The Forward Order Laws for $\{1, 2, 3\}$- and $\{1, 2, 4\}$-inverses of a Three Matrix Products

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Abstract. In this article, we study the forward order laws for $\{1, 2, 3\}$- and $\{1, 2, 4\}$-inverses of a product of three matrices by using the maximal and minimal ranks of the generalized Schur complement. The necessary and sufficient conditions for $A_1[1, 2, 3]A_2[1, 2, 3]A_3[1, 2, 3] \subseteq (A_1A_2A_3)[1, 2, 3]$ and $A_1[1, 2, 4]A_2[1, 2, 4]A_3[1, 2, 4] \subseteq (A_1A_2A_3)[1, 2, 4]$ are presented.

1. Introduction

Throughout this paper $\mathbb{C}^{m \times n}$ and $\mathbb{C}^m$ denote the set of $m \times n$ complex matrices and $m$-dimensional complex vectors, respectively. The identity matrix in $\mathbb{C}^{m \times n}$ is denoted by $I_n$ and $O_{m \times n}$ is the $m \times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For any matrix $A \in \mathbb{C}^{m \times n}$, let $r(A)$, $A^*$, $R(A)$ and $N(A)$ denote the rank, the conjugate transpose, the range space (or column space) and the null space of $A$, respectively.

The Moore-Penrose inverse of $A \in \mathbb{C}^{m \times n}$ denoted by $A^+$, is the unique element $X \in \mathbb{C}^{n \times m}$ which satisfies the following four Penrose equations [6]:

\begin{align}
(1) \ AXA &= A, & (2) \ XAX &= X, & (3) \ (AX)^* &= AX, & (4) \ (XA)^* &=XA. \tag{1.1}
\end{align}

Let $\emptyset \neq \mathcal{C} \subseteq \{1, 2, 3, 4\}$. Then $A_{\mathcal{C}}$ denotes the set of all matrices $X$ which satisfy (i) for $i \in \mathcal{C}$. Any $X \in A_{\mathcal{C}}$ is called an $\mathcal{C}$-inverse of $A$. As usually, $X$ is called a $\{1, 3\}$-inverse or a least squares $\mathcal{C}$-inverse of $A$ if it is an element of $A[1, 3]$ and $X$ is called a $\{1, 4\}$-inverse or a minimum norm $\mathcal{C}$-inverse of $A$ if it is an element of $A[1, 4]$.

Similarly, $X$ is called a $\{1, 2, 3\}$-inverse of $A$ if it is an element of $A[1, 2, 3]$ and $X$ is called a $\{1, 2, 4\}$-inverse of $A$ if it is an element of $A[1, 2, 4]$. The unique $\{1, 2, 3, 4\}$-inverse of $A$ is called the Moore-Penrose inverse of $A$. We refer the reader to [1, 13] for basic results on the generalized inverses.

Theory and computations of the reverse order laws for generalized inverses of matrix product are important subjects in many branches of applied science, such as non-linear control theory, matrix theory, matrix algebra, see [6, 8, 9, 13]. One of the core problems in reverse order laws is to find the necessary conditions for $A_1[1, 2, 3]A_2[1, 2, 3]A_3[1, 2, 3] \subseteq (A_1A_2A_3)[1, 2, 3]$ and $A_1[1, 2, 4]A_2[1, 2, 4]A_3[1, 2, 4] \subseteq (A_1A_2A_3)[1, 2, 4]$ are presented.

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and sufficient conditions for the reverse order laws for the generalized inverse of matrix product and it has attracted considerable attention, see [1, 2, 7, 9, 18, 19].

In 1996, Grevill [3] first gave a necessary and sufficient condition for the reverse order law \((AB)\dagger = B\dagger A\dagger\). Since then, more equivalent conditions for the reverse order laws for generalized inverses of matrix product have been derived. Hartwig [4] and Tian [10, 11] studied the reverse order laws for Moore-Penrose inverse of three and multiple matrix product, respectively. Using the Product Singular Value Decomposition (PSVD), Wei [14] and De Pierro and Wei [2], studied necessary and sufficient conditions for \(B[1]A[1] \subseteq (AB)[1]\) and \(B[1, 2]A[1, 2] \subseteq (AB)[1, 2]\). With the same method, Wei [2, 15], Wei and Guo [16] studied the equivalent conditions for the reverse order laws of \([1]-inverses, [1, 2]-inverses, [1, 3]-inverses and [1, 4]-inverses of multiple matrix products. During the recent years, Zheng and Xiong [18, 19] studied the reverse order laws for \([1, 2, 3]-inverses and [1, 2, 4]-inverses of multiple products. For other interesting results on this subject see [1, 2, 8, 9, 14].

