Order-Lipschitz Mappings Restricted with Linear Bounded Mappings in Normed Vector Spaces without Normalities of Involving Cones via Methods of Upper and Lower Solutions

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Abstract. In this paper, without assuming the normalities of cones, we prove some new fixed point theorems of order-Lipschitz mappings restricted with linear bounded mappings in normed vector space in the framework of \(w\)-convergence via the method of upper and lower solutions. It is worth mentioning that the unique existence result of fixed points in this paper, presents a characterization of Picard-completeness of order-Lipschitz mappings.

1. Introduction

Let \(P\) be cone of a normed vector space \((E, \|\cdot\|), D \subset E\) and \(\preceq\) the partial order on \(E\) introduced by \(P\). Recall that a mapping \(T : D \rightarrow E\) is called an order-Lipschitz mapping restricted with linear bounded mappings if there exist linear bounded mappings \(A, B : P \rightarrow P\) such that

\[-B(x - y) \leq Tx - Ty \leq A(x - y), \ \forall x, y \in D, \ y \preceq x. \] (1)

The research on fixed points of order-Lipschitz mappings was initiated by Krasnoselskii and Zabreiko [1]. Assuming that \(P\) is a normal solid cone of a Banach space \(E\), Krasnoselskii and Zabreiko [1] investigated the unique existence result of fixed points for order-Lipschitz mappings restricted with linear bounded mappings provided that \(A = B\) and \(\|A\| < 1\), which was then improved by Zhang and Sun [2] to the case that the spectral radius \(r(A) < 1\). Without assuming the solidness of \(P\), Sun [3] studied fixed points of order-Lipschitz mappings restricted with nonnegative real numbers via the method of upper and lower solutions.

Note that in [1-3], it is necessarily assumed that the cone is normal. Recently, without assuming the normality of the cone, Jiang and Li [4] proved the following fixed point result of order-Lipschitz mappings:

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restricted with vectors in Banach algebras in the framework of $w$-convergence.

**Theorem 1** ([4, Theorem 3]) Let $P$ be a solid cone of a Banach algebra $B$, $u_0, v_0 \in E$ with $u_0 \leq v_0$ and $T : D = [u_0, v_0] \rightarrow E$ an order-Lipschitz mapping restricted with vectors $A \in P$ and $B = \theta$. Assume that $u_0 \leq Tu_0, v_0 \leq v_0$ (i.e., $u_0$ and $v_0$ are a pair of lower and upper solutions of $T$), $r(A) < 1$ and $T$ is Picard-complete at $u_0$ and $v_0$. Then $T$ has a unique fixed point $u^* \in [u_0, v_0]$, and for each $x_0 \in [u_0, v_0]$, $x_n \xrightarrow{w} u^*$, where $\{x_n\} = O(T, x_0)$ and $O(T, x_0)$ denotes the Picard iteration sequence of $T$ at $x_0$ (i.e., $x_n = T^nx_0$ for each $n$).

In this paper, we shall consider fixed point theory of order-Lipschitz mappings restricted with linear bounded mappings in normed vector spaces (instead of, Banach spaces [1,2] and Banach algebras [3]) with non-normal cones. We first prove some fixed point theorems under the assumption that the order-Lipschitz mapping $T$ has only a lower solution or an upper solution. Furthermore, we investigate the unique existence of fixed points in the case that $T$ has a pair of lower and upper solutions. It is worth mentioning that the unique existence result of fixed points in this paper, presents a characterization of Picard-completeness of order-Lipschitz mappings (see Proposition 2). In addition, we present a suitable example which shows the usability of our theorems.

