Complete Heyting Algebra-Valued Convergence Semigroups

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Abstract. Considering a complete Heyting algebra $\mathcal{H}$, we introduce a notion of stratified $\mathcal{H}$-convergence semigroup. We develop some basic facts on the subject, besides obtaining conditions under which a stratified $\mathcal{H}$-convergence semigroup is a stratified $\mathcal{H}$-convergence group. We supply a variety of natural examples; and show that every stratified $\mathcal{H}$-convergence semigroup with identity is a stratified $\mathcal{H}$-quasi-uniform convergence space. We also show that given a commutative cancellative semigroup equipped with a stratified $\mathcal{H}$-quasi-uniform structure satisfying a certain property gives rise to a stratified $\mathcal{H}$-convergence semigroup via a stratified $\mathcal{H}$-quasi-uniform convergence structure.

1. Introduction

Inspired on the one hand by the enormous work done on classical convergence groups mostly within the framework of sequential convergence lead by J. Novák [42] and on the other, by the classical convergence group and its uniformization initiated by D. C. Kent and G. D. Richardson within the framework of filter-theoretic convergence (cf. [32–35, 45]), we generalized the notion of convergence groups into the context of frame-valued convergence groups [2] and later, lattice-valued convergence ring and its uniform convergence structures [1]. Furthermore, we studied enriched lattice-valued convergence groups in [3]. All of these works are based on lattice-valued filter-theoretic convergence structures attributed to G. Jäger undertaken since 2001 (cf. [21]), which originally initiated by U. Höhle and A. Šostak [19], and independently, by U. Höhle in [20].

One of the motivations of this work is stemmed from the fact that the category of classical convergence spaces [6–8, 10, 13, 16, 32, 33, 37, 38, 42, 46] is a better behaved category than the category of topological spaces; there are enormous quantity of research papers appeared over the years on topological semigroups alone, but unfortunately, we were able to find a very few papers devoted to convergence semigroups within the scope of filter-theoretic treatments (cf. [17, 18]); this is, however, in contrast to the category of convergence groups (cf. [32–35, 45]). This has stimulated our interest further to look for lattice-valued convergence semigroups based on whatever we found from existing papers on classical convergence semigroups [17, 18]. In this paper, we intend to develop a basic theory on lattice-valued convergence
semigroups; unlike previous findings [1–3], here classical results on semigroup theory are particularly used like their classical counterpart [9, 11, 14, 40, 41, 44, 49].

We organize our work as follows. In Section 2, we present various facts from existing articles which will be used throughout the text. Section 3 deals with, the main notion of stratified $H$-convergence semigroups along with various results. Here we show among others that a compact Hausdorff-separated stratified $H$-convergence semigroup contains an idempotent element; also, we show that the set of idempotent elements in a compact Hausdorff-separated stratified $H$-convergence semigroup is $\tau$-closed [30], and hence compact [27]. We give conditions for which a Hausdorff-separated stratified $H$-Choquet convergence semigroup produces a stratified $H$-convergence group. In section 4, we give results on ideals in stratified $H$-convergence semigroups. We provide in Section 5, a wide variety of examples including natural examples; one of the examples here states that a commutative cancellative semigroup equipped with a stratified $H$-quasi-uniform structure satisfying a certain property, produces a stratified $H$-convergence semigroup via stratified $H$-neighborhood system [19, 24]. A similar example is given in Section 6 but this time via stratified $H$-quasi-uniform convergence structure. In this section, the main result is to show that every stratified $H$-convergence semigroup with identity element is a stratified $H$-quasi-uniform convergence space.

2. Preliminaries

Throughout the text we consider $H = (H, \leq, \wedge)$, a complete Heyting algebra. This means that the lattice $H$ is a complete lattice, where finite meets are distributive over arbitrary joins; that is, for all $\alpha, \beta_i \in H$, $\alpha \wedge \bigvee_{i \in J} \beta_i = \bigvee_{i \in J} (\alpha \wedge \beta_i)$.

The set of all $H$-sets is denoted by $H^X = \{v : X \to H\}$. If $A \subseteq X$, then a constant $H$-set with value $\alpha \in H$ on $A$, is denoted by $\alpha_A$, and is defined as $\alpha_A(x) = \alpha$, if $x \in A$ and $\alpha_A(x) = \bot$, elsewhere.

In particular, $\tau_X(x) = \tau$, the characteristic function of $X$ and $\bot_X(x) = \bot$, the zero function. The residuated implication operation $\to : H \times H \to H$ is defined by: $\alpha \to \beta = \bigvee \{y \in H | \alpha \wedge y \leq \beta\}$. Then $\alpha \leq \beta \Rightarrow \alpha \land \beta \leq \gamma$.

Lemma 2.1. ([20]) Let $H = (H, \leq, \wedge)$ be a complete Heyting algebra. Then the following are satisfied for all $\alpha, \beta, \gamma, \delta \in H$:

(i) $\alpha \leq \beta \Rightarrow \alpha \to \gamma \geq \beta \to \gamma$ and $\gamma \to \alpha \leq \gamma \to \beta$;

(ii) $\alpha \to (\beta \land \gamma) = (\alpha \to \beta) \land (\alpha \to \gamma)$;

(iii) $(\bigvee_{i \in J} \alpha_i) \to \beta = \bigwedge_{i \in J} (\alpha_i \to \beta)$;

(iv) $(\alpha \to \beta) \land (\gamma \to \delta) \leq (\alpha \land \gamma) \to (\beta \land \delta)$.

Definition 2.2. ([19]) A map $F : H^X \to H$ is called an $H$-filter on $X$ if the conditions below are satisfied:

(HF1) $F(\tau_X) = \tau$, $F(\bot_X) = \bot$;

(HF2) if $v_1, v_2 \in H^X$ with $v_1 \leq v_2$, then $F(v_1) \leq F(v_2)$;

(HF3) $F(v_1) \land F(v_2) \leq F(v_1 \land v_2)$, $\forall v_1, v_2 \in H^X$;

(SH) An $H$-filter $F$ is called a stratified $H$-filter if $\forall \alpha \in H$, $\alpha \leq F(\alpha_X)$.

The set of all stratified $H$-filters on $X$ is denoted by $F^*_H(X)$. On $F^*_H(X)$, partial ordering $\leq$ is defined by: if $F, G \in F^*_H(X)$, then $F \leq G \iff F(v) \leq G(v)$, $\forall v \in H^X$. A maximal element in the partially ordered set $(F^*_H(X), \leq)$ is called a stratified $H$-ultrafilter. We denote $F^\omega_H(X)$ as the set of all stratified $H$-ultrafilters [19]. If $x \in X$, then $[x] \in F^*_H(X)$, called point stratified $H$-filter on $X$, and is defined as $[x](v) = v(x)$, for all $v \in H^X$.

If $f : X \to Y$ is a function, then $f^{-}\wedge : H^Y \to H^X$ is defined for any $\mu \in H^Y$ by $f^{-}(\mu) = \mu \circ f$; and $f^{-} : H^Y \to H^Y$ is defined by: $f^{-}(v)(y) = \bigvee \{(v)(x) | y \leq f(x), \forall v \in H^Y, y \in Y\}$. Moreover, if $F \in F^*_H(X)$, then the stratified $L$-filter $f^{-}(F) : H^Y \to H$ on $Y$ is defined for any $\mu \in H^Y$ by: $f^{-}(F)(\mu) = F(f^{-}(\mu)) = F(\mu \circ f)$. This is also true for $F \in F^*_H(Y)$, that is, in which case $f^{-}(F) \in F^*_H(X)$.

If $F \in F^*_H(Y)$, then $f^{-}(F) : H^X \to H$ defined by: $[f^{-}(F)](v) = \bigvee \{(F)(\mu) | \mu \in H^Y, f^{-}(\mu) \leq v\}$, for all $v \in H^X$, is a stratified $H$-filter on $X$ if and only if for all $\mu \in H^Y, f^{-}(\mu) = \bot_X = F(\mu) = \bot$. If, however,
If \(F \in \mathcal{F}_H^\dagger(X)\), then \(F \vartriangleleft G \in \mathcal{F}_H^\dagger(X)\) if and only if \(\forall \xi \in H^X: F \vartriangleleft G(\xi) = \forall \xi F(\nu(\xi) ; G(\xi) ; \mu(\xi) ; \nu(\xi)) \in \mathcal{F}(\nu(\xi)) \implies F(\nu(\xi)).\)

Lemma 2.3. ([22]) Let \(F \in \mathcal{F}_H^\dagger(X)\) and let \(f : X \rightarrow Y\). If \(\mathcal{U} \supseteq f^\circ(F)\) is a stratified \(H\)-ultrafilter on \(Y\), then there exists a \(G \in \mathcal{F}_H^\dagger(Y)\) such that \(G \supseteq f^\circ(F)\) and \(f^\circ(G) = \mathcal{U}\).

If \((X, \cdot)\) is a semigroup, and \(A, B \subseteq X\), then one defines \(A \cdot B\) (or simply \(AB\)) by \(A \cdot B = \{ab : a \in A, b \in B\}\). In particular, \([a]B\) is denoted just by \(aB\) and similarly, \(A[b]\) by \(Ab\) (cf. [9, 12, 44]).