In 2007, Xiong and Zheng [17] studied the forward order law for the generalized inverse of multiple matrix products, by using the maximal rank of generalized Schur complement. With the same thread, in this paper we obtain the necessary and sufficient conditions for one side inclusion

\[
A_1[1, 2, 3]A_2[1, 2, 3]A_3[1, 2, 3] \subseteq (A_1A_2A_3)[1, 2, 3]
\]

(1.2)

and

\[
A_1[1, 2, 4]A_2[1, 2, 4]A_3[1, 2, 4] \subseteq (A_1A_2A_3)[1, 2, 4].
\]

(1.3)

To our knowledge, there is no article yet discussing the forward order laws for these two generalized inverses in the literature.

The main tools of the later discussion are the following lemmas.

**Lemma 1.1** [12] Let \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times k}, C \in \mathbb{C}^{k \times n}\) and \(D \in \mathbb{C}^{k \times k}\). Then

\[
\max_{A[1,2,3]} r(D - CA[1,2,3])B = \min_{A[1,2,3]} \left\{ r \left( \begin{pmatrix} A' & A'B \\ C & D \end{pmatrix} \right) - r(A), r \left( \begin{pmatrix} A'B \\ D \end{pmatrix} \right) \right\}.
\]

(1.4)

**Lemma 1.2** [1] Let \(A \in \mathbb{C}^{m \times n}, X \in \mathbb{C}^{n \times m}\), then

\[
X \in A[1,2,3] \iff A'AX = A' \quad \text{and} \quad r(X) = r(A),
\]

(1.6)

\[
X \in A[1,2,4] \iff XAA' = A' \quad \text{and} \quad r(X) = r(A).
\]

(1.7)

**Lemma 1.3** [5] Let \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times k}\) and \(C \in \mathbb{C}^{p \times n}\), then

\[
r \left( \begin{pmatrix} A & B \\ \end{pmatrix} \right) = r(A) + r(E_A B) = r(B) + r(E_B A),
\]

(1.8)

\[
r \left( \begin{pmatrix} A \\ C \end{pmatrix} \right) = r(A) + r(CF_A) = r(C) + r(AF_C).
\]

(1.9)

where the projectors \(E_A = I - AA^\dagger, E_B = I - BB^\dagger, F_A = I - A^\dagger A, F_C = I - C^\dagger C\).
2. The necessary and sufficient conditions for inclusion (1.2).

In this section, we will present some necessary and sufficient conditions for one side include (1.2), by using the maximal and minimal ranks of some generalized Schur complement forms. Let

\[ S_{(A_i; A_2; A_3)} = S_{\mu} = (A_1 A_2 A_3)^T - (A_1 A_2 A_3) A_1 A_2 A_3 X_1 X_2 X_3 = \mu^* - \mu^* \mu X_1 X_2 X_3 \]  

(2.10)

where \( A_i \in \mathbb{C}^{n \times n} \), \( X_i \in \mathbb{C}^{1 \times 2} \), \( i = 1, 2, 3 \), and \( \mu = A_1 A_2 A_3 \). For the convenience of readers, we first give a brief outline of the basic idea. From the formula (1.6) in Lemma 1.2, we know that the inclusion (1.2) holds if and only if

\[ \mu^* \mu X_1 X_2 X_3 = \mu^* \text{ and } r(X_1 X_2 X_3) = r(\mu) \]

hold for any \( X_i \in A_i [1, 2, 3] \), \( i = 1, 2, 3 \), and \( \mu = A_1 A_2 A_3 \), which are respectively equivalent to the following two identities

\[ \max_{X_1, X_2, X_3} r(\mu^* - \mu^* \mu X_1 X_2 X_3) = 0 \]  

(2.11)

and

\[ \max_{X_1, X_2, X_3} r(X_1 X_2 X_3) = \min_{X_1, X_2, X_3} r(X_1 X_2 X_3) = r(\mu). \]  

(2.12)

To begin with, some useful results are introduced, which presenting the necessary and sufficient conditions for inclusion (1.2).