2. Preliminaries

Let $P$ be a cone of a normed vector space $E$. A cone $P$ induces partial order $\leq$ on $E$ by $x \leq y \iff y - x \in P$ for each $x, y \in X$. In this case, $E$ is called an ordered normed vector space. For each $u_0, v_0 \in E$ with $u_0 \leq v_0$, and set $[u_0, v_0] = \{u \in E : u_0 \leq u \leq v_0\}$, $[u_0, +\infty) = \{x \in E : u_0 \leq x\}$ and $(-\infty, v_0] = \{x \in E : x \leq v_0\}$. A cone $P$ is solid [5] if $\text{int} P \neq \emptyset$, where $\text{int} P$ denotes the interior of $P$. For each $x, y \in E$ with $y - x \in \text{int} P$, we write $x \ll y$. A cone $P$ of a normed vector space $E$ is normal [5] if there is a positive number $N$ such that $x, y \in E$ and $\theta \leq x \leq y$ implies that $\|x\| \leq N\|y\|$, and the minimal $N$ is called a normal constant of $P$. Note that a cone $P$ of a normed vector space $E$ is non-normal if and only if there exist $\{u_n\}, \{v_n\} \subset P$ such that $u_n + v_n \xrightarrow{||\cdot||} \theta \Rightarrow u_n \underset{||\cdot||}{\rightharpoonup} \theta$. Consequently, if $P$ is non-normal then the sandwich theorem in the sense of norm-convergence does not hold. While, it has been shown in [6] that the sandwich theorem in the sense of $w$-convergence still holds even if $P$ is non-normal.

Let $P$ be a solid cone of a normed vector space $E$. A sequence $\{x_n\} \subset E$ is $w$-convergent [6] if for each $\epsilon \in \text{int}P$, there exist a positive integer $n_0$ and $x \in E$ such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geq n_0$ (denote $x_n \xrightarrow{w} x$ and $x$ is called a $w$-limit of $\{x_n\}$). A sequence $\{x_n\} \subset E$ is $w$-Cauchy [4] if for each $\epsilon \in \text{int}P$, there exists a positive integer $n_0$ such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m, n \geq n_0$, i.e., $x_n - x_m \xrightarrow{w} \theta(m, n \rightarrow \infty)$. A subset $D \subset E$ is $w$-closed [4] if for each $\{x_n\} \subset D$, $x_n \xrightarrow{w} x$ implies $x \in D$.

**Lemma 1** ([4,6]) Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \leq v_0$. Then

(i) each sequence $\{x_n\} \subset E$ has a unique $w$-limit;

(ii) the partial order intervals $[u_0, v_0]$, $[u_0, +\infty)$ and $(-\infty, v_0]$ are $w$-closed;

(iii) for each $\{x_n\}, \{y_n\}, \{z_n\} \subset E$ with $x_n \leq y_n \leq z_n$ for each $n$, $x_n \xrightarrow{w} z$ and $z_n \xrightarrow{w} z$ imply $y_n \xrightarrow{w} z$, where $z \in E$.

**Lemma 2** ([6]) Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $x_n \in E$. Then $x_n \xrightarrow{||\cdot||} x$ implies $x_n \xrightarrow{w} x$. Moreover, if $P$ is normal then $x_n \xrightarrow{w} x \iff x_n \xrightarrow{||\cdot||} x$.

Let $P$ be a solid cone of a normed vector space $E$, $D \subset E, x_0 \in D$ and $T : E \rightarrow E$. If the Picard iteration sequence $O(T, x_0)$ is $w$-convergent provided that it is $w$-Cauchy, then $T$ is said to be Picard-complete at $x_0$. If $T$ is Picard-complete at each $x \in D$ then it is said to be Picard-complete on $D$.

**Remark 1** It is clear that if $O(T, x_0)$ is $w$-convergent then $T$ is certainly Picard-complete at $x_0$. In particular when $P$ is a normal cone of a Banach space $E$, each mapping $T : E \rightarrow E$ is Picard-complete on $E$ by Lemma
2.

Let \((E, \| \cdot \|)\) be a normed vector space, \(\{x_n\} \subset E\) and \(D \subset E\). If \(x_n \leq x_m\) or \(x_m \leq x_n\) for each \(m \neq n\) then \(\{x_n\}\) is said to be comparable. If there exists some \(c > 0\) such that \(\|x_n\| \leq c\) for each \(n\) then \(\{x_n\}\) is said to be bounded. For each mapping \(T : D \rightarrow E\), set

\[
T_{B_{\infty}} = \{ x \in D : O(T, x) \text{ is bounded and comparable} \}.
\]

**Proposition 1** For each \(\alpha > 1\) and each \(a, b \geq 0\) with \(a \leq b\), we have

\[
b^\alpha - a^\alpha \leq 2^{k_0} b^\alpha (b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}}),
\]

where \(k_0 = \min\{k \in \mathbb{N}_+ : \frac{n}{k} \leq 1\}\) and \(\mathbb{N}_+\) denotes the set of all positive integers.

