Schur-Convexity for a Mean of Two Variables with Three Parameters

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Abstract. Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for a mean of two variables with three parameters are investigated, and some mean value inequalities of two variables are established.

1. Introduction

Throughout the paper we assume that the set of $n$-dimensional row vector on the real number field by $\mathbb{R}^n$.

$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\}$.

In particular, $\mathbb{R}_1^1$ and $\mathbb{R}_1^1$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

In 2009, Kuang [1] defined a mean of two variables with three parameters as follows:

$$K(\omega_1, \omega_2, p; a, b) = \left[\omega_1 A(a^p, b^p) + \omega_2 G(a^p, b^p)\right]^{\frac{1}{p}}$$

where $A(a, b) = \frac{a+b}{2}$ and $G(a, b) = \sqrt{ab}$ respectively is the arithmetic mean and geometric mean of two positive numbers $a$ and $b$, parameters $p \neq 0$, $\omega_1, \omega_2 \geq 0$ with $\omega_1 + \omega_2 \neq 0$.

For simplicity, sometimes we will be $K(\omega_1, \omega_2, p; a, b)$ for $K(\omega_1, \omega_2, p)$ or $K(a, b)$.

In particular,

$$K\left(1, \frac{\omega}{2}, 1\right) = \frac{a + \omega \sqrt{ab} + b}{\omega + 2}$$

is the generalized Heron mean, which was introduced by Janous [2] in 2001.

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Our main results are as follows:

Theorem 1.1. (i) When \( \omega_1 \omega_2 \neq 0 \), if \( p \geq 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0 \), then \( K(\omega_1, \omega_2, p) \) is Schur-convex with \( (a, b) \in \mathbb{R}_+^2 \); if \( 1 \leq p < 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0 \), then \( K(\omega_1, \omega_2, p) \) is Schur-concave with \( (a, b) \in \mathbb{R}_+^2 \); if \( p < 1 \), then \( K(\omega_1, \omega_2, p) \) is Schur-convex with \( (a, b) \in \mathbb{R}_+^2 \).

(ii) When \( \omega_1 = 0, \omega_2 \neq 0 \), \( K(\omega_1, \omega_2, p) \) is Schur-convex with \( (a, b) \in \mathbb{R}_+^2 \).

(iii) When \( \omega_1 \neq 0, \omega_2 = 0 \), if \( p \geq 2 \), then \( K(\omega_1, \omega_2, p) \) is the Schur-convex with \( (a, b) \in \mathbb{R}_+^2 \); if \( p < 2 \), then \( K(\omega_1, \omega_2, p) \) is the Schur-concave with \( (a, b) \in \mathbb{R}_+^2 \).

Theorem 1.2. If \( p \geq 1 \), then \( K(\omega_1, \omega_2, p) \) is Schur-geometrically convex with \( (a, b) \in \mathbb{R}_+^2 \). If \( p < 0 \), then \( K(\omega_1, \omega_2, p) \) is Schur-geometrically concave with \( (a, b) \in \mathbb{R}_+^2 \).

Theorem 1.3. If \( p \geq -1 \), then \( K(\omega_1, \omega_2, p) \) is Schur-harmonically convex with \( (a, b) \in \mathbb{R}_+^2 \). If \( -2 < p < -1 \) and \( \omega_1(p + 1) + \omega_2(p + 1) \geq 0 \), then \( K(\omega_1, \omega_2, p) \) is Schur-harmonically convex with \( (a, b) \in \mathbb{R}_+^2 \). If \( p \leq -2 \) and \( \omega_1(p + 1) + \omega_2 = 0 \), then \( K(\omega_1, \omega_2, p) \) is Schur-harmonically concave with \( (a, b) \in \mathbb{R}_+^2 \).

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 2.1 ([3, 4]). Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).

(i) \( x \) is said to be majorized by \( y \) (in symbols \( x \preceq y \)) if \( \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i \) for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \), where \( x_1 \geq \cdots \geq x_n \) and \( y_1 \geq \cdots \geq y_n \) are rearrangements of \( x \) and \( y \) in a descending order.

(ii) \( \Omega \subseteq \mathbb{R}^n \) is called a convex set if \( (\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega \) for any \( x, y \in \Omega \), where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \).

(iii) let \( \Omega \subseteq \mathbb{R}^n, \varphi : \Omega \rightarrow \mathbb{R} \) is said to be a Schur-convex function on \( \Omega \) if \( x \preceq y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur-concave function on \( \Omega \) if and only if \(-\varphi\) is Schur-convex function.