If \((X, \cdot)\) is a group and \(F \in \mathcal{F}_H^\dagger(X)\), then \(F^{-1}\) is defined by \(\forall \nu \in H^X: F^{-1}(\nu) = F(\nu^{-1})\), where \(\nu^{-1} : X \rightarrow H, x \mapsto \nu(x)^{-1}\). Since for any \(\nu \in H^X, F(\nu) = F(\nu^{-1})\) and for any \(j : \nu : X \rightarrow x \mapsto x^{-1}, j\) is known as inversion mapping, we have \(F^{-1} = F(X)\). Also, if \(m : X \times X \rightarrow X, (x, y) \mapsto xy\), the semigroup or group operation on \(X\), then for any \(\nu_1, \nu_2 \in H^X\) and \(z \in X, m^{-1}(\nu_1 \otimes \nu_2)(z) = \bigvee_{x \in \nu_1, y \in \nu_2} (x, y) = \bigvee_{\nu_1 \supseteq \nu_2} \) \(\nu_1 \otimes \nu_2 \otimes \nu_2(x, y) = \bigvee_{\nu_1 \supseteq \nu_2} \nu_1 \otimes \nu_2 \otimes \nu_2(x, y)\).

If \((X, \cdot)\) is a semigroup, and \(F, G \in \mathcal{F}_H^\dagger(X)\), then the mapping \(F \circ G : H^X \rightarrow H^X\) is defined for any \(\nu \in H^X\) by: \(F \circ G(\nu)(a) = \bigvee_{\nu \vartriangleleft \nu_1} (F(\nu_1) \vartriangleleft G(\nu_1))\). It is shown in Proposition 3.3[2] (see also, Lemma 2.9 and Proposition 3.10 [3]) that \(F \circ G = m^\circ(F \times G) = \mathcal{F}(\nu(\xi))\) is a stratified \(H\)-filter on \(X\).

We just recall below the notion of stratified \(H\)-neighborhood system, for further details including the notion of stratified \(H\)-topology, we refer to [19] (see also [20]).

Definition 2.4. ([19]) A pair \((X, \mathcal{H} = (\mathcal{H}^\dagger)_{\nu \in X})\) is called a stratified \(H\)-neighborhood topological space, where \(\mathcal{H}\) is a family of stratified \(H\)-filters on a nonempty set \(X\) satisfying the following:

\[(\text{HN1}) \forall x \in X, \mathcal{H}^\dagger \supseteq \{x\} ; \]
\[(\text{HN2}) \forall x \in X, \forall \nu \in H^X, \mathcal{H}^\dagger(\nu) = \bigvee_{\nu(\xi) \in H^X, \xi(\nu) \in \mathcal{H}^\dagger(\nu)} (x \in X) .\]

A map \(f : (X_1, \mathcal{H}_1) \rightarrow (X_2, \mathcal{H}_2)\) between stratified \(H\)-neighborhood topological spaces is said to be \textit{continuous} at \(x \in X_1\) if \(\mathcal{H}_1^{\nu(\xi)} \supseteq f^\circ(\mathcal{H}_2^{\nu(\xi)})\). It is continuous if it is \textit{continuous} at each point of \(X\).

Definition 2.5. ([21, 25]) Let \(X\) be a nonempty set and \(\lim : (\mathcal{F}_H^\dagger(X) \rightarrow H^X\) a mapping satisfying the following:

\[(\text{LGC1}) \forall x \in X, \lim \mathcal{F}_H^\dagger(X) = \mathcal{T} ; \]
\[(\text{LGC2}) \forall F, G \in \mathcal{F}_H^\dagger(X), \forall x \in X, \lim \mathcal{F}_H^\dagger(X) \leq \lim \mathcal{G}_H^\dagger(X) \leq \lim \mathcal{F}(x), \forall x \in X, \]
then the pair \((X, \lim)\) is called a stratified \(H\)-generalized convergence space.

\(\bullet\) A stratified \(H\)-generalized convergence space is called stratified \(H\)-convergence space if it satisfies

\[(\text{LCS}) \forall F, G \in \mathcal{F}_H^\dagger(X), \forall x \in X, \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x) \leq \lim (\mathcal{F} \wedge \mathcal{G})(x) .\]

\(\bullet\) A stratified \(H\)-generalized convergence space \((X, \lim)\) is called stratified \(H\)-Choquet convergence space if it satisfies

\[(\text{LCC}) \forall F \in \mathcal{F}_H^\dagger(X), \forall x \in X, \lim \mathcal{F}(x) = \bigwedge_{\nu \in H^X} (\mathcal{H}^\dagger(\nu) \rightarrow \mathcal{F}(\nu)), \]
\[(\text{LP}) \forall F \in (\mathcal{F}_H^\dagger(X), \forall x \in X, \lim \mathcal{F}(x) = \bigwedge_{\nu \in H^X} (\mathcal{H}^\dagger(\nu) \rightarrow \mathcal{F}(\nu)).\]

A map \(f : (X, \lim) \rightarrow (X', \lim')\) between stratified \(H\)-generalized convergence spaces (resp. stratified \(H\)-convergence spaces, stratified \(H\)-pretopological convergence spaces, stratified \(H\)-Choquet convergence
spaces), is said to be continuous if and only if \( \forall F \in F_{H}^*(X), \forall x \in X : \lim F(x) \leq \lim' F^\circ(f)(x) \). Note that every stratified \( H \)-pretopological convergence space is a stratified \( H \)-convergence space (cf.[23]).

For a given source \( \left( f_{j} : X \rightarrow (X, \lim_{j}) \right)_{j \in J} \), the initial structure on \( X \) is defined in [21] for any \( F \in F_{H}^{*}(X) \) and \( x \in X \) by: \( \lim F(x) = \bigwedge_{j \in J} \lim_{j} f_{j}(F)(x) \).

Special examples of such a structure are subspaces and product spaces. As for subspace, consider \((X, \lim)\), a stratified \( H \)-generalized convergence space, \( A \subseteq X \) and \( i_{A} : A \hookrightarrow X, x \mapsto x \), an inclusion mapping, then the initial structure on \( A \) written as \( \lim_{A} \) is given for any \( F \in F_{H}(A) \) and \( x \in X \) by: \( \lim_{A} F(x) = \lim_{A}(F)(i_{A}(x)) = \lim F(x) \).

For product space, we just consider the projection mappings \( p_{r} : \prod_{i \in I} X_{i} \rightarrow X_{i} \). In particular, if \((X, \lim), (Y, \lim)\) are stratified \( H \)-generalized convergence spaces, then their product \( (X \times Y, \lim_{X} \times \lim_{Y}) \) is a stratified \( H \)-generalized convergence space [21], where \( \lim_{X} \times \lim_{Y} : F_{H}^{*}(X \times Y) \rightarrow H^{X \times Y} \) is defined for any \( F \in F_{H}^{*}(X \times Y) \) by:

\[
(\lim_{X} \times \lim_{Y}) F = pr_{X}^{-1}(\lim_{X} pr_{F}(F)) \wedge pr_{Y}^{-1}(\lim_{Y} pr_{F}(F)).
\]

**Definition 2.6.** ([25]) Let \((X, \lim)\) be a stratified \( H \)-generalized convergence space. Then \((X, \lim)\) is called Hausdorff-separated or \( T_{2} \)-space if and only if for all \( x, y \in X, \forall F \in F_{H}^{*}(X) \), \( \lim F(x) = \lim F(y) = \tau \) implies \( x = y \).

We recall a characterization of Hausdorff-separated space from [Lemma 4.4[25]]: A stratified \( H \)-pretopological convergence space \((X, \lim)\) is Hausdorff-separated if and only if \( \forall x, y \in X : \forall F \in F_{H}^{*}(X) \) implies \( x = y \).

**Definition 2.7.** ([27, 28, 30]) (1) If \((X, \lim)\) is a stratified \( H \)-generalized convergence space and \( A \subseteq X \), and \( x \in X \), then the limit-closure of \( A \), denoted by \( \overline{A} \), is defined as follows:

- \( x \in \overline{A} \) if there exists a \( F \in F_{H}^{*}(X) \) such that \( \lim F(x) = \tau \) and \( F(\tau) = \tau \).

- \( A \subseteq X \) is called \( \tau \)-closed in \((X, \lim)\) if for all \( F \in F_{H}^{*}(X) \), \( \lim F(x) = \tau \) and \( F(\tau) = \tau \) implies \( x \in A \).

Remark that \( A \) is \( \tau \)-closed if and only if \( \overline{A} \subseteq A \). Furthermore, note that as it is pointed out in [28], \( \tau \)-closedness of \( A \) can be characterized by stratified \( H \)-ultrafilters, in which case \( A \) is \( \tau \)-closed if and only if for all \( U \in F_{H}^{\omega}(X) \), \( \lim U(x) = \tau \) and \( U(\tau) = \tau \) implies \( x \in A \).

**Definition 2.8.** ([27]) A stratified \( H \)-generalized convergence space \((X, \lim)\) is called compact if and only if for all \( U \in F_{H}^{\omega}(X) \) there exists \( x \in X \) such that \( \lim U(x) = \tau \). If \( A \subseteq X \), then \( A \) is called compact if the subspace \((A, \lim_{A})\) of \( X \) is compact (see cf. [27]).

**Lemma 2.9.** ([27]) Let \((X, \lim)\) be a stratified \( H \)-generalized convergence space and \( A \subseteq X \). Then \( A \) is compact if and only if for every stratified \( H \)-ultrafilter \( U \in F_{H}^{\omega}(X) \) with \( U(\tau) = \tau \), there is an \( x \in A \) such that \( \lim U(x) = \tau \).