**Proof.** Direct calculation that

\[
b^\alpha - a^\alpha = (b^{\frac{\alpha}{\alpha+1}} + a^{\frac{\alpha}{\alpha+1}})(b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}})
\]

\[
= (b^{\frac{\alpha}{\alpha+1}} + a^{\frac{\alpha}{\alpha+1}})(b^{\frac{\alpha}{\alpha+1}} + a^{\frac{\alpha}{\alpha+1}})(b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}})
\]

\[
= 2^{k_0} b^\alpha b^{\frac{\alpha}{\alpha+1}} \cdots b^{\frac{\alpha}{\alpha+1}} (b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}}) = 2^{k_0} b^{\frac{\alpha}{\alpha+1}} b^{\frac{\alpha}{\alpha+1}} \cdots b^{\frac{\alpha}{\alpha+1}} (b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}})
\]

\[
\leq 2^{k_0} b^\alpha (b^{\frac{\alpha}{\alpha+1}} - a^{\frac{\alpha}{\alpha+1}}).
\]

\(\square\)

The following example will show that there exists some mapping \(T : D \rightarrow E\) such that it is Picard-complete on \(D\).

**Example 1** Let \(E = C^1_{\mathbb{R}}[0, 1]\) be endowed with the norm \(\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}\) and \(P = \{x \in E : x(t) \geq 0, \forall t \in [0, 1]\}\), where \(\|x\|_{\infty} = \max x(t)\) for each \(x \in C_{\mathbb{R}}[0, 1]\). Then \((E, \| \cdot \|)\) is a Banach space and \(P\) is a non-normal solid cone \([5]\). Let \((Tx)(t) = \int_0^t x(s)ds\) for each \(x \in E\) and each \(t \in [0, 1]\), where \(\alpha > 1\).

Let \(x_0(t) \equiv 1\) and \(\{x_n\} = O(T, x_0)\). Clearly, \(T : P \rightarrow P\) is a nondecreasing mapping and \((Tx_0)(t) = \int_0^t x_0(s)ds\) is \(t \leq 1 - x_0(t)\) for each \(t \in [0, 1]\), and so \(\{x_n\}\) is comparable. By induction, for each \(t \in [0, 1]\) and \(n \geq 2\) we have

\[
x_n(t) = (T^n x_0)(t) = \int_0^t x_{n-1}^n(s)ds
\]

\[
\leq (1 + \alpha)^{n-2} (1 + \alpha + \alpha^2)^{n-3} \cdots (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1})
\]

\[
\leq 1,
\]

and so

\[
\|x_n\| = \|x_n\|_{\infty} + \|x'_n\|_{\infty} = \|x_0\|_{\infty} + \|x'_0\|_{\infty} \leq 2,
\]

which together with \(\|Tx_0\| = 2\) implies that \(\{x_n\}\) is bounded. This shows \(T_{\infty} \neq \emptyset\).

For each \(x_0 \in T_{B_{\infty}}\), set \(\{x_n\} = O(T, x_0)\). Then \(\{x_n\} \subset P\) since \(T(P) \subset P\), and there exists \(c > 0\) such that \(\|x_n\| \leq c\) for each \(n\). For each \(\epsilon > 0\), there exists \(\epsilon \in \text{int}P\) such that \(\|\epsilon\| < \min\{\frac{\epsilon}{2}, \frac{\epsilon}{2\|x_0\|_{\infty}}\}\), where \(k_0\) is the one given in Proposition 1. Suppose that \(\{x_n\}\) is \(\text{tv}-\text{Cauchy}\), then there exists a positive integer \(n_0\) such that

\[
-\epsilon < x_n - x_m < \epsilon \quad \text{for each} \quad m > n \geq n_0, \text{i.e.,} \quad -\epsilon < x_n(t) - x_m(t) < \epsilon(t) \quad \text{for each} \quad t \in [0, 1].
\]