**Lemma 2.1** Let \( A_i \in \mathbb{C}^{m \times n} \), \( X_i \in A_i [1, 2, 3] \), \( i = 1, 2, 3 \), and \( \mu = A_1 A_2 A_3 \). Then

\[ \max_{X_1, X_2, X_3} r(\mu^* - \mu^* \mu X_1 X_2 X_3) = r\left( \mu^* - \mu^* A_3 A_2 A_1, \mu^* A_3 A_2 E_{A_1}, \mu^* A_3 E_{A_2}, \mu^* E_{A_3} \right) \]

\[ = r\left( \mu^* - \mu^* A_3 A_2 A_1, \mu^* A_3 A_2, \mu^* A_3, \mu^* \right) \]

\[ \sum_{i=1}^{3} r(A_i), \right) \]  

(2.13)

**Proof.** According to Lemma 1.3 and Lemma 1.1 (1.4) with \( A = A_3 \), \( B = I_m, C = \mu^* \mu X_1 X_2 \) and \( D = \mu^* \), we have

\[ \max_{X_3} r(\mu^* - \mu^* \mu X_1 X_2 X_3) = \min \left\{ r\left( A^*_2 A_3, A^*_2 \right) - r(A_3), r\left( A^*_3 \right) \right\} \]

\[ = \min \left\{ r\left( \mu^* \mu X_1 X_2 - \mu^* A_3, \mu^* \right) - r(A_3), r\left( A^*_3 \right) \right\} \]

\[ = r\left( \mu^* \mu X_1 X_2 - \mu^* A_3, \mu^* E_{A_1} \right) \]

\[ = r\left( \mu^* \mu X_1 X_2 \left( I_m, O \right) - \left( \mu^* A_3, -\mu^* E_{A_3} \right) \right), \]  

(2.14)

where the third equality hold as

\[ r\left( A^*_2 A_3, A^*_2 \right) = r\left( \mu^* \mu X_1 X_2 - \mu^* A_3, A^*_3 \right) \leq r\left( \mu^* \mu X_1 X_2 - \mu^* A_3, \mu^* \right) + r\left( A^*_3 \right) \]

and

\[ r\left( \mu^* \mu X_1 X_2 - \mu^* A_3 \right) \leq r(\mu) \leq r(A_3) \]
and
\[
\left(\frac{O}{\mu' \mu X_1 X_2 - \mu' A_3} A_3^* \mu\right) = \left(\frac{\mu' \mu X_1 X_2 - \mu' A_3 }{\mu' E_{A_3}} + r(A_3)\right).
\]

