On One Class Eigenvalue Problem with Eigenvalue Parameter in the Boundary Condition at One End-Point

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Abstract. In the present paper we investigate the spectrum of operator corresponding to eigenvalue problem with parameter dependent boundary condition. Trace formula for that operator is also established.

1. Introduction

In this paper we consider the boundary value problem with boundary condition depending on the first degree polynomial of spectral parameter. Namely, we consider Sturm-Liouville equation with unbounded coefficient. Note that in boundary condition some first degree polynomials appear before unknown function, as well as its derivative. Earlier studied problems for Sturm-Liouville operator equation where spectral parameter appears only before function or only before its derivative. For example, in \cite{1, 2, 3} the asymptotics of eigenvalues and self-adjoint extensions of minimal symmetric operators were studied. In \cite{4-10}, we have studied asymptotics of spectrum and established trace formulas for operators generated by regular and singular differential operator expressions and spectral parameter dependent boundary conditions.

Here we consider in space $L_2 (H, (0, 1))$ (where $H$ is separable Hilbert space) the following problem

\begin{align*}
Ly := -y'' (t) + Ay (t) + q (t) y (t) &= \lambda y (t) \quad (1) \\
y (0) &= 0 \quad (2) \\
y (1) (1 + \lambda) &= y' (1) ((t + 1 + \lambda) \quad (3)
\end{align*}

where $h$ is real number, $A = A'$, $A > E$ ($E$ is an identity operator in $H$) and has compact invers $A^{-1} \in \sigma_{ac}$. Clear that under stated conditions $A$ is discrete operator. Denote the eigenvectors of $A$ by $\varphi_1, \varphi_2, ...$. Let $\gamma_1 \leq \gamma_2 \leq ...$ be eigenvalues of operator $A$. Suppose that operator-valued function $q(t)$ is weakly measurable and for each $t$ is defined in $H$.

Fulton \cite{11} has considered the scalar Sturm-Liouville problem

\[-y'' (t) + q (t) y (t) = \lambda y (t)\]

\begin{itemize}
\item \textbf{Keywords.} Hilbert space, differential operator equation, self-adjoint operator, discrete spectrum, regularized trace.
\end{itemize}

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\[ \cos \alpha u(a) + \sin \alpha u'(a) = 0, \quad \alpha \in [0, \pi) \]

\[-\beta_1 u(b) - \lambda \beta_1' u(b) = -\beta_2 u'(b) - \lambda \beta_2' u'(b) \]

and given an operator-theoretic formulation of that problem, showing that one can associate a self-adjoint operator with it whenever the relation

\[ \beta = \begin{bmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{bmatrix} > 0 \]

holds. He obtained the expansion theorem and asymptotic formulas for eigenvalues and eigenfunctions.

In our case, obviously

\[ \beta = \begin{bmatrix} 1 & 1 \\ h & h + 1 \end{bmatrix} = h \]

and we take \( h > 0 \).

Introduce the space \( L_2 = L_2(H, (0, 1)) \oplus H \). Define in it scalar product of elements \( Y = (y(t), y_1) \), \( Z = (z(t), z_1) \in L_2, (y(t), z(t)) \in L_2(H, (0, 1)), y_1, z_1 \in H \) by

\[ (Y, Z) = \int_0^1 (y(t)z(t)) dt + \frac{1}{h} (y(1), z(1)) \]

where \((\cdot, \cdot)\) is scalar product in \( H \).

About \( q(t) \) we assume the next:

1) it is a bounded operator valued function \( \|q(t)\| \leq \text{const}, t \in [0, 1], q'(t) = q(t) \);
2) \( \sum_{k=1}^{\infty} |(q(t)q_k, q_k)| \leq \text{const}, \forall t \in [0, 1] \);
3) \( \int_0^1 (q(t)f, f) dt = 0, \text{for } f = \varphi_k, \quad k = 1, \infty. \)

Formulate (1)-(3) in case \( q(t) \equiv 0 \) in operator form. Thus define in \( L_2 \) operator \( L_0 \) as

\[ D(L_0) = \{ Y = (y(t), y(0) = 0, y_1 = y(1) - y'(1), 1y \in L_2(H, (0, 1))) \}. \]

\[ L_0Y = (-y''(t) + Ay(t), -(y(1) - (h + 1) y'(1))). \]

Denote by \( L \) operator defined in \( L_2 \) by \( L = L_0 + Q \), where \( QY = (q(t)y(t), 0) \).