Thus we have
$\|x_n - x_m\|_\infty = \max_{t \in [0,1]} |x_n(t) - x_m(t)| \leq \max_{t \in [0,1]} |e(t)| = \|e\|_\infty \leq |\|e\|| < \min\{\frac{1}{2}, (\frac{2k}{k+1})^{\frac{1}{2k}}\}$ for each $m, n \geq n_0$. Since $\{x_n\}$ is comparable, we have from $\{x_n\} \subset P$ and $0 < \frac{1}{k+1} \leq 1$ that $0 \leq |x_n(t) - x_m(t)| \leq |x_n(t) - x_n(t)|$ for each $m > n$ and each $t \in [0,1]$. Thus it follows from (2) and $(Tx_n)'(t) = x_{n-1}(t)$ that, for each $m > n \geq n_0 + 1$,

$$
\|x_n - x_m\| = \|x_n - x_m\|_\infty + \|(x_n - x_m)'\|_\infty
\leq \|x_n - x_m\|_\infty + 2^k e^a \max_{t \in [0,1]} |x_{n-1}(t) - x_{m-1}(t)|
\leq \|x_n - x_m\|_\infty + 2^k e^a \|x_{n-1}(t) - x_{m-1}(t)\|_\infty
\leq \|x_n - x_m\|_\infty + 2^k e^a \|x_{n-1}(t) - x_{m-1}(t)\|_\infty
\leq \frac{\|x_n - x_m\|_\infty}{\sqrt[k]{2}} + \frac{\|x_n - x_m\|_\infty}{\sqrt[k]{2}}
\leq \epsilon + \epsilon = \epsilon,
$$

which implies that $x_n \xrightarrow{\|\|} \theta$. Therefore, there exists $x^* \in P$ such that $x_n \xrightarrow{\|\|} x^*$, and hence $x_n \xrightarrow{\overline{\omega}} x^*$ by Lemma 2. This shows that $T$ is Picard-complete on $T_{\theta - C - \theta}$.

Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $D \subset E$. A mapping $T : D \to E$ is $w$-continuous at $x_0 \in D$ if for each $\{x_n\} \subset E, x_n \xrightarrow{w} x_0$ implies $Tx_n \xrightarrow{w} Tx_0$. If $T$ is $w$-continuous at each $x \in D$ then $T$ is said to be $w$-continuous on $D$.

**Lemma 4** Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $A : E \to E$ a linear bounded mapping with $A(P) \subset P$. Then $A$ is $w$-continuous on $E$.

**Proof.** Let $x \in E$ and $\{x_n\}$ be a sequence in $E$ such that $x_n \xrightarrow{w} x$. For each $e \in \text{int}P$, it is clear that $\frac{e}{m} \in \text{int}P$ for each $m$, and hence there exists $n_m$ such that $-\frac{e}{m} \ll x_n - x \ll \frac{e}{m}$ for each $n \geq n_m$. Note that $A$ is a linear mapping with $A(P) \subset P$, then $-\frac{e}{m} \ll Ax_n - Ax \ll \frac{e}{m}$ for each $n \geq n_m$. It is clear that $\frac{e}{m} \xrightarrow{\|\|} \theta(m \to \infty)$ since $A$ is a bounded mapping, and hence $\frac{e}{m} \xrightarrow{\theta} \theta(m \to \infty)$ by Lemma 2. Moreover, by (iii) of Lemma 1 we obtain $Ax_n - Ax \xrightarrow{\theta} \theta$, i.e., $A$ is continuous at $x$. 

3. Main results

We first state and prove some existence results of fixed points of order-Lipschitz mappings in normed vector spaces without assumption of the normality of the cone.

**Theorem 2** Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0 \in E$ and $T : D = [u_0, +\infty) \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \theta$, where $\theta$ denotes the zero mapping. Assume that $u_0 \leq Tu_0$, $r(A) < 1$ and $T$ is Picard-complete at $u_0$. Then $T$ has a fixed point $u^* \in [u_0, +\infty)$. Moreover, let $u \in [u_0, +\infty)$ be a fixed point of $T$ such that it is comparable to $u^*$ (i.e., $u \leq u^*$ or $u^* \leq u$), then $u = u^*$.

