frac{1}{2} |3h|.
\]
This implies that \( \sup_{h \in \mathcal{R}_1} h_0 E[(u(x), h)] \circ \varphi^{-1} \geq \frac{3}{2} \), where \( \mathcal{R}_1 \) is the closed unit ball of \( \mathcal{R} \). It is sufficient put \( r = \frac{3}{2} - \epsilon \), for \( \epsilon \) arbitrary. Then, by Theorem 2.2 we deduce that \( T^*_u \) on \( L^1(\Sigma) \) has the Hyers-Ulam stability.

References