Our aim is to obtain asymptotic formulae for eigenvalue distribution and establish trace formula for operator \( L \).

2. Asymptotic formulae for eigenvalues

It can be easily verified that \( L_0 \) is self adjoint positive-definite operator.

Since \( A \) is self adjoint operator in virtue of spectral expansion of \( A \) we get the next eigenvalue problem for scalar function \( y_k(t) = (y(t), \varphi_k) (y(t) \in L_2(H, (0, 1))) \)

\[ -y_k''(t) + \gamma_k y_k(t) = \lambda y_k(t) \]

\[ y_k(0) = 0 \]

\[ - (y_k(1) - (h + 1) y_k'(1)) = \lambda (y_k(1) - y_k'(1)) \]

Solution of problem (4), (5) is \( y_k(t) = \sin \sqrt{\lambda - \gamma_k} t. \) Obviously, the eigenvalues of \( L_0 \) and the problem (1)-(3) are the same. They are obtained from equation

\[ - (\sin \sqrt{\lambda - \gamma_k} - (h + 1) \sqrt{\lambda - \gamma_k} \cos \sqrt{\lambda - \gamma_k}) = \lambda (\sin \sqrt{\lambda - \gamma_k} - \sqrt{\lambda - \gamma_k} \cos \sqrt{\lambda - \gamma_k}) \]
By taking
\[ \sqrt{\lambda - \gamma_k} = z \]
we can put (7) in form
\[ \sin z - (h + 1)z \cos z = \left( z^2 + \gamma_k \right) (z \cos z - \sin z) \] (8)

Eigenvectors of operator \( L_0 \) are
\[ \Psi_n = C_n [ \sin (\alpha_{k,n,m}) \varphi_k \beta - \sin \alpha_{k,n,m} + \alpha_{k,n,m} \cos \alpha_{k,n,m} ] \varphi_k \] (9)

where \( \alpha_{k,n,m} \) are roots of (8). Note that \( \alpha_{k,n} \) are imaginary numbers. \( C_n \) are coefficients. Normalizing that vectors we get the next orthonormal eigen-vectors:

\[ \Psi_n = \sqrt{\frac{4\pi_{k,n,m}}{K_{k,n,m}}} \left( \sin(\alpha_{k,n,m}) \varphi_k \beta - \sin \alpha_{k,n,m} + \alpha_{k,n,m} \cos \alpha_{k,n,m} \right) \varphi_k \]

where \( K_{k,n,m} = 2\alpha_{k,n,m} - h \sin 2\alpha_{k,n,m} + 2\alpha_{k,n,m} - 2\alpha_{k,n,m} \cos 2\alpha_{k,n,m} - 4\alpha_{k,n,m}^2 \sin 2\alpha_{k,n,m} + 2\alpha_{k,n,m}^2 + 2\alpha_{k,n,m}^2 \cos 2\alpha_{k,n,m} \).

Now find the roots of equation (8). Firstly we will investigate is there any imaginary root of that equation. For this reason taking in (8) \( z = iy, y > 0 \), we have
\[ \frac{-shy}{i} - (h + 1)iy \cosh y = (\gamma_k - y^2) \left( iy \cosh y + \frac{shy}{i} \right) \]

or expanding into series
\[ \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} + (h + 1) y \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} = (\gamma_k - y^2) \left( \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} - y \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} \right) \]

Simplifying we have
\[ \sum_{n=0}^{\infty} \frac{(h + 1)(2n + 1) - 1 - \gamma_k + \gamma_k (2n + 1)}{(2n + 1)!} y^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{2n+3}}{(2n)!} - \sum_{n=0}^{\infty} \frac{y^{2n+3}}{(2n + 1)!} \]

\[ yh + \sum_{n=0}^{\infty} \frac{y^{2n+3}(h (2n + 3) + (\gamma_k + 1) (2n + 2))}{(2n + 3)!} - \sum_{n=0}^{\infty} \frac{(2n - 1) y^{2n+3}}{(2n + 1)!} = 0. \]