**Proof.** Since $A : P \to P$ is a linear bounded mapping with $r(A) < 1$, the inverse of $I - A$ exists, denote it by $(I - A)^{-1}$. Moreover by Neumann’s formula,

$$
(I - A)^{-1} = \sum_{n=0}^\infty A^n = I + A + A^2 + \cdots + A^n + \cdots,
$$

(3)
which implies that $(I - A)^{-1} : P \to P$ is a linear bounded mapping. It follows from $r(A) < 1$ and Gelfand’s formula that there exist a positive integer $n_0$ and $\beta \in (r(A), 1)$ such that

$$
\|A^n\| \leq \beta^n, \quad \forall \ n \geq n_0.
$$

(4)

Set $\{u_n\} = O(T, u_0)$. Note that (1) holds for $D = [u_0, +\infty)$, then $B = \emptyset$ implies that $T$ is nondecreasing on $[u_0, +\infty)$. Thus by $u_0 \leq Tu_0$, 

$$
u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1}, \quad \forall \ n.
$$

(5)

Moreover by (1),

$$
\theta \leq u_{n+1} - u_n \leq A(u_n - u_{n-1}) \leq \cdots \leq A^n(u_1 - u_0), \quad \forall \ n,
$$

and so by (3),

$$
\theta \leq u_m - u_n = \sum_{i=n}^{m-1}(u_{i+1} - u_i) \leq \sum_{i=n}^{m-1}A^i(u_1 - u_0)
\leq A^n\sum_{i=0}^{m-n-1}A^i(u_1 - u_0) \leq A^n(I - A)^{-1}(u_1 - u_0), \quad \forall \ m > n.
$$

It follows from (4) that $A^n(I - A)^{-1}(u_1 - u_0) \xrightarrow{n} \theta$ since $\|A^n(I - A)^{-1}(u_1 - u_0)\| \leq \|A^n\|\|I - A\|^{-1}(u_1 - u_0)\| \leq \beta^n\|I - A\|^{-1}(u_1 - u_0)\|$ for each $n \geq n_0$, and hence by Lemma 2,

$$
A^n(I - A)^{-1}(u_1 - u_0) \xrightarrow{n} \theta,
$$

which together with (4) and (iii) of Lemma 1 implies that

$$
u_m - u_n \xrightarrow{n} \theta(m > n \to \infty),
$$

(6)
i.e., $\{u_n\}$ is a $\theta$-Cauchy sequence. Since $T$ is Picard-complete at $u_0$, there exists $u' \in E$ such that

$$
u_n \xrightarrow{n} u'.
$$

(7)

Note that $u_m \in [u_n, +\infty)$ for each $m \geq n$ by (5), then by (ii) of Lemma 1,

$$
u_n \leq u', \quad \forall \ n.
$$

(8)

Moreover by the nondecreasing property of $T$ on $[u_0, +\infty)$,

$$
u_{n+1} \leq Tu', \quad \forall \ n,
$$

(9)

which together with (7) and (ii) of Lemma 1 implies that

$$
u' \leq Tu'.
$$

(10)

Thus it follows from (5), (8) and (9) that

$$
\theta \leq Tu' - u_{n+1} = Tu' - Tu_n \leq A(u' - u_n), \quad \forall \ n.
$$

(11)

Letting $n \to \infty$ in (11), by (7), (iii) of Lemma 1 and Lemma 3, we obtain $u_{n+1} \xrightarrow{n} Tu'$. Moreover by (i) of Lemma 1, we get $u' = Tu'$. Let $u' \in [u_0, +\infty)$ be another fixed point of $T$ such that it is comparable to $u'$. We may assume that $u' \leq x'$ (the proof of the other case $x \leq u'$ is similar). Then by (5), we get

$$
\theta \leq x' - u' = T^n x' - T^n u' \leq A^n(u - u') \quad \text{for each} \ n.
$$

(This together with (4), (iii) of Lemma 1 and Lemma 2 implies $x' = u'$.)

\[Q.E.D.\]
In analogy to Theorem 2, we have the following fixed point result.