**Lemma 2.10.** ([26, 27]) Let \((X, \lim)\) and \((Y, \lim')\) be stratified \( H \)-generalized convergence spaces, and \( f : (X, \lim) \rightarrow (Y, \lim') \) be continuous. If \( A \subseteq X \) is compact, then \( f(A) \) is a compact subset of \((Y, \lim')\).

**Definition 2.11.** ([9, 12]) Let \((X, \cdot)\) be a semigroup. A subsemigroup \( T \) of a semigroup \( X \) is a non-empty subset \( T \) of \( X \) such that \( TT \subseteq T \). A subgroup of a semigroup \( X \) is a nonempty subset \( A \) of \( X \) such that \( xA = Ax = A \) for each \( x \in A \). An element \( e \) in \( X \) is called an idempotent if and only if \( e^2 = e \). The set of all idempotents is denoted by \( E(X) \). If \( X \) contains an idempotent \( e \), then \( [e] \) is a subgroup of \( X \), and it is contained in a maximal subgroup. We denote \( H(e) \) the maximal subgroup of \( X \) containing the idempotent \( e \).

**Definition 2.12.** ([9, 12]) A semigroup \((X, \cdot)\) is called right simple if \( aX = X \) for all \( a \in X \); equivalently, for all \( a, b \in X \) there exists \( x \in X \) such that \( ax = b \); it is called left simple if \( Xa = X \) for all \( a \in X \); equivalently, for all \( a, b \in X \) there exists \( x \in X \) such that \( xa = b \). It is called simple if it is both left simple and right simple.

A semigroup \((X, \cdot)\) is called right group if and only if \( X \) is both left cancellative, i.e., \( zx = zy \) implies that \( x = y \), and right simple.
3. Complete Heyting Algebra-Valued Convergence Semigroups

Definition 3.1. A triple \((X, \cdot, \lim)\) is called a stratified \(\mathcal{H}\)-generalized convergence semigroup (resp. stratified \(\mathcal{H}\)-convergence semigroup, stratified \(\mathcal{H}\)-pretopological semigroup, stratified \(\mathcal{H}\)-Choquet convergence semigroup) if the following are fulfilled:

1. (LCSG1) \((X, \cdot)\) is a semigroup;
2. (LCSG2) \((X, \lim)\) is a stratified \(\mathcal{H}\)-generalized convergence space (resp. stratified \(\mathcal{H}\)-convergence space, stratified \(\mathcal{H}\)-pretopological space, stratified \(\mathcal{H}\)-Choquet convergence space);
3. (LCSG3) the multiplication \(m : X \times X \to X, (x, y) \mapsto xy\) is continuous; equivalently, for all \(F, G \in \mathcal{F}_x(X)\), for all \(x, y \in X\) we have \(\lim F(x) \land G(y) \leq \lim (F \circ G)(xy)\).

Lemma 3.2. Let \((X, \cdot, \lim)\) be a stratified \(\mathcal{H}\)-generalized convergence semigroup, and \(a \in X\). Then both the left and the right translations by \(a, ^a_\partial : X \to x, x \mapsto ax, \) and \(\partial_a : X \to x, x \mapsto xa\) are continuous.

Proof. Let \(a \in X\). Then for any \(x \in X\) and \(F \in \mathcal{F}_x^\partial(X)\), \(\lim F(x) = \lim [a](ax) \land \lim F(x) \leq \lim ([a] \circ F)(ax) \leq \lim [^a_\partial(F)](a(x))\); in fact, upon using stratification \((\mathcal{H})\), for any \(v \in X, ([a] \circ F)(x) = \cup_{n \in \mathbb{N}, x \in F} [a](v_1) \land F(v_2) = \cup_{n \in \mathbb{N}, x \in F} [v_1(a) \land F(v_2) \leq \cup_{n \in \mathbb{N}, x \in F} [v_1(a) \land F(v_2) \leq [v_1(a) \land F(v_2) \leq \lim [^a_\partial(F)](a(x)).

Similarly, the continuity of \(\partial_a\) follows from the observation: \(\lim F(x) = \lim F(x) \land \lim [a](ax) \leq \lim ([F \circ [a]])(xa) \leq \lim [^a_\partial(F)](a(x)). \)

Lemma 3.3. Let \((X, \cdot, \lim)\) be a Hausdorff-separated stratified \(\mathcal{H}\)-generalized convergence semigroup and \(A, B\) are subsets of \(X\). Then the following holds:

(a) If \(B\) is \(\tau\)-closed in \((X, \lim)\), then \(\{x \in X : xA \subseteq B\}\) is \(\tau\)-closed.
(b) If \(B\) is compact, then \(\{x \in X : A \subseteq xB\}\) is \(\tau\)-closed.
(c) If \(B\) is compact, then \(\{x \in X : xA \subseteq Bx\}\) is \(\tau\)-closed.

Proof. (a) Denote \(C := \{x \in X : xA \subseteq B\}\) with \(B\) is \(\tau\)-closed in \((X, \lim)\), and let \(x \in \overline{C}^\lim\). Then there exists \(F \in \mathcal{F}_x^\partial(X)\) such that \(\lim F(x) = \tau\) with \(F(T_C) = \tau\). Let \(a \in A\). Then it follows from the continuity of \(x \mapsto ax\) that \(\lim (F \circ [a])(xa) = \tau\). Now note that due to Lemma 2.6[29], as \(x\) is \(\tau\)-closed, one can deduce \(F(T_{[x]}) = \tau\), and since \(T_x \cdot T_A = T_{xA} \subseteq T_B\), we obtain: \(\{F \circ [a]\}(T_B) = \{F \circ [a]\}(T_B) \geq F(T_{[x]} ) \land T_A(a) = T, i.e., (F \circ [a])(T_B) = T\). As \(B\) is \(\tau\)-closed, we have \(xA \subseteq B\). This means that \(x \in C\).

(b) Let \(K := \{x \in X : xA \subseteq Bx\}\), and \(B\) be a compact subset of \(X\). We show \(K\) is \(\tau\)-closed. For, let \(x \in \overline{K}^\lim\). Then there exists a \(F \in \mathcal{F}_x^\partial(X)\) such that \(\lim F(x) = \tau\) and \(\lim F(T_K) = \tau\). Let \(U \in \mathcal{F}_x^\partial(X)\) such that \(U \supseteq \tau\). Take \(a \in A\), then \([a]\) is \(\tau\)-closed, and so, \(\lim F(a) = \tau\) but then \(\lim U(a) = \tau\). Since \(B\) is compact, \(xB\) compact subset of \(X\), and hence there exists \(z \in xB\) such that \(\lim U(z) = \tau\). As \(z \in xB\), there exists \(b \in B\) such that \(z = xb\). Then \(\lim U(b) = \tau\). But \(x\) is Hausdorff-separated so, \(a = xb\). This means \(A \subseteq Bx\) and hence \(x \in K\).

(c) Set \(F := \{x \in X : xA \subseteq Bx\}\), and let \(x \in \overline{K}^\lim\). Then there exists a \(F \in \mathcal{F}_x^\partial(X)\) such that \(\lim F(x) = \tau\) and \(\lim F(T_F) = \tau\). If \(y \in A\), then there is an \(a \in A\) such that \(y = xa\). Choose \(U \in \mathcal{F}_x^\partial(X)\) such that \(U \supseteq \tau\). Then since \(B\) is compact, there exists \(z \in B\) such that \(\lim U(z) = \tau\). Then due to continuity of \(x \mapsto xz\), \(\lim U(xz) = \tau\). Also, then we have \(\lim U(xa) = \tau\). Since \(X\) is Hausdorff-separated, it follows from \(\lim U \circ F(xz) = \lim U \circ F(xa) = \tau\) implies \(xz = xa\), but then \(y = xz\), showing \(xA \subseteq Bx\), hence \(x \in F\). \(\square\)

In classical theory of topological semigroups, the following result is known as Ellis’s Lemma[14](see also [41]). This result has been incorporated for classical convergence semigroup in [17].

Theorem 3.4. Let \((X, \cdot, \lim)\) be a compact Hausdorff-separated stratified \(\mathcal{H}\)-generalized convergence semigroup. Then it contains an idempotent element.

Proof. Let \(S\) denote the set of \(\tau\)-closed subsemigroups of \(X\), i.e., \(S = \{A \subseteq X : AA \subseteq A, A^\lim = A\}\). Since it follows from Lemma 3.5[27] that \(X\) itself is a \(\tau\)-closed subsemigroup, we have \(X \in S\), so \(S \neq \emptyset\). If \(S\) is partially ordered by reverse inclusion, then by Zorn’s lemma, there exists a minimal element \(K \in S\). Let
y ∈ K. Then since K is \(\tau\)-closed subgroup of X, by Corollary 3.4[27], K is compact, and by continuity of the mapping \(f : X → X, x ↦ yx\), yK is compact subset of X, and therefore, again by Lemma 3.5[27], yK is a \(\tau\)-closed subgroup of X such that yK ⊆ K. That yK is a subgroup follows from \((yK)(yK) ⊆ yK\), which is true because of the fact that \((yx_1)yx_2 = y(x_1y_2) = x_3 ∈ yK, x_1, x_2, x_3 ∈ K\). Hence by minimality of K, we get yK = K. Similarly, K\(y\) = K. Thus, K is a subgroup of X. If \(e\) is the identity of K, then again by minimality, K = \{e\}.

The following lemma in classical topological semigroups is known as Swelling Lemma (see Lemma 1.9[9]). We generalize it in the context of compact Hausdorff-separated stratified \(H\)-generalized convergence semigroup, in which case, a part of the classical proof will remain as it is.