From the last
\[ yh + \sum_{n=0}^{\infty} \frac{y^{2n+3} (2nh + 3h + 2ny_k + 2y_k + 2n + 2 - (2n + 3) (2h + 2) (2n - 1))}{(2n + 3)!} = 0. \] (10)

Consider the function
\[ f(z) = 2zh + 3h + 2y_k + 2 \gamma_k + 2z + 2 - 8z^3 - 16z^2 - 2z + 6. \]

Since \( f(0) = 3h + 2 \gamma_k + 8 > 0 \), \( f(z) \to -\infty \) as \( z \to +\infty \), and \( f(z) \) is continuous on \((0, +\infty)\), then it has roots on positive semiaxis. There is only one sign change of terms of \( f(z) = -8z^3 - 16z^2 + 2zh + 2y_k + 8 + 3h + 2 \gamma_k. \)
Thus, function $f(z)$ has only one positive root by Descartes principle. Denote it by $z = M$. Note that $f(z) > 0$ when $z < M$ and $f(z) < 0$ when $z > M$. Therefore, coefficients of series (10) are positive when $n < [M]$ and negative when $n > [M]$. Thus there is only one change of sign of coefficients (10). But obviously equation

$$-\sinh y + (h+1) \cosh y = (\gamma_k - y^2) (\sinh y - y \cosh y)$$

or

$$\cosh y = \frac{\gamma_k - y^2 + 1}{(h+1) y + y (\gamma_k - y^2)}$$

has no any root for great $y$ values. Left hand side of (11) goes to 1, while right hand side goes to 0 when $y \to \infty$. Consequently we get that eigenvalues corresponding to that roots are

$$\lambda_k = \gamma_k + a_{k,0}$$

where $a_{k,0} = iy$ and form some bounded set.

Now we shall look for real roots of equation (8). Rewrite it in the form

$$\cosh z = \frac{z^2 + \gamma_k + 1}{z^2 + \gamma_k z + (h+1) z^2}.$$  

From (13) denoting real roots by $\alpha_m$ we get

$$\alpha_m = \frac{\pi}{2} + \pi m + O\left(\frac{1}{m}\right)$$

for large $m$ values. Eigenvalues of $L_0$ corresponding to that roots are

$$\lambda_{k,m} = \gamma_k + \left[\pi m + \frac{\pi}{2} + O\left(\frac{1}{m}\right)\right]^2.$$  

Thus we have proved the next theorem.

**Theorem 1.** Eigenvalues of operator $L_0$ form the next two sequence

$$\lambda_k = \gamma_k + a_{k,0}^2,$$  

(12)

and

$$\lambda_{k,m} = \gamma_k + \left[\pi m + \frac{\pi}{2} + O\left(\frac{1}{m}\right)\right]^2,$$

(15)

when $m \to \infty$.

**Lemma 1.** If eigenvalues of $A \gamma_k \sim a \cdot k^\alpha$ ($\alpha > 0, a > 0$), then eigenvalues of operator $L_0$ have the next asymptotics at large $n$

$$\lambda_n \sim C n^{2/\alpha},$$

where $C$ is some constant.

**Proof.** Firstly find asymptotics of distribution function $N(\lambda)$ of operator $L_0$. We have

$$N(\lambda) \equiv \sum_{\lambda_k < \lambda} 1 + \sum_{\gamma_k < \lambda} 1 \equiv N_1(\lambda) + N_2(\lambda)$$

By using formulas (12) and (15) we get

$$N_1(\lambda) \sim C_1 \lambda^{2\alpha}$$

and

$$N_2(\lambda) \sim C_2 \lambda^{2/\alpha}. $$
But if \( \alpha > 0 \) then \( \frac{1}{2} < \frac{2\alpha}{\alpha^2} \), thus \( N(\lambda) \sim N_2(\lambda) \). From which by using similar arguments as in [11] we get the statement of Lemma 1.

Now we will calculate the first regularized trace of operator \( L \).