Again by Lemma 1.3 and Lemma 1.1 (1.4) with \(A = A_2, B = (I_m, O, C = \mu' \mu X_1\) and \(D = (\mu' A_3, -\mu' E_{A_3})\), we obtain
\[
\begin{align*}
\max_{X_2, X_3} r(\mu' - \mu' \mu X_1 X_2 X_3) &= \max_{X_2} r\left(\mu' \mu X_1 X_2 (I_m, O) - \left(\mu' A_3, -\mu' E_{A_3}\right)\right) \\
&= \min\left\{r\left(A_2^* A_2 \frac{\mu' \mu X_1 \mu' A_3}{\mu' E_{A_3}} - \mu' E_{A_3}\right) - r(A_2), r\left(A_3^* \frac{E}{\mu' A_3} - \mu' E_{A_3}\right)\right\} \\
&= r\left(\mu' \mu X_1 - \mu' A_3 A_2 \frac{A_2^*}{\mu' A_3} - \mu' E_{A_3}\right) - r(A_2) \\
&= r\left(\left(\mu' \mu X_1 - \mu' A_3 A_2, \mu' A_3, -\mu' E_{A_3}\right) \cdot F(O, A_2^*, O)\right) \\
&= r\left(\mu' \mu X_1 (I_m, O, O) - \left(\mu' A_3 A_2, -\mu' A_3 E_{A_2}, \mu' E_{A_3}\right)\right),
\end{align*}
\]
where the third equality hold as
\[
r\left(A_2^* A_2 \frac{\mu' \mu X_1 A_3}{\mu' E_{A_3}} - \mu' E_{A_3}\right) = \left(\begin{array}{c} O \\ \mu' \mu X_1 - \mu' A_3 A_2 \frac{A_2^*}{\mu' A_3} - \mu' E_{A_3}\end{array}\right) \leq \left(\begin{array}{c} O \\ \mu' \mu X_1 - \mu' A_3 A_2 \frac{A_2^*}{\mu' A_3} - \mu' E_{A_3}\end{array}\right) + r\left(A_3^* \frac{E}{\mu' A_3} - \mu' E_{A_3}\right)
\]
and
\[
r\left(\mu' \mu X_1 - \mu' A_3 A_2 \frac{A_2^*}{\mu' A_3} - \mu' E_{A_3}\right) = r(\mu' \mu X_1 - \mu' A_3 A_2) \leq r(\mu) \leq r(A_2)
\]
and
\[
r\left(\mu' \mu X_1 - \mu' A_3 A_2 \frac{A_2^*}{\mu' A_3} - \mu' E_{A_3}\right) = r(\mu' \mu X_1 - \mu' A_3 A_2) + r\left(\mu' A_3 A_2, \mu' A_3, -\mu' E_{A_3}\right) \cdot F(O, A_2^*, O).
\]
Again by Lemma 1.3 and Lemma 1.1 (1.4) with \(A = A_1, B = (I_m, O, O), C = \mu' \mu\) and \(D = (\mu' A_3 A_2, -\mu' A_3 E_{A_2}, \mu' E_{A_3})\), we have
\[
\begin{align*}
\max_{X_1, X_3} r(\mu' - \mu' \mu X_1 X_2 X_3) &= \max_{X_1} r\left(\mu' \mu X_1 (I_m, O, O) - \left(\mu' A_3 A_2, -\mu' A_3 E_{A_2}, \mu' E_{A_3}\right)\right) \\
&= \min\left\{r\left(A_1^* A_1 \frac{\mu' \mu X_1 A_3}{\mu' E_{A_3}} - \mu' E_{A_3}\right) - r(A_1), r\left(A_3^* A_3 \frac{E}{\mu' A_3} - \mu' E_{A_3}\right)\right\} \\
&= r\left(\mu' \mu - \mu' A_3 A_2 A_1 \frac{A_1^*}{\mu' A_3 A_2} - \mu' A_3 E_{A_2} \frac{E}{\mu' E_{A_3}} - r(A_1)\right) \\
&= r\left(\mu' \mu - \mu' A_3 A_2 A_1, \mu' A_3 A_2, -\mu' A_3 E_{A_2}, \mu' E_{A_3}\right) \cdot F(O, A_1^*, O, O) \\
&= r\left(\mu' \mu - \mu' A_3 A_2 A_1, \mu' A_3 A_2 E_{A_2}, \mu' A_3 E_{A_2}, \mu' E_{A_3}\right),
\end{align*}
\]
where the third equality hold as
\[
\begin{bmatrix}
A_1^* & A_1^* & A_1^* \\
\mu^* & \mu^* & \mu^* \\
A_2 & A_2 & A_2 \\
O & -\mu^* & -\mu^*
\end{bmatrix}
= \begin{bmatrix}
A_1^* & A_1^* & A_1^* \\
\mu^* & \mu^* & \mu^* \\
A_2 & A_2 & A_2 \\
O & -\mu^* & -\mu^*
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
O & A_1^* & A_1^* \\
\mu^* - \mu^* & -\mu^* & -\mu^* \\
E_{A_1} & E_{A_1} & E_{A_1} \\
O & O & O
\end{bmatrix}
= \begin{bmatrix}
O & A_1^* & A_1^* \\
\mu^* - \mu^* & -\mu^* & -\mu^* \\
E_{A_1} & E_{A_1} & E_{A_1} \\
O & O & O
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
O & A_1^* \\
\mu^* - \mu^* & -\mu^* \\
E_{A_1} & E_{A_1} \\
O & O
\end{bmatrix}
= \begin{bmatrix}
O & A_1^* \\
\mu^* - \mu^* & -\mu^* \\
E_{A_1} & E_{A_1} \\
O & O
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
O & A_1^* \\
\mu^* - \mu^* & -\mu^* \\
E_{A_1} & E_{A_1} \\
O & O
\end{bmatrix}
= \begin{bmatrix}
O & A_1^* \\
\mu^* - \mu^* & -\mu^* \\
E_{A_1} & E_{A_1} \\
O & O
\end{bmatrix}
\]