**Lemma 3.5.** Let (\(X, \cdot\), \(lim\)) be a compact Hausdorff-separated stratified \(H\)-generalized convergence semigroup. If \(A\) is a \(\tau\)-closed subset of \(X\), \(t ∈ X\) and \(A ⊆ tA\), then \(A = tA\).

**Proof.** Let \(K := \{x ∈ X : tA ⊆ xA\}\). Then clearly \(T\) is a subgroup of \(X\) and in view of Preceding Lemma 3.3(c), we have \(K\) is \(\tau\)-closed, and hence by Theorem 3.4, it contains an idempotent \(e\). Hence the rest of the proof, mainly, algebraic part follows from the proof of the classical Swelling Lemma 1.9[9]. □

**Theorem 3.6.** Let (\(X, \cdot\), \(lim\)) be a compact Hausdorff-separated stratified \(H\)-pretopological convergence semigroup. Then

(a) The set of all idempotents \(E(X)\) of \(X\) is \(\tau\)-closed, and hence compact.

(b) Every maximal subgroup \(H(e)\) of \(X\) is \(\tau\)-closed and hence compact.

**Proof.** (a) Let \(x ∈ \overline{E(X)}\) and \(x^2 \neq x\). Then by Lemma 4.4[25], \(R^2 \vee R^2 \not∈ F_H^T(X)\). This implies that there are \(v_1, v_2 \in \overline{H^X}\) such that \(v_1 \wedge v_2 = \perp_X\) and \(R^2(v_1) \wedge R^2(v_2) \neq \perp\) ...

(b) For the fact that \(H(e) = \{x ∈ eX : (\exists y ∈ eX) xy = yx = e\}\) is the largest subgroup of \(X\) having \(e\) as its identity follows from [40]. We only show that \(H(e)\) is \(\tau\)-closed. To this end, let \(x ∈ \overline{H(e)}\). Then there exists \(F \in \overline{F_H^T(X)}\) such that \(lim F(x) = \tau\) and \(F(\tau_{H(e)}) = \tau\). Choose \(U \in \overline{U_{H^X}}\) such that \(U \geq F.\) As \(X\) is compact, there exists \(y \in X\) such that \(lim U(y) = \tau\). Because of continuity of the mapping \(m : X × X → X, (x, y) ↦ xy\), \(lim U(x) \leq lim(F \circ U)(xy) = lim F \circ U(xy) = \tau, y \in X\). Since \(q : X → X, x ↦ e\) is continuous, it follows from [26] that \(eX\) is compact subset of \(X\), and by Lemma 3.5[27], \(eX\) is \(\tau\)-closed; as \(H(e) ⊆ eX\), we have \(U(\tau_{eX}) = \tau\), therefore, \(y \in eX\). Also, note that \(lim F \circ U(xy) = \tau\). Thus \(lim F \circ U(xy) = lim F \circ U(e) = \tau\), but as \(X\) is Hausdorff-separated, \(xy = e\); similarly, \(yx = e\) implying that \(x ∈ H(e)\), showing that \(\overline{H(e)} \subseteq H(e)\). So, \(H(e)\) is \(\tau\)-closed, and hence by Corollary 3.4[27], it is compact. □

**Theorem 3.7.** Let \((X, \cdot, lim)\) and \((Y, \cdot, lim')\) be compact Hausdorff-separated stratified \(H\)-convergence semigroups and \(f : X → Y\) be a surjective continuous semigroup-homomorphism. Then \(f(E(X)) = E(Y)\).

**Proof.** We only check \(E(Y) \subseteq f(E(X))\). For, let \(e ∈ E(Y)\). Then since \(e\) is \(\tau\)-closed by Lemma 2.6[28]; so, by Corollary 2.13[30], \(f^{-1}(e)\) is \(\tau\)-closed which in view of Corollary 3.4[27] is compact, and then a
subsemigroup of $X$. Consequently, by Theorem 3.4, it contains an idempotent element, say $e'$. So, $f(e') = e$. Therefore, $E(Y) \subseteq f(E(X))$. □

**Proposition 3.8.** Let $(X, \cdot, \lim)$ be a stratified $\mathcal{H}$-convergence semigroup, and $A$ be a subsemigroup of $X$. Then $\overline{A}$ is a subsemigroup of $X$.

Proof. Let $x, y \in \overline{A}$. Then there are $F, G \in F^*_1(X)$ such that $\lim F(x) = \tau$, $\lim G(y) = \tau$ and $G(\tau_A) = \tau$. Since $m : X \times X \to X$, $(x, y) \mapsto xy$ is continuous, $\lim F(x) \land \lim G(y) \subseteq \lim (F \circ G)(xy)$ implying that $\lim F \circ G(xy) = \tau$. Since $A$ is a subsemigroup, we have $\overline{A}A \subseteq A$ and hence $\overline{A}A \circ \overline{A} = \overline{A}A \subseteq \overline{A}A$ which implies that $F \circ G(\tau_A) = \lim F(\nu_1) \land G(\nu_2) = \nu_1, \nu_2 \in X, \nu_1 \land \nu_2 \leq \tau_A \Rightarrow F(\nu_1) \land G(\nu_2) = \tau$. But in view of Proposition 3.3[2], we know $F \circ G \in F^*_A(X)$, by setting $H := F \circ G \in F^*_A(X)$, we have $\lim H(xy) = \tau$ and $H(\tau_A) = \tau$ which implies that $xy \in \overline{A}$. □

**Corollary 3.9.** Let $(X, \cdot, \lim)$ be a stratified $\mathcal{H}$-convergence semigroup and $A \subseteq X$ be a maximal subsemigroup of $X$. Then $A$ is $\tau$-closed.

Proof. By Proposition 3.8, $\overline{A} \subseteq X$ is a subsemigroup of $X$. Due to Lemma 2.7(2)[30], $A \subseteq \overline{A}$, which by maximality of $A$ coincides with $\overline{A}$. Hence $A$ is $\tau$-closed. □

**Theorem 3.10.** Let $(X, \cdot, \lim)$ be a Hausdorff-separated stratified $\mathcal{H}$-convergence semigroup and $G$ a subgroup of $X$. Then $\overline{G}$ is a subsemigroup of $X$ with identity.

Proof. Due to Proposition 3.8, it follows immediately that $\overline{G}$ is a subsemigroup of $X$. To show the remaining part, let $e \in G$ be the identity element of $G$, and assume $x \in \overline{G}$. Then there is $F \in F^*_A(X)$ such that $\lim F(x) = \tau$ and $F(\tau_G) = \tau$. By continuity of the map $(e, x) \mapsto ex$, we have $\tau = \lim[e] \land \lim F(x) \leq \lim[e] \land F(ex)$ implies $\lim[e] \lor F(ex) = \tau$. Now as $\tau_G \circ \tau_G = \tau_G \circ \tau_G = \tau_G$, we have $[e] \lor F(ex) = \lim[e] \lor (F(v_1) \land F(v_2)) : v_1, v_2 \in X, v_1 \lor v_2 \leq \tau_G \Rightarrow \lim[e] \lor F(v_1) \land F(v_2) = \tau$. Also, for any $v \in X$, we have $\lim[e] \lor F(v) = \lim[e] \lor F(v_1) \land F(v_2) \lor v_1, v_2 \in X, v_1 \lor v_2 \leq v \Rightarrow \lim[e] \lor F(v) = \tau \land F(v) = \tau$ which implies that $\lim e \lor F \in \tau$. Thus we arrive at $\lim[e] \lor F)(ex) = \tau$ and also, $\lim[e] \lor F)(x) = \tau$. Since $[e] \lor F \in F^*_A(X)$, and $X$ is Hausdorff, we have $ex = x$. Similarly, $xe = x$, which in conjunction with Lemma 2.7(2)[30] ensures that $e \in \overline{G}$. □

**Theorem 3.11.** Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathcal{H}$-convergence semigroup and $Y$ a subgroup of $X$. Then $\overline{Y}$ is also a subgroup of $X$.

Proof. In view of Proposition 3.8, it suffices to show that, if $x \in \overline{Y}$, then there exists $x^{-1} \in \overline{Y}$ such that $x^{-1}x = x^{-1}x = e$. Let $y \in \overline{Y}$. Then there exists $F \in F^*_A(X)$ such that $\lim F(y) = \tau$ and $F(\tau_Y) = \tau$. Clearly, $j^\tau(F) = F^{-1} \in F^*_A(X)$. Choose $U \supseteq F^{-1}$. Since $(X, \lim)$ is compact, there exists $h \in X$ such that $\lim U(h) = \tau$. As before, $Y$ is a subgroup of $X$, we have $U(\tau_Y) \supseteq F^{-1}(\tau_Y) \supseteq F(\tau_Y) = \tau$, i.e., $U(\tau_Y) = \tau$. This together with $\lim U(h) = \tau$ imply that $h \in \overline{Y}$. Now we have $\lim U \circ F(hy) \geq \lim U(h) \land \lim F(y) = \tau$, implying $\lim U \circ F(hy) = \tau$. Since $j^\tau(F) \supseteq U$, then in view of Lemma 2.3, there exists $G \in F^*_A(Y)$ such that $G \supseteq F$ and $j^\tau(G) = U$. Then it follows that $U \circ F = j^\tau(G) \supseteq G \supseteq [e]$ which yields that $\lim e(hy) \geq \lim U \circ F(hy) = \tau$, i.e., $\lim e(hy) = \tau$ but as we know $\lim e(hy) = \tau$ implies $hy = e$ because of Hausdorffness. Similarly, $yh = e$, and hence $h = y^{-1} \in Y$. This proves $\overline{Y}$ is a subgroup of $X$. □

**Corollary 3.12.** Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathcal{H}$-convergence semigroup and $A$ be a $\tau$-closed subsemigroup of $X$, then $A$ is a subgroup of $X$. □
Theorem 3.13. Let \((X, \cdot, \lim)\) be a compact Hausdorff-separated stratified \(\mathbb{H}\)-Choquet convergence semigroup such that \((X, \cdot)\) is algebraically a group. Then \((X, \cdot, \lim)\) is a stratified \(\mathbb{H}\)-convergence group.