Definition 2.2 ([5, 6]). Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n \).

(i) \( \Omega \subseteq \mathbb{R}_+^n \) is called a geometrically convex set if \( (x_1^\alpha y_1, \ldots, x_n^\alpha y_n) \) \( \in \Omega \) for any \( x, y \in \Omega \), where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \).

(ii) let \( \Omega \subseteq \mathbb{R}_+^n, \varphi : \Omega \rightarrow \mathbb{R}_+ \) is said to be a Schur-geometrically convex function on \( \Omega \) if \( (\ln x_1, \ldots, \ln x_n) \in \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur-geometrically concave function on \( \Omega \) if and only if \(-\varphi\) is Schur-geometrically convex function.

Definition 2.3 ([7, 8]). Let \( \Omega \subseteq \mathbb{R}_+^n \).

(i) A set \( \Omega \) is said to be a harmonically convex set if \( \frac{xy}{\lambda x + (1-\lambda)y} \in \Omega \) for every \( x, y \in \Omega \) and \( \lambda \in [0, 1] \), where \( xy = \sum_{i=1}^n x_i y_i \) and \( \frac{1}{\lambda} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \).

(ii) A function \( \varphi : \Omega \rightarrow \mathbb{R}_+ \) is said to be a Schur harmonically convex function on \( \Omega \) if \( \frac{1}{\lambda} < \frac{1}{y} \) implies \( \varphi(x) \leq \varphi(y) \). A function \( \varphi \) is said to be a Schur harmonically concave function on \( \Omega \) if and only if \(-\varphi \) is a Schur harmonically convex function.
Lemma 2.4 ([3, 4]). Let $\Omega \subset \mathbb{R}^n$ be convex set, and has a nonempty interior set $\Omega^0$. Let $\varphi : \Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\varphi$ is the Schur – convex(Schur – concave) function, if and only if it is symmetric on $\Omega$ and if

\[
(x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0)
\]

holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 2.5 ([5, 6]). Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior $\Omega^0$. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on $\Omega$ and differentiable on $\Omega^0$. Then $\varphi$ is a Schur geometrically convex (Schur geometrically concave) function if and only if $\varphi$ is symmetric on $\Omega$ and

\[
(x_1 - x_2) \left( \frac{x_1}{x_1^2} \frac{\partial \varphi}{\partial x_1} - \frac{x_2}{x_2^2} \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0)
\]

holds for any $x = (x_1, \ldots, x_n) \in \Omega^0$.

Lemma 2.6 ([7, 8]). Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric harmonically convex set with a nonempty interior $\Omega^0$. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on $\Omega$ and differentiable on $\Omega^0$. Then $\varphi$ is a Schur harmonically convex (Schur harmonically concave) function if and only if $\varphi$ is symmetric on $\Omega$ and

\[
(x_1 - x_2) \left( \frac{x_1}{x_1^2} \frac{\partial \varphi}{\partial x_1} - \frac{x_2}{x_2^2} \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0)
\]

holds for any $x = (x_1, \ldots, x_n) \in \Omega^0$.

Lemma 2.7 ([9]). Let $a \leq b, u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$. If $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$, then

\[
(u(t_2), v(t_2)) < (u(t_1), v(t_1)) < (a, b).
\]

Lemma 2.8. Let

\[
f(x) = \omega_1 (p + 1)x^\frac{p}{2} + 1 + \omega_2 \frac{p}{2} x - \omega_2 \frac{p}{2}, \quad x \in [1, \infty)
\]

where $\omega_1, \omega_2 \geq 0, \omega_1^2 + \omega_2^2 \neq 0$. If $-2 < p < -1$ and $\omega_1 (p + 1) + \omega_2 (\frac{p}{2} + 1) \geq 0$, then $f(x) \geq 0$, if $p \leq -2$ and $\omega_1 (p + 1) + \omega_2 = 0$, then $f(x) \leq 0$.

Proof. If $-2 < p < -1$ and $\omega_1 (p + 1) + \omega_2 (\frac{p}{2} + 1) \geq 0$, then

\[
g(x) = \omega_1 (p + 1)x^\frac{p}{2} + 1 + \omega_2 \frac{p}{2} x
\]

\[
\geq \omega_1 (p + 1)x^\frac{p}{2} + 1 + \omega_2 \frac{p}{2} x^\frac{p}{2} \geq x^\frac{p}{2} \left[ \omega_1 (p + 1) + \omega_2 \frac{p}{2} + 1 \right]
\]

\[
\geq 0,
\]

and then $f(x) = g(x) - \omega_2 \frac{p}{2} \geq 0$.