Combing (2.16) with the formula (1.9) in Lemma 1.3, we finally have
\[
\max_{x_1,x_2,x_3} r(\mu' - \mu'X_1X_2X_3)
= \begin{bmatrix}
\mu' & \mu' & \mu' \\
A_2 & A_2 & A_2 \\
O & O & O \\
O & O & O \\
O & O & O
\end{bmatrix}
- \sum_{i=1}^{3} r(A_i).
\]

Next Lemma gives the expression in the ranks of the known matrices for
\[
\max_{x_1,x_2,x_3} r(X_1X_2X_3).
\]

Lemma 2.2 Let \(A_i \in \mathbb{C}^{m \times n}, X_i \in A_i(1,2,3), i = 1,2,3\). Then
\[
\max_{x_1,x_2,x_3} r(X_1X_2X_3) = \min \left\{ r(A_1), r(A_2), r(A_3) \right\}.
\]

Proof. By the formula (1.4) in Lemma 1.1 with \(A = A_3, B = I_n, C = X_1X_2\) and \(D = O\), we have
\[
\max_{x_3} r(X_1X_2X_3)
= \min \left\{ r \begin{bmatrix} A_3^* & A_3^* \end{bmatrix} - r(A_3), r \begin{bmatrix} A_3^* \end{bmatrix} \right\}
= \min \left\{ r \begin{bmatrix} O & A_3^* \end{bmatrix} - r(A_3), r(A_3) \right\}
= \min \left\{ r(X_1X_2), r(A_3) \right\}.
\]
Again by the formula (1.4) in Lemma 1.1 with $A = A_2$, $B = I_m$, $C = X_1$ and $D = O$, we have

$$\max_{X_1, X_3} r(X_1X_2X_3)$$

$$= \min \left\{ \max_{X_1} r(X_1X_2), \ r(A_3) \right\}$$

$$= \min \left\{ \min \left\{ r\left( A_2^* A_1 \begin{bmatrix} A_2^* \\ X_1 \end{bmatrix} O \right) - r(A_2), \ r\left( A_2^* \begin{bmatrix} A_1 \\ O \end{bmatrix} \right) \right\}, \ r(A_3) \right\}$$

$$= \min \left\{ \min \left\{ r\left( \begin{bmatrix} O \\ X_1 \end{bmatrix} A_2^* \begin{bmatrix} A_1 \\ O \end{bmatrix} \right) - r(A_2), \ r\left( A_2^* \begin{bmatrix} A_1 \\ O \end{bmatrix} \right) \right\}, \ r(A_3) \right\}$$

$$= \min \left\{ \min \left\{ r(X_1), \ r(A_2) \right\}, \ r(A_3) \right\}.$$  

(2.19)

Since $X_1 \in A_1 \{1, 2, 3\}$, we have $r(X_1) = r(A_1)$. Then by (2.19), we have

$$\max_{X_1, X_2, X_3} r(X_1X_2X_3) = \min \left\{ r(A_1), \ r(A_2), \ r(A_3) \right\}. \quad \square$$

We now give the expression in the ranks of the known matrices for the following minimal rank problem:

$$\min_{X_1, X_2, X_3} \ r(X_1X_2X_3).$$

Lemma 2.3 Let $A_i \in \mathbb{C}^{m \times m}$, $X_i \in A_i \{1, 2, 3\}$, $i = 1, 2, 3$. Then

$$\min_{X_1, X_2, X_3} \ r(X_1X_2X_3)$$

$$= r(A_3) - r\left( A_3 A_2 E_{A_1} \right)$$

$$= r\left( A_3 \begin{bmatrix} A_2^* & A_3 A_2 \end{bmatrix} \right) - r\left( A_3 \begin{bmatrix} A_2^* & A_3 \end{bmatrix} \right) - r\left( A_1 \begin{bmatrix} A_1 \ O \\ O \end{bmatrix} \right) + \sum_{i=1}^3 r(A_i) - r\left( A_3 \begin{bmatrix} A_2^* & A_3 \end{bmatrix} \right) - r\left( A_1 \begin{bmatrix} A_1 \ O \\ O \end{bmatrix} \right).$$  