Proof. Let \(j : X \rightarrow X, x \mapsto x^{-1}, F \in \mathcal{F}_{\mathbb{H}}(X)\) and \(x \in X\). Then \(j^\circ(F) \in \mathcal{F}_{\mathbb{H}}(X)\). Let \(\mathcal{U} \in \mathcal{F}_{\mathbb{H}}^e(X)\) such that \(\mathcal{U} \supseteq j^\circ(F)\). Since \((X, \cdot, \lim)\) is compact, there exists \(y \in X\) such that \(\lim \mathcal{U}(y) = \tau\), and due to continuity of semigroup operation, we have \(\mathcal{T} = \lim \{x\}\) \(\lim \mathcal{U}(yx) \leq \lim F(x) \land \lim U(y) \leq \lim F \circ U(xy)\) implying that \(\lim F \circ U(xy) = \tau\). Now it ensures from \(\mathcal{U} \supseteq j^\circ(F)\) in conjunction with Lemma 2.3 (see Lemma 3.7[22]) that there exists \(\mathcal{G} \in \mathcal{F}_{\mathbb{H}}^e(X)\) such that \(\mathcal{G} \supseteq F\) implying \(j^\circ(\mathcal{G}) = \mathcal{U}\). This yields that \(\mathcal{G} \circ \mathcal{G} \leq \mathcal{G} \circ \mathcal{G} \leq [e]\). But then \(\mathcal{F} \circ \mathcal{U} = \mathcal{F} \circ \mathcal{G} \leq [e]\). So, \(\lim \{x\}(yx) \geq \lim \mathcal{F} \circ \mathcal{U}(xy) = \tau\) implying \(\lim \{x\}(xy) = \tau\). But we know that \(\lim \{e\} = \tau\), hence \(xy = e\), since \((X, \lim)\) is Hausdorff-separated. Similarly, one obtains \(yx = e\) which implies \(y = x^{-1}\). Now as \((X, \lim)\) is a stratified \(\mathbb{H}\)-Choquet convergence space, we have \(\lim \mathcal{F}(x) \leq \tau = \bigwedge_{\mathcal{U} \in \mathcal{F}_{\mathbb{H}}^e(X)} \lim \mathcal{U}(y) = \lim j^\circ(F)(y)\), i.e., \(\lim \mathcal{F}(x) \leq \lim j^\circ(F)(x^{-1})\), proving that \(j\) is continuous. Hence \((X, \cdot, \lim)\) is a stratified \(\mathbb{H}\)-convergence group. \(\square\)

Proposition 3.14. If \((X, \cdot, \lim)\) is a compact Hausdorff-separated stratified \(\mathbb{H}\)-convergence semigroup, then each maximal subgroup is \(\tau\)-closed.

Proof. Let \((X, \cdot, \lim)\) be a compact stratified \(\mathbb{H}\)-convergence semigroup and \(Y\) a maximal subgroup of \(X\). Then in view of Theorem 3.11, \(Y_{\lim}\) is a subgroup of \(X\) and \(Y \subseteq Y_{\lim} \subseteq X\). Since \(Y_{\lim}\) is a subgroup of \(X\) and \(Y\) is maximal, we have \(Y_{\lim} = Y\). This shows that \(Y\) is \(\tau\)-closed. \(\square\)

Definition 3.15. Let \(X\) be a right group. For each idempotent \(e\) of \(X\), let \(\varphi_e : X \rightarrow X\) be defined by \(\varphi_e(x) = (ex)^{-1}\). Then \((X, \cdot, \lim)\) is called a stratified \(\mathbb{H}\)-convergence right group if and only if \(\varphi_e\) is continuous for every idempotent \(e\) in \(X\).

Theorem 3.16. Let \((X, \cdot, \lim)\) be a compact Hausdorff-separated stratified \(\mathbb{H}\)-Choquet convergence semigroup. If \(X\) is right simple, then \((X, \cdot, \lim)\) is a stratified \(\mathbb{H}\)-convergence right group.

Proof. By a result in Clifford-Preston [12] in conjunction with Theorem 3.4, \(X\) is a right group since it is right simple and contains an idempotent. For each idempotent \(e\) of \(X\), let \(\varphi_e : X \rightarrow X\) be defined by \(\varphi_e(x) = (ex)^{-1}\). Let \(\mathcal{F} \in \mathcal{F}_{\mathbb{H}}^e(X)\) and \(x \in X\). Then \(\varphi_e^\circ(F) \in \mathcal{F}_{\mathbb{H}}^e(X_e)\). Let \(\mathcal{U}\) be an ultrafilter on \(X_e\) such that \(\mathcal{U} \supseteq \varphi_e^\circ(F)\). Since \(X_e\) is compact, because of \(X\) is compact, and \(X = X_e\), as \(X\) is a right group. Then there exists \(y \in X_e\) such that \(\lim \mathcal{U}(y) = \tau\). Now \(\lim \{\mathcal{U} \circ \mathcal{F}\}(yx) = \tau\) and also, \(\lim \{\mathcal{U} \circ \mathcal{F}\}(y) = \tau\) implying \(yx = e\) since \(X\) is Hausdorff-separated. Thus \(y = (ex)^{-1}\). Now as \((X, \lim)\) is stratified \(\mathbb{H}\)-Choquet convergence space, we have \(\lim \mathcal{F}(x) \leq \tau = \bigwedge_{\mathcal{U} \in \mathcal{F}_{\mathbb{H}}^e(X)} \lim \mathcal{U}(y) = \lim \varphi_e^\circ(F)(y)\). Hence \(\lim \mathcal{F}(x) \leq \lim \varphi_e^\circ(F)(x)(ex)^{-1}\). \(\square\)

Definition 3.17. Let \((X, \cdot, \lim)\) be a semigroup and for any \(x \in X\), let \(\mathcal{O}(x) = \{x, x^2, x^3, \ldots\}\). Define \(\Gamma(x) = \mathcal{O}(\lim x)\).

Remark 3.18. Note that \(\mathcal{O}(x)\) is a subsemigroup of \(X\) and \(\Gamma(x)\) is also a subsemigroup of \(X\) by Proposition 3.11.

Proposition 3.19. Let \((X, \cdot, \lim)\) be a Hausdorff-separated stratified \(\mathbb{H}\)-generalized convergence semigroup. Then \((X, \cdot, \lim)\) has an idempotent if and only if there exists \(x \in X\) such that \(\Gamma(x)\) is compact.

Proof. Let \((X, \cdot, \lim)\) have an idempotent \(e\). Then \(\mathcal{O}(e) = \{e\}\) which is compact. Next, let there be an \(x \in X\) such that \(\Gamma(x)\) is compact. Then by a classical result on semigroups \(\Gamma(x)\) has a kernel \(\mathcal{K}_r\) and \(\Gamma(x)\) is commutative. Consequently, it is a group and the identity of the kernel \(\mathcal{K}\) is the desired idempotent. \(\square\)

4. Ideals in Complete Heyting Algebra-Valued Convergence Semigroups

Definition 4.1. ([9, 12, 44]) A nonempty subset \(A\) of a semigroup \((X, \cdot)\) is called left ideal if \(XA \subseteq A\); it is called right ideal if \(AX \subseteq A\). It is called ideal if it is both left and right ideal.
Definition 4.2. ([9, 12, 44]) Let $(X, \cdot)$ be a semigroup and $a \in X$. Let $J(a) = \{a\} \cup aX \cup aX \cup aS$, $L(a) = \{a\} \cup Xa$ and $R(a) = \{a\} \cup aX$, whence $J(a)$ is the smallest ideal of $X$ containing $a$; and $L(a)$ and $R(a)$ are the smallest left and right ideals of $X$ containing $a$. Furthermore, if $A$ is a subset of a semigroup $X$, then $L(A) = A \cup aX$, $R(A) = A \cup Xa$ and $J(A) = A \cup Xa \cup aX \cup aXa$.

If $A \subseteq X$, then $J_0(A)$ is defined to be the empty set if $A$ contains no ideal of $X$ and $J_0(A)$ is the union of all ideals contained in $A$ in the contrary case. Similarly, $L_0(A)$ and $R_0(A)$ are defined.

Proposition 4.3. Let $(X, \cdot, \lim)$ be a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and $A$ be an ideal of $X$. Then $\overline{A}_\text{lim}$ is an ideal of $X$.