**Theorem 3** Let $P$ be a solid cone of a normed vector space $(E, \| \cdot \|)$, $v_0 \in E$ and $T : D = (-\infty, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$. Assume that $Tv_0 \leq v_0$, $r(A) < 1$ and $T$ is Picard-complete at $v_0$. Then $T$ has a fixed point $u^* \in (-\infty, v_0]$. Moreover, let $v \in (-\infty, v_0]$ be a fixed point of $T$ such that it is comparable to $v^*$.

**Proof.** Note that (1) holds for $D = (-\infty, v_0]$, then $B = \hat{\theta}$ implies that $T$ is nondecreasing on $(-\infty, v_0]$. Set $\{v_n\} = O(T, v_0)$, from $Tv_0 \leq v_0$ we get

$$v_{n+1} \leq v_n \leq \cdots \leq v_1 \leq v_0, \forall n. \tag{12}$$

In analogy to the proof of Theorem 2, by (1), (3) and (12), we get

$$\begin{align*}
\theta &\leq v_n - v_m = \sum_{i=n}^{m-1} (v_i - u_{i+1}) \leq \sum_{i=n}^{m-1} A^i(v_0 - v_1) \\
&= A^m \sum_{i=0}^{m-n-1} A^i(v_0 - v_1) \leq A^m(I - A)^{-1}(v_0 - v_1), \forall m > n,
\end{align*}$$

which together with (4), (iii) of Lemma 1 and Lemma 3 implies that

$$v_n - v_m \xrightarrow{w} \theta(m > n \to \infty). \tag{13}$$

Thus by the Picard-completeness of $T$ at $v_0$, there exists $v' \in E$ such that

$$v_n \xrightarrow{w} v'. \tag{14}$$

In analogy to (8)-(10), by (12) and (ii) of Lemma 1 we get

$$Tv' \leq v' \leq v_n, \forall n. \tag{15}$$

Thus by (12) and (15), we get

$$\theta \leq v_{n+1} - Tv' = Tv_n - Tv' \leq A(v_n - v'), \forall n. \tag{16}$$

Letting $n \to \infty$ in (16), by (iii) of Lemma 1 and Lemma 3, we obtain $v_{n+1} \xrightarrow{w} Tv'$. Moreover by (i) of Lemma 1, we get $u^* = Tv'$. The rest proof is totally similar to that of Theorem 2, we omit it here. \qed

In the case that $T$ has a pair of lower and upper solutions, we obtain the unique existence theorem of fixed points as follows.

**Theorem 4** Let $P$ be a solid cone of a normed vector space $(E, \| \cdot \|)$, $u_0, v_0 \in E$ with $u_0 \leq v_0$ and $T : D = [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$. Assume that $u_0 \leq Tu_0, Tv_0 \leq v_0$, $r(A) < 1$ and $T$ is Picard-complete at $u_0$ and $v_0$. Then $T$ has a unique fixed point $u^* \in [u_0, v_0]$, and for each $x_0 \in [u_0, v_0]$, $x_n \xrightarrow{w} u^*$, where $\{x_n\} = O(T, x_0)$.

**Proof.** The existence of fixed points immediately follows from Theorems 2 and 3. Thus it suffices to show the uniqueness of fixed point. Following the proof of Theorems 2 and 3, we know that $T : [u_0, v_0] \to [u_0, v_0]$ is nondecreasing, and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \forall n. \tag{17}$$

By (1) and (17),

$$\theta \leq v_{n+1} - u_{n+1} \leq A(v_n - u_n), \forall n. \tag{18}$$
Letting \( n \to \infty \) in (18), by (7), (14), (iii) of Lemma 1 and Lemma 3, we obtain \( u^* = \nu^* \).