(2.20)

Proof. Using the formula (1.5) in Lemma 1.1 with $A = A_1$, $B = X_2X_3$, $C = I_m$ and $D = O$, we have

$$\min_{X_1} \ r(X_1X_2X_3)$$

$$= r\left( A_1^* A_1 \begin{bmatrix} A_1^* X_2X_3 \\ I_m \end{bmatrix} O \right) + r(A_1^* X_2X_3) - r\left( A_1 \begin{bmatrix} A_1 \ O \\ O \end{bmatrix} \right)$$

$$= r(A_1^* X_2X_3).$$  

(2.21)
By Lemma 1.3 and the formula (1.5) in Lemma 1.1 with $A = A_2$, $B = X_3$, $C = A_1^*$ and $D = O$, we have

$$\min_{X_2} \ r(X_1 X_2 X_3) = \min_{X_2} \ r(A_1^* X_2 X_3)$$

$$= r \left( A_1^* A_2 \ A_1^* X_3 \ O \right) + r \left( A_1^* X_3 \ O \right) - r \left( A_2 \ O \ A_1^* X_3 \ O \right)$$

$$= r \left( A_1^* A_2, \ A_1^* X_3 \right) \cdot F(A_1^*, \ O) + r(A_1) - r(A_2)$$

$$= r \left( A_1^* A_2 E_{A_i}, \ A_1^* X_3 \right) + r(A_1) - r(A_2)$$

$$= r \left( A_1^* X_3 \ (I_{m_r} \ O) - (O, \ -A_1^* A_2 E_{A_i}) \right) + r(A_1) - r(A_2) \quad (2.22)$$

Again by Lemma 1.3 and the formula (1.5) in Lemma 1.1 with $A = A_3$, $B = (I_{m_r} \ O)$, $C = A_2^*$ and $D = (O, \ -A_2^* A_2 E_{A_i})$, we have

$$\min_{X_3} \ r(X_1 X_2 X_3)$$

$$= \min_{X_3} \ r \left( A_2^* X_3 \ (I_{m_r} \ O) - (O, \ -A_2^* A_2 E_{A_i}) \right) + r(A_1) - r(A_2)$$

$$= r \left( A_2^* A_3 \ A_2 \ O \ -A_2^* A_2 E_{A_i} \right) + r(A_3) - r(A_2)$$

$$= r \left( A_2 + r(A_2) + r(A_3) + r(A_2 E_{A_i}) - r \left( A_2^* \ A_2 E_{A_i} \right) \right)$$

$$= r(A_3) - r \left( A_3 E_{A_i}, \ A_3 A_2 E_{A_i} \right) \quad (2.23)$$

Combing (2.23) with the formula (1.9) in Lemma 1.3, we finally have

$$\min_{X_1, X_2, X_3} \ r(X_1 X_2 X_3)$$

$$= \min_{X_1, X_2, X_3} \ r(A_3 E_{A_i}, \ A_3 A_2 E_{A_i})$$

$$= \sum_{i=1}^{3} \ r(A_i) - r \left( A_2^* \ A_2 \ O \ A_1^* \ A_1^* \right). \quad \square$$

From Lemmas 2.1, 2.2 and 2.3, we immediately obtain the following theorem by equations (2.11) and (2.12).

**Theorem 2.1** Let $A_i \in C^{m \times m}$, $i = 1, 2, 3$, and $\mu = A_1 A_2 A_3$. Then the following statements are equivalent:

1. $A_1 = A_2^* A_2 A_3 A_2 A_1, \ A_2^* A_2 A_3 A_2 A_1 \subseteq (A_1 A_2 A_3)(1, 2, 3)$;

2. $r(\mu^* \mu - \mu^* A_3 A_2 A_1, \ \mu^* A_3 A_2 A_1, \ \mu^* A_3 A_2 A_1, \ \mu^* E_{A_i}) = 0$ and

$$r(\mu) = \min \{ r(A_1), \ r(A_2), \ r(A_3) \} = \sum_{i=1}^{3} r(A_i) - r \left( A_2^* \ A_2 \ O \ A_1^* \ A_1^* \right).$$
We now state more equivalent conditions for one side inclusion (1.2) without proofs since they are easy.