If $p \leq -2$ and $\omega_1 (p + 1) + \omega_2 = 0$, then

\[
f'(x) = \omega_1 (p + 1)\frac{p}{2} x^\frac{p}{2} + 1 + \omega_2 \frac{p}{2} + 1,
\]
and 

\[ f''(x) = \omega_1 \frac{p}{2}(p + 1)(\frac{p}{2} + 1)x^{p - 1} \leq 0, \]

so \( f'(x) \) is decreasing, but

\[ f'(1) = \omega_1(p + 1)(\frac{p}{2} + 1) + \omega_2(\frac{p}{2} + 1) = \left(\frac{p}{2} + 1\right)[\omega_1(p + 1) + \omega_2] = 0, \]

then \( f'(x) \leq 0 \), so \( f(x) \) is decreasing, furthermore

\[ f(1) = \omega_1(p + 1) + \omega_2(\frac{p}{2} + 1) - \omega_2 \frac{p}{2} = \omega_1(p + 1) + \omega_2 = 0, \]

thus \( f(x) \leq 0 \). \( \square \)

3. Proofs of Main results

From the definition of \( K(\omega_1, \omega_2, p) \), we have

\[ K(\omega_1, \omega_2, p) = \left( \frac{\omega_1 \frac{x^p + y^p}{2} + \omega_2 x^p + \omega_2 y^p}{\omega_1 + \omega_2} \right)^\frac{1}{\frac{p}{2} + 1}. \]

It is clear that \( K(\omega_1, \omega_2, p) \) is symmetric with \((a, b) \in \mathbb{R}^2\).

Write \( m(a, b) := \left( \frac{\omega_1(a^p + b^p) + 2\omega_2 a^p b^p}{2(\omega_1 + \omega_2)} \right)^{\frac{1}{p}}. \)

**Proof.** [Proof of Theorem 1.1] (i) When \( \omega_1 \omega_2 \neq 0 \),

\[ \frac{\partial K}{\partial a} = m(a, b) \left( \frac{\omega_1 a^{p-1} + \omega_2 a^{p-1} b^{p-1}}{\omega_1 + \omega_2} \right), \]

\[ \frac{\partial K}{\partial b} = m(a, b) \left( \frac{\omega_1 b^{p-1} + \omega_2 a^{p-1} b^{p-1}}{\omega_1 + \omega_2} \right), \]

and then

\[ \Delta_1 := (a - b) \left( \frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) = \frac{a - b}{2(\omega_1 + \omega_2) m(a, b)} \left[ \omega_1 (a^{p-1} - b^{p-1}) - \omega_2 (a - b) a^{p-1} b^{p-1} \right]. \]

Without loss of generality, we may assume that \( a \geq b \), then \( z := \frac{a}{b} \geq 1 \), and then

\[ \Delta_1 = \frac{a - b}{2(\omega_1 + \omega_2) m(a, b) b^{p-1}} f(z), \]

where

\[ f(z) = \omega_1(z^{p-1} - 1) - \omega_2(z - 1)z^{p-1}, \quad z \geq 1. \]

\[ f'(z) = \omega_1(p - 1)z^{p-2} - \omega_2(z - 1)z^{p-2} = z^{p-2} \left[ \omega_1(p - 1)z^2 - \omega_2 \frac{p}{2} z + \omega_2 \frac{p}{2} - 1 \right]. \]
If \( p \geq 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0 \), then
\[
\omega_1(p-1)z^{p-1} - \omega_2 \frac{p}{2} z = z \left[ \omega_1(p-1)z^{p-1} - \omega_2 \frac{p}{2} \right] \geq z \left[ \omega_1(p-1) - \omega_2 \frac{p}{2} \right].
\]
Notice that
\[
\omega_1(p-1) - \omega_2 \frac{p}{2} \geq 0 \implies p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0,
\]
we have \( f(z) \geq 0 \), for \( z \in [1, \infty) \), but \( f(1) = 0 \), so \( f(z) \geq 0 \), further \( \Delta_1 \geq 0 \). By Lemma 1, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-convex with \((a, b) \in \mathbb{R}^2_+\).