For each \( x_0 \in [u_0, v_0] \), set \( \{x_n\} = O(T, x_0) \). By the nondecreasing property of \( T \) on \([u_0, v_0]\),
\[
u_n \leq x_n \leq v_n.
\] (19)

Letting \( n \to \infty \) in (19), by \( u^* = \nu^*, (7) \), (14) and (iii) of Lemma 1, we get
\[
x_n \overset{w}{\to} u^*.
\] (20)

Let \( x^* \in [u_0, v_0] \) be another fixed point of \( T \). Set \( \{y_n\} = O(T, x^*) \), then \( y_n \equiv x^* \) and hence \( y_n \overset{w}{\to} x^* \). On the other hand, in analogy to (20) we obtain \( y_n \overset{w}{\to} u^* \). Thus by (i) of Lemma 1, we have \( x^* = u^* \). This shows that \( u^* \) is the unique fixed point of \( T \). \( \square \)

**Remark 2** It is clear that Theorem 4 is still valid in the case that \( E \) is a Banach algebra. Thus Theorem 1 is a particular case of Theorem 4 in Banach algebras with \( A \in P \).

**Remark 3** It follows from Theorem 4 and Remark 1 that \( T \) is Picard-complete on \([u_0, v_0]\) provided that \( T \) is Picard complete at \( u_0 \) and \( v_0 \) since for each \( x_0 \in [u_0, v_0] \), the Picard iteration sequence \( \{x_n\} \) is weakly convergent. Thus we have the following characterization of Picard-completeness of order-Lipschitz mappings.

**Proposition 2** Let \( P \) be a solid cone of a normed vector space \( (E, \| \cdot \|) \), \( u_0, v_0 \in E \) with \( u_0 \leq v_0 \) and \( T : [u_0, v_0] \to E \) an order-Lipschitz mapping restricted with linear bounded mappings \( A : P \to P \) and \( B = \hat{0} \). Assume that \( u_0 \leq Tu_0, Tv_0 \leq v_0 \) and \( r(A) < 1 \). Then the following two statements are equivalent:

(i) \( T \) is Picard-complete on \([u_0, v_0]\);

(ii) \( T \) has a unique fixed point \( u^* \in [u_0, v_0] \), and for each \( x_0 \in [u_0, v_0] \), the Picard iteration sequence \( \{x_n\} \) weakly converges to \( u^* \).

**Example 2** Let \( E \) and \( P \) be the same ones as those in Example 1. Let \( u_0 = \theta, v_0(t) \equiv 1 \) and \( (Tx)(t) = \int_0^t x^2(s)ds \) for each \( x \in E \) and each \( t \in [0, 1] \).

Clearly, \( u_0 \leq Tu_0, (Tv_0)(t) = t \leq 1 = v_0(t) \) for each \( t \in [0, 1] \), and \( u_n = u_0 \) for each \( n \). By Example 1, \( T \) is Picard-complete at \( u_0 \) and \( v_0 \). For each \( x, y \in [u_0, v_0] \) with \( y \leq x \) and each \( t \in [0, 1] \), we have
\[
0 \leq (Tx)(t) - (Ty)(t) = \int_0^t (x(s) - y(s))(x(s) + y(s))ds \leq 2A(x - y)(t),
\]
where \( (Ax)(t) = \int_0^t x(s)ds \) for each \( x \in P \) and \( t \in [0, 1] \). For each \( x \in E \) and \( t \in [0, 1] \), by induction we get
\[
(A^n x)(t) \leq \frac{1}{t^{n-1}} \leq \frac{1}{t^{n-1}} \leq \frac{1}{t^{n-1}},
\]
and so \( \|A^n x\|_\infty \leq \frac{1}{t^{n-1}} \). On the other hand, we have \( \|[A^n x]'\|_\infty = \|[A^{n-1} x]'\|_\infty \leq \frac{1}{(n-1)!} \) since \( (A^n x)'(t) = (A^{n-1} x)'(t) \). Thus \( \|A^n x\|_\infty + \|A^n x\|_\infty \leq \frac{1}{n!} + \frac{1}{(n-1)!} \) and \( \|A^n\| \leq \frac{1}{n!} + \frac{1}{(n-1)!} \). By Gelfand’s formula, we obtain \( 0 \leq r(A) = \lim_{n \to \infty} \sqrt[1]{1 + \frac{1}{(n-1)!}} \leq \lim_{n \to \infty} \frac{1}{1} \sqrt[1]{1 + \frac{1}{(n-1)!}} = 0 \), and Hence \( r(2A) = 0 \). This shows that all the assumptions of Theorem 5 are satisfied, and hence \( T \) has a unique fixed point \( u_0 = \theta \).