Proof. Let $A$ be an ideal, $x \in X$ and $z \in \overline{A}_\text{lim}$. We show that $xz \in \overline{A}_\text{lim}$. From $z \in \overline{A}_\text{lim}$ it follows that there exists $F \in F^a_\mathbb{H}(X)$ such that $F(z) = \top$ and $F(\tau_A) = \top$. Now since $\psi_x : X \to X, z \mapsto xz$ is continuous, we have $\lim F(z) \leq \lim \psi_x^\infty(F)(\psi_x(z))$ which implies that $\lim \psi_x^\infty(F)(xz) = \lim \psi_x^\infty(F)(\psi_x(z)) = \top$, and $\psi_x^\infty(F) \in F^a_\mathbb{H}(X)$. Now $\psi_x^\infty(F)(\tau_A) = F(\psi_x^\infty(\tau_A)) = F(\tau_A \circ \psi_x) = \top$, since $\tau_A \circ \psi_x(z) = \tau_A(xz) = \top$ and $xz \in A$ being $A$ a left ideal of $X$. This proves that $xz \in \overline{A}_\text{lim}$, meaning $\overline{A}_\text{lim}$ is a left ideal of $X$. Similarly, we can prove that it is right ideal. Hence it is an ideal of $X$. \hfill $\square$

Lemma 4.4. If $(X, \cdot, \lim)$ is a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and $A \subseteq X$ is $\tau$-closed, then $J_0(A)$, $L_0(A)$ and $R_0(A)$ are $\tau$-closed.

Proof. We only prove the case for $J_0(A)$. Let $J_0(A) \neq \emptyset$. Since $J_0(A)$ is the largest ideal of $X$ contained in $A$, i.e., $J_0(A) \subseteq A$, we have by Lemma 2.7(3)[30], $\overline{J_0(A)}_{\text{lim}} \subseteq \overline{A}_{\text{lim}}$. By Lemma 4.3, $\overline{J_0(A)}_{\text{lim}}$ is an ideal of $X$. Now since $A$ is $\tau$-closed, we have $\overline{A}_{\text{lim}} = A$, and so, $\overline{J_0(A)}_{\text{lim}} \subseteq J_0(A)$. This proves that $\overline{J_0(A)}_{\text{lim}}$ is $\tau$-closed. \hfill $\square$

Proposition 4.5. Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and let $A$ be a compact subset of $X$. Then $L(A)$, $R(A)$ and $J(A)$ are all compact.

Proof. Since $X$ and $A$ are compact, $L(A) = A \cup Xa$ is compact. Similarly, other parts follow. \hfill $\square$

Proposition 4.6. Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup with identity. Then $J(a)$ is compact for each $a \in X$. The same holds for $L(a)$ and $R(a)$.

Proof. Since $X$ is compact, $\{a\}$ is compact being $\tau$-closed subset of Hausdorff-separated space $X$; $aX$, $Xa$ and $XaX$ are compact under continuous mappings. \hfill $\square$

Definition 4.7. ([30]) Let $\mathcal{E}$ be a class of stratified $\mathbb{H}$-generalized convergence spaces $(E, \lim_\mathcal{E})$ which contains a space with at least two points. A stratified $\mathbb{H}$-generalized convergence space $(X, \lim)$ is called $\mathcal{E}$-connected if for any $(E, \lim_\mathcal{E}) \in \mathcal{E}$, every continuous mapping $f : (X, \lim) \to (E, \lim_\mathcal{E})$ is constant.

Proposition 4.8. Let $(X, \cdot, \lim)$ be an $\mathcal{E}$-connected stratified $\mathbb{H}$-convergence semigroup with identity and let $A$ be an $\mathcal{E}$-connected subset of $X$. Then $L(A)$, $R(A)$ and $J(A)$ are all $\mathcal{E}$-connected.

Proof. In view of Lemma 5.1[30], $Xa$ being the continuous image of $\mathcal{E}$-connected sets $X$ and $A$ under $m : X \times X \to X, (x, y) \mapsto xy$ is $\mathcal{E}$-connected. Hence exploiting Lemma 5.5[30], we deduce that $L(A) = A \cup Xa$ is $\mathcal{E}$-connected. Similar arguments show that $R(A)$ and $J(A)$ are $\mathcal{E}$-connected. \hfill $\square$

5. Examples: Stratified $\mathbb{H}$-Convergence Semigroups, Approach Convergence Semigroups and Probabilistic Convergence Semigroups

Example 5.1. Let $(X, \cdot)$ be a semigroup and $(X, \lim_\mathcal{E})$ be an indiscrete stratified $\mathbb{H}$-convergence space, where $\lim \mathcal{F}(x) = \top \forall \mathcal{F} \in F^a_\mathbb{H}(X)$ and $x \in X$ [24]. Then $(X, \cdot, \lim_\mathcal{E})$ is an indiscrete stratified $\mathbb{H}$-convergence semigroup.
Example 5.2. Let \((X, \cdot)\) be a semigroup and \((X, \lim_\nu)\) be a discrete stratified \(\mathcal{H}\)-convergence space, where

\[
\lim_\nu \mathcal{F}(x) = \begin{cases} 
\top, & \text{if } \mathcal{F} \geq [x]; \\
\bot, & \text{if } \mathcal{F} \nolongeq [x].
\end{cases}
\]

Then \((X, \cdot, \lim_\nu)\) is a discrete stratified \(\mathcal{H}\)-convergence semigroup.

Definition 5.3. A triple \((X, \cdot, \mathcal{N} = (\mathcal{H}^x)_{x \in X})\) is called a stratified \(\mathcal{H}\)-neighborhood topological semigroup if the following conditions are satisfied:

1. \((\mathrm{HTSG1})\) \((X, \cdot)\) is a semigroup;
2. \((\mathrm{HTSG2})\) \((X, \mathcal{N})\) is a stratified \(\mathcal{H}\)-neighborhood topological space;
3. \((\mathrm{HTSGM})\) the mapping \(m : (X \times X, \mathcal{N} \times \mathcal{N}) \rightarrow (X, \mathcal{N})\), \((x, y) \mapsto xy\) is continuous, where the product stratified \(\mathcal{H}\)-neighborhood system \(\mathcal{N}_1 \times \mathcal{N}_2\) on \(X \times X\) is given for any \(v \in \mathcal{H}^{X \times X}\) by: \(\mathcal{N}_1 \times \mathcal{N}_2(v) = \mathcal{H}_1 ^x \mathcal{N}_1(v_1) \land \mathcal{H}_1 ^x \mathcal{N}_2(v_2) : v_1, v_2 \in \mathcal{H}^x, v_1 \circ v_2 \leq v \rightarrow \mathcal{F} \land \mathcal{G} \circ \mathcal{F}(v)\).

Proposition 5.4. Every stratified \(\mathcal{H}\)-neighborhood topological semigroup is a stratified \(\mathcal{H}\)-convergence topological semigroup.

Proof. Let \((X, \cdot, \mathcal{N})\) be a stratified \(\mathcal{H}\)-neighborhood topological semigroup, we show that \((X, \cdot, \lim_\nu)\) is a stratified \(\mathcal{H}\)-convergence semigroup. Let \(m : (X \times X, \mathcal{N} \times \mathcal{N}) \rightarrow (X, \mathcal{N}), (x, y) \mapsto xy\) be continuous, and \(\mathcal{F}, \mathcal{G} \in \mathcal{F}^H(X)\). Then for any \(x, y \in X\), we have

\[
\lim_\mathcal{F} \mathcal{G}(xy) = \bigwedge_{v \in \mathcal{H}^x} (\mathcal{H}^y(v) \rightarrow (\mathcal{F} \land \mathcal{G})(v))
\]

which gives

\[
\mathcal{F} \lor \mathcal{G}(xy) \leq \bigwedge_{v \in \mathcal{H}^x} (\mathcal{H}^y(v) \rightarrow (\mathcal{F} \land \mathcal{G})(v))
\]

that is, \(\lim_\nu \mathcal{F}(x) \land \lim_\nu \mathcal{G}(y) \leq \lim (\mathcal{F} \land \mathcal{G})(xy)\).

Definition 5.5. ([15]) Let \((X, \cdot, \mathcal{U})\) be a nonempty set and \(\mathcal{U}\) a stratified \(\mathcal{H}\)-filter on \(X \times X\). If \(\mathcal{U}\) satisfies the properties below, then it is called stratified \(\mathcal{H}\)-quasi-uniformity on \(X\).

1. \((\mathrm{QUS1})\) \(\mathcal{U}(d) \leq \bigwedge_{x \in \mathcal{X}} d(x, x), \forall d \in \mathcal{H}^{x \times x}\);
2. \((\mathrm{QUS2})\) \(\mathcal{U}(d) \leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d], \forall d \in \mathcal{H}^{x \times x}\),

where \(d_1 \circ d_2(x, y) = \bigwedge_{x \in \mathcal{X}} d_1(x, z) \land d_2(z, y), \forall (x, y) \in X \times X\).

The pair \((X, \mathcal{U})\) is called stratified \(\mathcal{H}\)-quasi-uniform space.

It follows from [15] that given a stratified \(\mathcal{H}\)-quasi-uniformity on \(X\) one can obtain a stratified \(\mathcal{H}\)-neighborhood system for each \(x \in X\) and \(v \in \mathcal{H}^x\): \(\mathcal{N}_x^v(v) = \bigvee [\mathcal{U}(d)|d \in \mathcal{H}^{x \times x}, d(x, -) \leq v]\), where \(d(x, -) : X \rightarrow \mathcal{H}, y \mapsto d(x, -)(y) = d(x, y)\). This stratified \(\mathcal{H}\)-neighborhood system then yields a stratified \(\mathcal{H}\)-convergence structure on \(X\): \(\lim_\mathcal{U} \mathcal{F}(x) = \bigwedge_{v \in \mathcal{H}^x} (\mathcal{N}_x^v(v) \rightarrow (\mathcal{F} \land \mathcal{G})(v))\) [24].