If \( 1 \leq p < 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0 \), then
\[
\omega_1(p-1)z^{p-1} - \omega_2 \frac{p}{2} z = z \left[ \omega_1(p-1)z^{p-1} - \omega_2 \frac{p}{2} \right] \leq z \left[ \omega_1(p-1) - \omega_2 \frac{p}{2} \right].
\]
Notice that
\[
\omega_1(p-1) - \omega_2 \frac{p}{2} \geq 0 \implies p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0,
\]
we have \( f(z) \leq 0 \), for \( z \in [1, \infty) \), but \( f(1) = 0 \), so \( f(z) \leq 0 \), further \( \Delta_1 \leq 0 \). By Lemma 2.4, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-concave with \((a, b) \in \mathbb{R}^2_+\).

If \( p < 1 \), then
\[
f(z) = \omega_1(z^{p-1} - a) - \omega_2(z - 1)z^{p-1} \leq \omega_1(1 - a) - \omega_2(z - 1)z^{p-1} \leq 0,
\]
and then \( \Delta_1 \leq 0 \). By Lemma 2.4, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-concave with \((a, b) \in \mathbb{R}^2_+\).

(ii) When \( \omega_1 = 0, \omega_2 \neq 0 \), \( K(\omega_1, \omega_2, p) = \sqrt{ab} \), then
\[
\Delta_1 := (a - b) \left( \frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) = -\frac{1}{2} \frac{(a - b)^2}{\sqrt{ab}} \leq 0.
\]
By Lemma 2.4, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-concave with \((a, b) \in \mathbb{R}^2_+\).

(iii) When \( \omega_1 \neq 0, \omega_2 = 0 \), \( K(\omega_1, \omega_2, p) = \left( \frac{x_{ab}^p}{2} \right)^{\frac{1}{p}} \), then
\[
\Delta_1 := (a - b) \left( \frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) = \frac{1}{2} (a - b)(a^p + b^p)^{\frac{1}{p}} - (a^{p-1} + b^{p-1}).
\]
If \( p \geq 2 \), then \( a - b \) and \( a^{p-1} - b^{p-1} \) has the same sign, so \( \Delta_1 \geq 0 \). By Lemma 2.4, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-convex with \((a, b) \in \mathbb{R}^2_+\). If \( p < 2 \), then \( a - b \) and \( a^{p-1} - b^{p-1} \) has the opposite sign, so \( \Delta_1 \leq 0 \). By Lemma 2.4, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-concave with \((a, b) \in \mathbb{R}^2_+\).

The proof of Theorem 1.1 is complete. \( \square \)

Proof. [Proof of Theorem 1.2] It is easy to see that
\[
\frac{a}{\partial a} = m(a, b) \left( \frac{\omega_1 a^p + \omega_2 a^p b^p}{\omega_1 + \omega_2} \right),
\]
\[
\frac{b}{\partial b} = m(a, b) \left( \frac{\omega_1 b^p + \omega_2 a^p b^p}{\omega_1 + \omega_2} \right),
\]
and then
\[
\Delta_2 := (a - b) \left( \frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) = \frac{(a - b)m(a, b)\omega_1(a^p - b^p)}{2(\omega_1 + \omega_2)}.
\]
If \( p \geq 0 \), then \( a - b \) and \( a^p - b^p \) has the same sign, so \( \Delta_2 \geq 0 \). By Lemma 2.5, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-geometrically convex with \((a, b) \in \mathbb{R}^2 \). If \( p < 0 \), then \( a - b \) and \( a^p - b^p \) has the opposite sign, so \( \Delta_2 \leq 0 \). By Lemma 2.5, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-geometrically concave with \((a, b) \in \mathbb{R}^2 \).

The proof of Theorem 1.2 is complete. \( \blacksquare \)

**Proof.** [Proof of Theorem 1.3] It is easy to see that

\[
\frac{\partial K}{\partial a} = \frac{m(a, b)}{2(\omega_1 + \omega_2)} \left( \omega_1 a^{p+1} + \omega_2 a^{p+1} \right),
\]

\[
\frac{\partial K}{\partial b} = \frac{m(a, b)}{2(\omega_1 + \omega_2)} \left( \omega_1 b^{p+1} + \omega_2 a^{p+1} \right),
\]

and then

\[
\Delta_3 := (a - b) \left( \frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) = \frac{m(a, b)}{2(\omega_1 + \omega_2)} f(x, y),
\]

where

\[
f(a, b) := \omega_1(a - b)(a^{p+1} - b^{p+1}) + \omega_2 a^{p} b^{p} (a - b)^2.
\]

If \( p \geq -1 \), then \( a - b \) and \( a^{p+1} - b^{p+1} \) has the same sign, so \( f(a, b) \geq 0 \), and then \( \Delta_3 \geq 0 \). By Lemma 2.6, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-harmonically convex with \((a, b) \in \mathbb{R}^2 \).