**Corollary 2.1** Let $A_i \in \mathbb{C}^{m \times m}$, $i = 1, 2, 3$, and $\mu = A_1 A_2 A_3$. Then the following statements are equivalent:

1. $A_1[1, 2, 3] A_2[1, 2, 3] A_3[1, 2, 3] \subseteq (A_1 A_2 A_3)[1, 2, 3]$;
2. $\mu^* A_3 A_2 A_1 = O$ and $\mu A_3 A_2 E_{A_1} = O$ and $\mu A_3 E_{A_2} = O$ and $\mu E_{A_3} = O$ and
   \[ r(\mu) = \min \{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r \begin{pmatrix} A_3 & A_3 A_2 \\ O & A_1^* \end{pmatrix} \]
3. $\mu^* = \mu A_3 A_2 A_1$ and $N(\mu A_3 A_2) \supseteq N(A_1^*)$ and $N(\mu A_3) \supseteq N(A_2^*)$ and $N(\mu) \supseteq N(A_3^*)$ and
   \[ r(\mu) = \min \{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r \begin{pmatrix} A_3 & A_3 A_2 \\ O & A_1^* \end{pmatrix} \]

**Corollary 2.2** Let $A_i \in \mathbb{C}^{m \times m}$, $i = 1, 2$. Then the following statements are equivalent:

1. $A_1[1, 2, 3] A_2[1, 2, 3] \subseteq (A_1 A_2)[1, 2, 3]$;
2. $r\left( (A_1 A_2)^* A_1 A_2 - (A_1 A_2)^* A_2 A_1, (A_1 A_2)^* A_2 E_{A_1}, (A_1 A_2)^* E_{A_2} \right) = 0$ and
   \[ r(A_1 A_2) = \min \{r(A_1), r(A_2)\} = \sum_{i=1}^{2} r(A_i) - r \begin{pmatrix} A_2^* \\ A_1^* \end{pmatrix} \]
3. $(A_1 A_2)^* A_1 A_2 - (A_1 A_2)^* A_2 A_1 = O$ and $(A_1 A_2)^* A_2 E_{A_1} = O$ and $(A_1 A_2)^* E_{A_2} = O$ and
   \[ r(A_1 A_2) = \min \{r(A_1), r(A_2)\} = \sum_{i=1}^{2} r(A_i) - r \begin{pmatrix} A_2^* \\ A_1^* \end{pmatrix} \]
4. $(A_1 A_2)^* A_1 A_2 = (A_1 A_2)^* A_2 A_1$ and $N((A_1 A_2)^* A_2) \supseteq N(A_1^*)$ and $N((A_1 A_2)^* A_2) \supseteq N(A_2^*)$ and
   \[ r(A_1 A_2) = \min \{r(A_1), r(A_2)\} = \sum_{i=1}^{2} r(A_i) - r \begin{pmatrix} A_2^* \\ A_1^* \end{pmatrix} \]
5. $\begin{pmatrix} (A_1 A_2)^* A_1 A_2 - (A_1 A_2)^* A_2 A_1 \\ O \end{pmatrix} \begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} = \sum_{i=1}^{2} r(A_i)$ and
   \[ r(A_1 A_2) = \min \{r(A_1), r(A_2)\} = \sum_{i=1}^{2} r(A_i) - r \begin{pmatrix} A_2^* \\ A_1^* \end{pmatrix} \]

Notice that $X A A^* = A^*$ and $r(X) = r(A)$ are equivalent to the equations $A A^* X^* = A^*$ and $r(X^*) = r(A^*)$, respectively. This implies that, by the formula (1.6) and (1.7) in Lemma 1.2, $X \in A[1, 2, 4]$ if and only if $X^* \in A[1, 2, 3]$. So we can get the necessary and sufficient conditions for (1.3) by a similar approach in the previous section and hence provide the following results without the proof.

**Theorem 2.2** Let $A_i \in \mathbb{C}^{m \times m}$, $i = 1, 2, 3$, and $\mu = A_1 A_2 A_3$. Then the following statements are equivalent:

1. $A_1[1, 2, 4] A_2[1, 2, 4] A_3[1, 2, 4] \subseteq (A_1 A_2 A_3)[1, 2, 4]$;
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