Proposition 5.6. Let \((X, \cdot, \mathcal{U})\) be a commutative and cancelation semigroup and let \(\mathcal{U}\) be a stratified \(\mathcal{H}\)-quasi-uniformity on \(X\) such that for all \(d \in \mathcal{H}^{x \times x}\) and for all \(x, y, z \in X\), the property that \(d(x, y) \leq d(xy, yz)\) holds. Then \((X, \cdot, \lim_\mathcal{U})\) is a stratified \(\mathcal{H}\)-convergence semigroup.

Proof. Let \(x, y \in X\) and \(v \in \mathcal{H}^x\). Then we have

\[
\mathcal{N}_x^v\mathcal{U}(v) = \mathcal{H}^y(v)
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x, -) \leq v]
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2)
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]

\[
\leq \bigvee [\mathcal{U}(d_1) \land \mathcal{U}(d_2) : d_1 \circ d_2 \leq d]|d \in \mathcal{H}^{x \times x}, d(x_1, x_2) \leq v(x_1, x_2), \forall x_1, x_2 \in X
\]
\[ \leq \bigvee \{\nu^y_S(d_1(x, -)) \land \nu^y_G(d_2(y, -)) \big| d_1(x, -), d_2(y, -) \in \mathcal{H}_X, d_1(x, -) \times d_2(y, -)(x', y') \leq m^-(\nu)(x', y'), x', y' \in X \big\} \]
\[ = \nu^y_{\mathcal{H}}(\kappa_{\mathcal{H}}(x, y)). \]

In fact, upon using the given property, one obtains: for any \(x', y' \in X, d_1(x, -) \times d_2(y, -)(x', y') = d_1(x, x') \land d_2(y, y') \leq d_1(xy, x'y') \land d_2(x'y', y') \leq d(xy, x'y') \leq \nu(xy)(y'), \) note that \(X\) is a semigroup, and so \(x', y' \in X.\)

Thus, we have \( \nu^y_{\mathcal{H}}(x, y) \leq \nu^y_{\mathcal{H}}(\kappa_{\mathcal{H}})(m^-(\nu)), \) showing that \((x, y) \mapsto xy\) is continuous, and hence \( (X, \mathcal{H} = \nu^y_{\mathcal{H}}) \) is a stratified \(H\)-neighborhood topological semigroup, which in conjunction with Proposition 5.4 proves that \((X, \mathcal{H} \cap \nu_{x,y})\) is a stratified \(H\)-convergence semigroup.

**Definition 5.7.** ([5, 37, 38]) An ultra approach convergence structure \(\lambda\) on a set \(X\) is a function \(\lambda : F(X) \to [0, \infty]^X\) which satisfies the following conditions:

- (uALS1) \(\lambda (x)(x) = 0, \forall x \in X, \) where \(x = \{A \subseteq X | x \in A\} \in F(X).\)
- (uACS2) If \(F, G \in F(X)\) with \(F \leq G,\) then \(\lambda(G) \leq \lambda(F).\)
- (uACS3) \(\forall F, G \in F(X), \lambda(F \land G) = \lambda(F) \vee \lambda(G).\)

The pair \((X, \lambda)\) is called an ultra approach convergence space.

A mapping \(f : (X, \lambda) \to (X', \lambda')\) between ultra approach convergence spaces is called a contraction if and only if \( \forall F \in F(X) \) and \(x \in X, \lambda'((f)(x)) \leq \lambda(F)(x).\)

**Definition 5.8.** (see also, [38]) Let \((X, \cdot)\) be a semigroup and \((X, \lambda)\) be an ultra approach convergence space. Then the triple \((X, \cdot, \lambda)\) is called an ultra approach convergence semigroup if and only if the following conditions are fulfilled:

- (uACCM) \(\forall F, G \in F(X), x, y \in X: \lambda(F \circ G)(xy) \leq \lambda(F)(x) \land \lambda(G)(y)\)

If \(F \in F_{[0,1]}^+(X),\) then \(\Phi_F\) is a filter defined by: \(\Phi_F = \{A \subseteq X | F(1_A) = 1\},\) (cf. [29]).

Now let \(S : [0, 1] \to [0, \infty]\) be a strictly decreasing surjective mapping such that \(S(0) = 0,\) which is also order reversing and satisfies that \(S\left(\bigvee_{i \in I} A_i\right) = \bigvee_{i \in I} S(A_i)\) and \(S\left(\bigwedge_{i \in I} A_i\right) = \bigwedge_{i \in I} S(A_i).\) For this map \(S,\) there exists inverse \(S^{-1} : [0, \infty] \to [0, 1]\) which is strictly decreasing and surjective.

**Proposition 5.9.** ([29]) If \((X, \lambda)\) is a convergence approach space, then \((X, \operatorname{lim}_\lambda)\) is a stratified \([0,1]-convergence\) space, where \(\operatorname{lim}_\lambda F(x) = S^{-1}(\lambda(\Phi_{F\cap G})(x)), \forall F \in F_{[0,1]}^+(X).\)

**Proposition 5.10.** ([29]) If \(f : (X, \lambda) \to (X', \lambda')\) is a contraction, then \(f : (X, \operatorname{lim}_\lambda) \to (X', \operatorname{lim}_{\lambda'})\) is continuous.

**Proposition 5.11.** If \((X, \cdot, \lambda)\) is an ultra approach convergence semigroup, then \((X, \cdot, \operatorname{lim}_\lambda)\) is a \([0,1]-convergence\) semigroup, where for any \(F \in F_{[0,1]}^+(X), \operatorname{lim}_\lambda F(x) = S^{-1}(\lambda(\Phi_{F\cap G})(x)).\)

**Proof.** Only we need to prove that if \(F, G \in F_{[0,1]}^+(X),\) and \(x, y \in X,\) then \(\operatorname{lim}_\lambda F(x) \land \operatorname{lim}_\lambda G(y) \leq \operatorname{lim}_\lambda (F \circ G)(x, y),\) i.e., the multiplication, \(m : X \times X \to X, (x, y) \mapsto xy\) is continuous. We have

\[ \lim_\lambda (F \circ G)(xy) = S^{-1}(\lambda(\Phi_{F \circ G})(xy)). \]

\[ \leq S^{-1}(\lambda(\Phi_F)(x)) \land S^{-1}(\lambda(\Phi_G)(y)) \] (as both \(S^{-1}\) and \(\lambda\) are order reversing)

\[ \leq S^{-1}(\lambda(\Phi_F)(x) \lor \lambda(\Phi_G)(y)) \] (since \(S^{-1}\) is order reversing, applying (uACCM))

\[ = S^{-1}(\lambda(\Phi_F)(x) \land S^{-1}(\lambda(\Phi_G)(y)) \land \operatorname{lim}_\lambda F(x) \land \operatorname{lim}_\lambda G(y). \]

In fact, if \(F, G \in F_{[0,1]}^+(X),\) then we have: \(\Phi_F \circ \Phi_G \leq \Phi_{F \circ G}.\)

Indeed, if \(A \in \Phi_F \circ \Phi_G,\) then there are \(F \in \Phi_F\) and \(G \in \Phi_G\) such that \(F \cdot G \subseteq A.\) These mean that there are \(F, G\) with \(F(1_A) = 1\) and \(G(1_C) = 1\) such that \(F \cdot G \subseteq A.\) Since \(1_F : 1_G = 1_{F \cdot G} \leq 1_A,\) we have \(1 = F(1) \land G(1) \leq \bigvee [F(v_1) \land G(v_2) : v_1 \circ v_2 \leq 1_A] = F \circ G(1_A), i.e., F \circ G(1_A) = 1,\) which yields that \(A \in \Phi_{F \circ G}.\) \(\square\)
Remark 5.12. Lowen-Windels approach semigroup (see [38, Proposition 5.1]) is a convergence approach semigroup (see Definition 6.4, and [38, Proposition 6.5]) according to their notions of approach space and approach convergence space [39]. However, if we consider \((X, +, \lambda)\) (cf. [38, Definition 6.4]), as convergence approach semigroup, then replacing \(\circ\) by \(\oplus\), and making some notational readjustments, one can deduce that the triple \((X, +, \lambda_1)\) is an example of stratified \([0,1]\)-convergence semigroup.

\[\
\text{Definition 5.13.} \quad \text{(21)} \quad \text{Let (} X, \text{lim}_X \text{) and (} Y, \text{lim}_Y \text{) be stratified \(\mathbb{H}\)-convergence spaces. If } \mathcal{C}(X, Y) = \{f \mid f : (X, \text{lim}_X) \to (Y, \text{lim}_Y) \text{ is continuous}\}, \text{then the convergence structure of continuous convergence is defined for any } \mathcal{F} \in \mathcal{F}_{\mathcal{H}}^1(X, Y) \text{ and } f \in \mathcal{C}(X, Y) \text{ by} \\
\lim_{\mathcal{F}} f = \bigwedge_{G \in \mathcal{F}} \bigwedge_{x \in X} (\text{lim}_X G(x) \to \text{lim}_Y \mathcal{E}(\mathcal{F} \times \mathcal{G})(f(x)))
\]

Proposition 5.14. \( \text{Let (} X, \text{lim} \text{) be a stratified \(\mathbb{H}\)-convergence space and (} Y, \text{lim}' \text{) be a Hausdorff-separated stratified \(\mathbb{H}\)-convergence semigroup. Then (} \mathcal{C}(X, Y), \text{lim} \cdot \text{lim}' \text{) is a Hausdorff-separated stratified \(\mathbb{H}\)-convergence semigroup.} \)

\[\
\text{Proof.} \quad \text{This follows from Proposition 4.11[3] in conjunction with Corollary 5.3[27].} \quad \square
\]