Without loss of generality, we may assume that \( a \geq b \), then \( z := \frac{a}{b} \geq 1 \), and then

\[
f(a, b) = b^{p+2}(z - 1)g(z),
\]

where

\[
g(z) = \omega_1(z^{p+1} - 1) + \omega_2 z^p (z - 1),
\]

\[
g'(z) = z^{p-2} s(z),
\]

where

\[
s(z) = \omega_1(p + 1)(z^{p+1} + \omega_2 \frac{p+1}{2} z - \omega_2 \frac{p}{2}).
\]

If \(-2 < p \leq -1 \) and \( \omega_1(p + 1) + \omega_2 (\frac{p}{2} + 1) \geq 0 \), from Lemma 2.8, it follows \( s(z) \geq 0 \), and then \( g'(z) \geq 0 \), but \( g(1) = 0 \), so \( g(z) \geq 0 \) and \( f(a, b) \geq 0 \). Thus \( \Delta_3 \geq 0 \), by Lemma 2.6, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-harmonically convex with \((a, b) \in \mathbb{R}^2 \).

If \( f \leq -2 \) and \( \omega_1(p + 1) + \omega_2 = 0 \), from Lemma 2.8, it follows \( s(z) \leq 0 \), and then \( g'(z) \leq 0 \), but \( g(1) = 0 \), so \( g(z) \leq 0 \) and \( f(a, b) \leq 0 \). Thus \( \Delta_3 \leq 0 \), by Lemma 2.6, it follows that \( K(\omega_1, \omega_2, p) \) is Schur-harmonically concave with \((a, b) \in \mathbb{R}^2 \).

The proof of Theorem 1.3 is complete. \( \blacksquare \)

4. Applications

**Theorem 4.1.** Let \((a, b) \in \mathbb{R}^2 \), \( u(t) = tb + (1 - t)a, v(t) = ta + (1 - t)b \). Assume also that \( \frac{1}{2} \leq t_2 \leq t_1 \leq 1 \) or \( 0 \leq t_1 \leq t_2 \leq \frac{1}{2} \).

If \( \omega_1 \omega_2 \neq 0, p \geq 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0 \), then we have the bound

\[
K \left( \omega_1, \omega_2, p; \frac{a + b}{2}, \frac{a + b}{2} \right) \leq K \left( \omega_1, \omega_2, p; u(t_2), v(t_2) \right)
\]

\[
\leq K \left( \omega_1, \omega_2, p; u(t_1), v(t_1) \right) \leq K(\omega_1, \omega_2, p; a, b) \leq G(\omega_1, \omega_2, p; a, b).
\]

(5)

If \( \omega_1 \omega_2 \neq 0, 1 \leq p < 2 \) and \( p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0 \), then inequalities in (5) are all reversed.
Proof. From Lemma 2.7, we have

\[
\left(\frac{a + b}{2}, \frac{a + b}{2}\right) < (u(t_2), v(t_2)) < (u(t_1), v(t_1)) < (a, b),
\]

and it is clear that \((a, b) < (a + b - \varepsilon, \varepsilon)\), where \(\varepsilon\) is enough small positive number.

If \(\omega_1\omega_2 \neq 0, p \geq 2\) and \(p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0\), by Theorem 1, and let \(\varepsilon \to 0\), it follows that (5) are holds. If \(\omega_1\omega_2 \neq 0, 1 \leq p < 2\) and \(p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0\), then inequalities in (5) are all reversed.

The proof is complete. \(\square\)

Theorem 4.1 enable us to obtain a large number of refined inequalities by assigning appropriate values to the parameters \(\omega_1, \omega_2, p, t_1\) and \(t_2\).