Since \( Q \) is a bounded operator in \( L_2(H, (0, 1) \oplus H) \), operator \( L \) is discrete (spectrum is discrete). So, its eigenvalues can be arranged in ascending order. Denote them by \( \mu_n, (n = 1, \infty) \), \( \mu_1 \leq \mu_2 \leq \ldots \).

From boundedness of \( Q \) it follows that
\[
\mu_n \sim Cn^{\frac{2\alpha}{\alpha^2}}, n \to \infty.
\]
(16)

In virtue of asymptotics (16) in similar way as in [13] we get
\[
\lim_{m \to \infty} \frac{1}{n_m} \sum_{n=1}^{n_m} [\mu_n - \lambda_n - (Q\Psi_n, \Psi_n)] = 0,
\]
(17)

where \( n_m \) is some subsequence of natural numbers.

**Lemma 2.** Series
\[
\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} 4\alpha_{k,m} h \int_0^{\frac{1}{2}} \sin^2 \alpha_{k,m} t q_k(t) \, dt
\]
where \( K_{k,m} = 2\alpha_{k,m} h - h \sin 2\alpha_{k,m} + 2\alpha_{k,m} - 2\alpha_{k,m} \cos 2\alpha_{k,m} - 4\alpha_{k,m} \sin 2\alpha_{k,m} + 2\alpha_{k,m}^3 + 2\alpha_{k,m}^3 \cos 2\alpha_{k,m}, \) and \( q_k(t) = (q(t)\phi_k, \phi_k) \), is absolutely convergent.

**Proof.** In virtue of formulas (14) and (12) the summund for big \( m \) values is equivalent to
\[
O\left(\frac{1}{m^2}\right) \int_0^1 \left| q_k(t) \right| \, dt.
\]
Now validity of statement follows from assumptions 1) and 2).

In virtue of Lemma 2 and expression (9)
\[
\lim_{m \to \infty} \frac{1}{n_m} \sum_{n=1}^{n_m} (Q\Psi_n, \Psi_n) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} 2\alpha_{k,m} h \int_0^{\frac{1}{2}} \cos(2\alpha_{k,m} t) q_k(t) \, dt
\]

Consider the finite sum
\[
S_N = \sum_{m=0}^{N} 2\alpha_{k,m} h \cos(2\alpha_{k,m} t)
\]
(18)

For evaluating \( \lim_{N \to \infty} S_N \) choose a function of complex variable whose poles are roots of equation (8), and residues of that function at poles are terms of sum (18). Taking the function
\[
g(z) = \frac{2zh \cos 2zt}{\sin z - \left((h + 1)\cos z + (h + 1)z \sin z\right) \left(\cos z - \sin z\right)^2}
\]
we can see that it has such properties. Hence its poles are \( \alpha_{k,m} \).

Evaluate residues of \( g(z) \) at poles. Thus
\[
\text{res}_{z=\alpha_{k,m}} g(z) = \left(\frac{(\cos z - (h + 1) \cos z + (h + 1)z \sin z) (\cos z - \sin z)}{(\cos z - \sin z)^2} - \frac{(\sin z - (h + 1) \cos z + (h + 1)z \sin z) (\cos z - \sin z)}{(\cos z - \sin z)^2} \right)_{z=\alpha_{k,m}}
\]
and \(B\) where it follows that

\[
\int_{z=\alpha_m} (\alpha_{k,n} \cos \alpha_{k,n} - \sin \alpha_{k,m})^2
\]

\[
= 2\alpha_{k,m} \cos 2\alpha_{k,m} t
\]

\[
= \frac{2\alpha_{k,m} h \cos (2\alpha_{k,m} t)}{A_{k,m}}
\]

\[
- h \alpha_{k,m} + h \cos \alpha_{k,m} \sin \alpha_{k,m} - 2\alpha_{k,m} \cos \alpha_{k,m} \sin \alpha_{k,m} \cos \alpha_{k,m} - 2\alpha_{k,m} \sin^2 \alpha_{k,m}
\]