However, none of the results in [1-4] is applicable here since the cone \( P \) is non-normal and there does not exist \( A \in P \) such that (1) is satisfied.

Note that in Theorems 2-4 it is assumed that \( B = \hat{0} \), which together with (1) implies the nondecreasing property of \( T \). In what follows, we shall consider the case that \( B \neq \hat{0} \).

**Corollary 1** Let \( P \) be a solid cone of a normed vector space \( (E, \| \cdot \|) \), \( u_0, v_0 \in E \) with \( u_0 \leq v_0 \) and \( T : D = [u_0, v_0] \to E \) an order-Lipschitz mapping restricted with linear bounded mappings \( A, B : P \to P \). Assume that \( u_0 \leq Tu_0, Tv_0 \leq v_0 \), \( A, B \) are commutative (i.e., \( AB = BA \)), \( (I + B)^{-1} \) is invertible (i.e., \( (I + B)^{-1} \) exists), \( r(\tilde{A}) < 1 \), and \( \tilde{T} \) is Picard-complete at \( u_0 \) and \( v_0 \), where \( \tilde{A} = (I + B)^{-1}(A + B) \) and \( \tilde{T} = (I + B)^{-1}(T + B) \), then \( T \) has a unique fixed point in \([u_0, v_0]\).
Proof. By $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ and (1),

$$u_0 \preceq \tilde{T}u_0, \quad \tilde{T}v_0 \preceq v_0, \quad \theta \preceq \tilde{T}x - \tilde{T}y \preceq \tilde{A}(x - y), \quad \forall \ x, y \in [u_0, v_0], \ y \preceq x.$$

Note that $r(\tilde{A}) < 1$ and $\tilde{T}$ is Picard-complete at $u_0$ and $v_0$, then by Theorem 4, $\tilde{T}$ has a unique fixed point $u^* \in [u_0, v_0]$. Thus we have $Tu^* + Bu^* = u^* + Bu^*$ and so $Tu^* = u^*$. Let $x \in [u_0, v_0]$ be another fixed point of $T$, then $Tx = x$ and hence $\tilde{T}x = x$. Moreover by the uniqueness of fixed point of $\tilde{T}$ in $[u_0, v_0]$, we get $x = u^*$. Hence $u^*$ is the unique fixed point of $T$ in $[u_0, v_0]$.

In analogy to Corollary 1, we obtain the following fixed point result corresponding to Theorems 2 and 3.

**Corollary 2** Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0 \in E$ and $T : D = [u_0, +\infty) \to E$ (resp. $T : D = (-\infty, u_0] \to E$) an order-Lipschitz mapping restricted with linear bounded mappings $A, B : P \to P$. Assume that $u_0 \preceq Tu_0$ (resp. $Tu_0 \preceq u_0$), $A, B$ are commutative, $I + B$ is invertible, $r(\tilde{A}) < 1$, and $\tilde{T}$ is Picard-complete at $u_0$. Then $T$ has a fixed point in $[u_0, +\infty)$ (resp. $(-\infty, u_0]$).

In particular when $T$ is an order-Lipschitz mapping restricted with nonnegative real numbers, we have the following fixed point result by Corollary 1.

**Corollary 3** Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with nonnegative real numbers $A \in [0, 1)$ and $B \in [0, +\infty)$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$, $\tilde{T}$ is Picard-complete at $u_0$ and $v_0$, where $\tilde{T}u = \frac{Tu + Bu}{1 + B}$ for each $u \in [u_0, v_0]$, then $T$ has a unique fixed point $u^* \in [u_0, v_0]$.

**Remark 4** Note that in Theorems 2-4 and Corollaries 1-3, $E$ is not confined to a Banach space, i.e., $E$ needs not to be complete.

**References**