For example, putting \(\omega_1 = \omega_2 = 1\) in (5), we can get

**Corollary 4.2.** Let \(p \geq 2\). Then for \((a, b) \in \mathbb{R}_+^2\), we have

\[
A(p, b^p) + G(p, b^p) \geq 2(A(a, b))^p. \tag{6}
\]

Putting \(p = \frac{1}{2}, \omega_1 = 2, \omega_2 = 1\) and \(t_1 = \frac{3}{2}, t_2 = \frac{1}{2}\) in (5), we can get

**Corollary 4.3.** Let \((a, b) \in \mathbb{R}_+^2\). Then

\[
\frac{a + b}{2} \geq \frac{1}{36} \left[ \sqrt[3]{a + 3b} + \sqrt[3]{(a + 3b)(3a + b)} + \sqrt{3a + b} \right]^2 \geq \frac{1}{9} \left( \sqrt[4]{a} + \sqrt[4]{ab} + \sqrt{b} \right)^2. \tag{7}
\]

**Theorem 4.4.** Let \((a, b) \in \mathbb{R}_+^2\). If \(p \geq 0(0, 0)\), we have

\[
G(a, b) \leq (\geq)K(\omega_1, \omega_2, p; a, b). \tag{8}
\]

**Proof.** Since \((\log \sqrt[4]{ab}, \log \sqrt[4]{ab}) < (\log a, \log b)\), if \(p \geq 0(0, 0)\), by Theorem 1.2, it follows

\[
G(a, b) = K(\omega_1, \omega_2, p; \sqrt[4]{ab}, \sqrt[4]{ab}) \leq (\geq)K(\omega_1, \omega_2, p; a, b).
\]

The proof is complete. \(\square\)

For example, putting \(\omega_1 = \omega_2 = 1\) in (8), we can get

**Corollary 4.5.** Let \((a, b) \in \mathbb{R}_+^2\). If \(p \geq 0(0, 0)\), then

\[
A(p, b^p) + G(p, b^p) \leq (\geq)2(G(a, b))^p. \tag{9}
\]

**Theorem 4.6.** Let \((a, b) \in \mathbb{R}_+^2\). If \(p \geq -1\) or \(-2 < p < -1\) and \(\omega_1(p + 1) + \omega_2(\frac{p}{2} + 1) \geq 0\), then

\[
H(a, b) \leq K(\omega_1, \omega_2, p; \frac{ab}{tb + (1 - t)a} + \frac{ab}{ta + (1 - t)b}) \leq K(\omega_1, \omega_2, p; a, b). \tag{10}
\]

where \(H(a, b) = \frac{2}{\omega_1 + \omega_2}\) is the harmonic mean.

If \(p \leq -2\) and \(\omega_1(\frac{p}{2} + 1) + \omega_2 = 0\), then inequalities in (10) are all reversed.

**Proof.** By Lemma 2.7, we have

\[
\left(\frac{a^{-1} + b^{-1}}{2}, \frac{a^{-1} + b^{-1}}{2}\right) < (a^{-1} - (1 - t)b^{-1}, tb^{-1} + (1 - t)a^{-1}) < (a^{-1}, b^{-1}).
\]
Corollary 4.7. Let $p \geq -1$ or if $-2 < p < -1$ and $\omega_1(p + 1) + \omega_2(\frac{p}{2} + 1) \geq 0$, then by Theorem 1.3, it follows

$$H(a, b) = K \left( \frac{\omega_1 \omega_2 p}{a^{1+b^{-1}} + b^{1+a^{-1}}}, \frac{2}{a^{1+b^{-1}} + b^{1+a^{-1}}} \right) \leq K \left( \frac{\omega_1 \omega_2 p}{\omega_1^a \omega_2^b (1-t) a^t + (1-t) b^t}, \frac{2}{\omega_1^a \omega_2^b (1-t) a^t + (1-t) b^t} \right) \leq K(a_1, a_2; p; a, b).$$

If $p \leq -2$ and $\omega_1(\frac{p}{2} + 1) + \omega_2 = 0$, then inequalities in (10) are all reversed.

The proof is complete. □

Putting $\omega_1 = \omega_2 = 1$ in (10), we can get

Corollary 4.7. Let $(a, b) \in \mathbb{R}^2$. If $p \geq -1$ or $-\frac{3}{4} < p < -1$, then

$$A(a^p, b^p) + G(a^p, b^p) \geq 2(H(a, b))^p. \quad (11)$$

If $p = -4$, then the inequality in (11) is reversed.

References