\[
= \frac{-2h \alpha_{k,m} + h \sin 2\alpha_{k,m} - 2\alpha_{k,m} \cos \alpha_{k,m} \sin \alpha_{k,m} + 4\alpha_{k,m} \sin 2\alpha_{k,m} + 2\alpha_{k,m} \cos 2\alpha_{k,m}}{4\alpha_{k,m} \cos (2\alpha_{k,m} t)}
\]

where \(A_{k,m} = (-h \cos \alpha_{k,m} + h \alpha_{k,m} \sin \alpha_{k,m} + \alpha_{k,m} \sin \alpha_{k,m})(\alpha_{k,m} \cos \alpha_{k,m} - \sin \alpha_{k,m}) - \alpha_{k,m}^2 \cos \alpha_{k,m} + 4\alpha_{k,m} \sin \alpha_{k,m} \cos \alpha_{k,m} - 2\alpha_{k,m} \sin^2 \alpha_{k,m} \). As it is seen from (19) residues of \(g(z)\) at poles are terms of sum in (18).

But roots of equation \(z \cos z - \sin z = 0\) are also poles of \(g(z)\).

Denote them by \(\beta_n\). They have the asymptotics \(\beta_n \sim \frac{\pi}{2} + \pi n\) for large \(n\). Thus

\[
\text{res } g(z) = \frac{2\beta_n h \cos (2\beta_n t)}{z - \beta_n - \beta_n (h + 1) \cos \beta_n - (\beta_n + \gamma_k)(\beta_n \cos \beta_n - \sin \beta_n) \beta_n \sin \beta_n}
\]

in virtue of relation

\[
\beta_n \cos \beta_n - \sin \beta_n = 0,
\]

it follows that

\[
\text{res } g(z) = \frac{2\beta_n h \cos (2\beta_n t)}{(z - \beta_n - \beta_n (h + 1) \cos \beta_n - (\beta_n + \gamma_k)(\beta_n \cos \beta_n - \sin \beta_n) \beta_n \sin \beta_n)} = \frac{2\beta_n h \cos (2\beta_n t)}{-\beta_n \cos \beta_n (\beta_n \sin \beta_n)} = \frac{4 \cos (2\beta_n t)}{\beta_n \sin 2\beta_n}.
\]

We must select for each \(k\) a contour which includes all \(\alpha_{k,m} (m = 0, N)\) values, so integration of \(g(z)\) along that contour will yield the sum in (19) by Cauchy theorem.

For that purpose take a rectangular contour \(C_N\) with vertices at points \(A_N \pm i B, \pm i B\), where \(A_N = \pi (N + 1)\), and \(B > |\alpha_{k,0}|\).

Let \(C_N\) by pass the origin and \(-\alpha_{k,0}\) along semicircle from the left, and imaginary numbers \(\alpha_{k,0} (n = 1, M_0)\) from the right. Since \(g(z)\) is odd function of argument \(z\) the integral along left-hand side of contour vanishes. Consider integral along semicircle by-pasing zero from the left:

\[
i = \int_{z=\alpha_m} g(z) dz
\]

\[
= \int_{z=\alpha_m} \frac{2zh \cos 2zt dz}{(z - (h + 1) \cos z)(z \cos z - \sin z)(z^2 + \gamma_k)(z \cos z - \sin z)^2} = \int_{z=\alpha_m} \frac{2zh [1 - \frac{1}{2} z + ...]}{F(z)},
\]

where \(F(z) = \left[z - \frac{1}{2} + ... (h + 1) z \left(1 - \frac{1}{2} + ...ight) - \left(z - \frac{1}{2} + ...ight)\right] - \frac{1}{2} z^2 + ... \)

When \(r \to 0\)

\[
\int_{z=\alpha_m} g(z) dz \sim \int_{z=\alpha_m} \frac{2zh [1 - 2z^2 + ...]}{-2\gamma_k z^2} dz.
\]
so
\[
\lim_{r \to 0} \int_{z = re^{i\phi}} g(z) \, dz = \lim_{r \to 0} \int_{\frac{\pi}{2}}^{\pi/2} \frac{h \, r \, e^{i\phi} \, d\phi}{-\gamma_k \, r \\ & \gamma_k} = -\frac{h}{\gamma_k} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = -\frac{\pi \gamma}{\gamma_k},
\]
(20)

Since for big \(z\) values
\[
g(z) \sim \frac{\cos 2zt}{z^3 \cos^2 2z}
\]
taking \(z = u + vi\) it is easily seen that \(g(z)\) will be of order \(e^{2\pi t(z-1)}\). That is why integral along upper and lower parts of contour vanishes as \(B \to \infty\).

On the right hand side of contour when \(N \to \infty\), we have
\[
\lim_{B \to \infty} \frac{1}{2\pi i} \int_{A_N - iB}^{A_N + iB} g(z) \, dz \sim \lim_{B \to \infty} \frac{1}{2\pi i} \int_{-iB}^{+iB} \frac{\cos (2\pi (N + 1) t + 2i\nu)}{(A_N + iv)^3 (1 + \cosh 2\nu)} \, dv \sim \frac{1}{A_N^3 \pi} \int_{-\infty}^{+\infty} \frac{\cosh (2\nu)}{(1 + \cosh 2\nu)} \, dv \to 0.
\]

Therefore,
\[
\frac{1}{2\pi i} \lim_{N \to \infty} \int_{C_N} g(z) \, dz \qquad q_k(t) \, dt = \lim_{N \to \infty} \int_{0}^{1} \left[ S_N(t) + T_N(t) - \frac{h}{\gamma_k} \right] q_k(t) \, dt = \lim_{N \to \infty} \int_{0}^{1} \left[ S_N(t) + T_N(t) \right] q_k(t) \, dt
\]
(21)

where
\[
T_N(t) = \sum_{n=1}^{N} \frac{4 \cos (2\beta_n t)}{\beta_n \sin 2\beta_n}.
\]

For evaluating \(\lim_{N \to \infty} T_N(t)\) in accordance with above arguments select function of complex variable
\[
G(z) = \frac{\cos 2zt}{(z \cos z - \sin z) \cos z}
\]

Obviously poles of that function are zeros of \(\cos z\), i.e., \(\frac{\pi}{2} + \pi n = \delta_n\) and \(\beta_n\).

\[
\text{res} \, G(z) \at \, z = \beta_n = \frac{2 \cos 2\beta_n t}{(\cos \beta_n - \beta_n \sin \beta_n - \cos \beta_n) \cos \beta_n} = -\frac{2 \cos 2\beta_n t}{\beta_n \sin \beta_n \cos \beta_n} = -\frac{4 \cos 2\beta_n t}{\beta_n \sin 2\beta_n}
\]
(22)

and
\[
\text{res} \, G(z) \at \, z = \delta_n = \frac{\cos (2\delta_n t)}{\sin^2 \left( \frac{\pi}{2} + \pi n \right)} = \cos (\pi + 2\pi n) t = \cos (2n + 1) \pi t.
\]
(23)

Selecting corresponding contour in the way similar to one done above and extending it to infinity we can show that limit of \(G(z)\) along it vanishes. Denote by \(M_N(t)\) the sum
\[
\sum_{n=1}^{N} \cos (2n + 1) \pi t.
\]
Thus,

\[
\lim_{N \to \infty} \int_0^1 S_N(t) f_k(t) \, dt = -\lim_{N \to \infty} \int_0^1 T_N(t) g_k(t) \, dt = -\lim_{N \to \infty} \int_0^1 M_N(t) g_k(t) \, dt = \frac{1}{4} [q_k(1) - q_k(0)]
\]  

(24)

Summing the last for \(k\), when \(k = 1, \infty\) we will have from (17)

\[
\lim_{m \to \infty} \sum_{n=1}^m [\mu_n - \lambda_n] = \lim_{m \to \infty} \sum_{n=1}^m (Q \Psi_n, \Psi_n) = \frac{1}{4} [trq(1) - trq(0)]
\]

Thus we have proved the next theorem.

**Theorem 2.** Under the conditions 1)-3) and Lemma 1, the next trace formula for operator \(L\) is true

\[
\lim_{m \to \infty} \sum_{n=1}^m [\mu_n - \lambda_n] = \frac{1}{4} [trq(1) - trq(0)]
\]

References