Definition 5.16. \( \text{A triple } (X, \cdot, \mathcal{C} = (c_x)_{x \in X}) \text{ is called a stratified convergence semigroup under } \cdot \text{ if and only if the following conditions are satisfied:} \\
(\text{PGC1}) \quad \forall x \in X \quad c_x(x) = 1. \\
(\text{PGC2}) \quad \forall x \in X \quad \forall F, G \in \mathcal{F}(X) \text{ with } F \leq G \text{ implies } c_x(F) \leq c_x(G). \\
(\text{PGCM}) \quad \forall F, G \in \mathcal{F}(X), \forall x, y \in X \quad c_x(F) \land c_y(G) \leq c_{xy}(F \land G). \\
\text{A mapping } f : (X, \mathcal{C}^X) \to (Y, \mathcal{C}^Y) \text{ is called continuous between probabilistic limit spaces } (X, \mathcal{C}^X) \text{ and } (Y, \mathcal{C}^Y) \text{ if and only if for all } x \in X \text{ and for all } F \in \mathcal{F}(X), c^x(F) \leq c^y(f(F)).
\]

Proposition 5.17. \( \text{If } (X, \cdot, \mathcal{C} = (c_x)_{x \in X}) \text{ is a probabilistic convergence semigroup under } \cdot \text{ then } \text{lim}_\mathcal{C} F(x) = c_x(\Phi_F) \text{ for any } F \in \mathcal{F}^1_{[0,1]}(X) \text{ and } x \in X. \)

\[\
\text{Proof.} \quad \text{The pair } (X, \text{lim}_\mathcal{C}) \text{ is a stratified } [0,1]-\text{convergence space. Let } \mathcal{F}, \mathcal{G} \in \mathcal{F}^1_{[0,1]}(X) \text{ and } x, y \in X \text{, lim}_\mathcal{C} F(x) \land \text{lim}_\mathcal{G} y \leq \text{lim}_\mathcal{C} (F \land G)(xy). \text{ Let } F, G \in \mathcal{F}^1_{[0,1]}(X) \text{ and } x, y \in X. \text{ Then we have} \\
\text{lim}_\mathcal{C} (F \land G)(xy) = c_{xy}(\Phi_{F \land G}) \geq c_{xy}(\Phi_F \land \Phi_G) \geq c_x(\Phi_F) \land c_y(\Phi_G)(by(\text{PGCM})) = \text{lim}_\mathcal{C} F(x) \land \text{lim}_\mathcal{G} G(y). \quad \square
\]

6. Stratified \(\mathbb{H}\)-QUCS-Uniformization of Stratified \(\mathbb{H}\)-Convergence Semigroups

Definition 6.1. \( \text{(24)} \quad \text{Let } X \text{ be a non-empty set. A map } \mathfrak{U} : \mathcal{F}_{\mathcal{H}}^1(X \times X) \to \mathbb{H} \text{ is called a stratified } \mathbb{H}\text{-quasi-uniform convergence structure if and only if } \forall F, G \in \mathcal{F}_{\mathcal{H}}^1(X \times X) \\
(\text{HQUA1}) \quad \forall x \in X \quad \mathfrak{U}(((x, x))) = \tau. \\
(\text{HQUA2}) \quad F \leq G \text{ implies } \mathfrak{U}(F) \leq \mathfrak{U}(G). \\
(\text{HQUA3}) \quad \mathfrak{U}(F) \land \mathfrak{U}(G) \leq \mathfrak{U}(F \land G). \\
(\text{HQUA4}) \quad \mathfrak{U}(F) \land \mathfrak{U}(G) \leq \mathfrak{U}(F \circ G), \text{ whenever } F \circ G \text{ exists, are fulfilled. Then the pair } (X, \mathfrak{U}) \text{ is called a stratified } \mathbb{H}\text{-quasi-uniform convergence space.} \)

If, moreover, there is a stratified \(\mathbb{H}\)-filter \(\mathcal{V} \in \mathcal{F}_{\mathcal{H}}^1(X \times X) \text{ such that} \\
(\text{PHQUA}) \quad \forall F \in \mathcal{F}_{\mathcal{H}}^1(X \times X), \mathfrak{U}(F) = \bigwedge_{\mathcal{V} \in \mathcal{V}} (\mathcal{V}(\eta) \to \mathcal{F}(\eta)), \\
\text{then the pair } (X, \mathfrak{U}) \text{ is called a principal stratified } \mathbb{H}\text{-quasi-uniform convergence space.} \)

A map \( f : (X, \mathfrak{U}) \to (Y, \mathfrak{U}') \) is called quasi-uniformly continuous if \( \forall F \in \mathcal{F}_{\mathcal{H}}^1(X \times X), \mathfrak{U}(F) \leq \mathfrak{U}'((f \times f)\eta)(F)). \)
Let $\mathbb{H} = (\mathbb{H}, \leq, \wedge, \vee)$ be a complete Heyting algebra and $(X, \cdot)$ be a semigroup with identity element $e$. If $v \in \mathbb{H}^3$, define a map $\Upsilon_v^r : X \times X \rightarrow \mathbb{H}$ for any $x, y \in X$ by $\Upsilon_v^r(x, y) = x \circ v(y)$, where $x \circ v(y) = \vee_{z \in \mathbb{H}^2} v(z)$ (analogously, define $\Upsilon_v^l : X \times X \rightarrow \mathbb{H}$ for any $x, y \in X$ by $\Upsilon_v^l(x, y) = v \circ x(y)$, where $v \circ x(y) = \vee_{z \in \mathbb{H}^2} v(z)$).

Also, for any $\mathcal{F} \in \mathcal{F}_H^2(X)$, we define a map $\Upsilon_{\mathcal{F}}^r : \mathcal{H}^{X \times X} \rightarrow \mathbb{H}$ defined by

$$\Upsilon_{\mathcal{F}}^r(d) = \vee \{\mathcal{F}(v) : v \in \mathbb{H}^3, \Upsilon_{\mathcal{F}}^l \leq d\}$$

(analogously, for any $\mathcal{F} \in \mathcal{F}_H^2(X)$, we define a map $\Upsilon_{\mathcal{F}}^l : \mathcal{H}^{X \times X} \rightarrow \mathbb{H}$ defined by

$$\Upsilon_{\mathcal{F}}^l(d) = \vee \{\mathcal{F}(v) : v \in \mathbb{H}^3, \Upsilon_{\mathcal{F}}^r \leq d\}$$

**Theorem 6.2.** Every stratified $\mathbb{H}$-convergence semigroup with identity element $e$ gives rise to a stratified $\mathbb{H}$-quasi-uniform convergence space. That is, every stratified $\mathbb{H}$-convergence semigroup with identity element $e$ is $\mathbb{SH}$-$\text{QUCS}$-uniformizable.

The proof of this theorem follows from the following construction, for details see Theorem 5.6[3]. If we define the map $\mathcal{U}^l : \mathcal{F}_H^2(X \times X) \rightarrow \mathbb{H}$ for any $\mathcal{G} \in \mathcal{F}_H^2(X \times X)$ by

$$\mathcal{U}^l(\mathcal{G}) = \vee \{\lim \mathcal{F}(e) : \mathcal{F} \in \mathcal{F}_H^2(X), \Upsilon_{\mathcal{F}}^l \leq \mathcal{G}\},$$

then the pair $(X, \mathcal{U}^l)$ is a stratified left $\mathbb{H}$-quasi-uniform convergence space (analogously, $(X, \mathcal{U}^r)$ is also a stratified right $\mathbb{H}$-quasi-uniform convergence space, where

$$\mathcal{U}^r(\mathcal{G}) = \vee \{\lim \mathcal{F}(e) : \mathcal{F} \in \mathcal{F}_H^2(X), \Upsilon_{\mathcal{F}}^r \leq \mathcal{G}\}.$$ 

**Definition 6.3.** A triple $(X, \cdot, \mathcal{U})$ is called a stratified $\mathbb{H}$-quasi-uniform convergence semigroup if the following conditions are fulfilled:

(QUCG1) $(X, \cdot)$ is a semigroup;

(QUCG2) $(X, \mathcal{U})$ is a stratified $\mathbb{H}$-quasi-uniform convergence space;

(QUCG3) The mapping $m : X \times X \rightarrow X, (x, y) \mapsto xy$ is quasi-uniformly continuous, where the product stratified $\mathbb{H}$-quasi-uniform convergence structure $\mathcal{U} \times \mathcal{U}$ on $X \times X$ is given by: $\mathcal{U} \times \mathcal{U}(\mathcal{F}) = \mathcal{U}((\mathcal{F}^1 \times \mathcal{F}^2)^{\mathcal{U}})$, $\forall \mathcal{F} \in \mathcal{F}_H^2((X \times X) \times (X \times X))$.

**Proposition 6.4.** Every principal stratified $\mathbb{H}$-quasi-uniform convergence semigroup is a stratified $\mathbb{H}$-convergence semigroup.

**Proof.** This follows from Proposition 7.6[1].

**Proposition 6.5.** Let $(X, \cdot, \mathcal{U})$ be a commutative and cancelation semigroup, and let $\mathcal{U}$ be a stratified $\mathbb{H}$-quasi-uniformity on $X$ such that for all $x, y, z \in X$, the property that $d(x, y) \leq d(xz, yz)$ holds. Then $(X, \cdot, \lim_{\mathcal{U}d})$ is a stratified $\mathbb{H}$-convergence semigroup.

**Proof.** It follows from Lemma 5.5[24] that $(X, \lim_{\mathcal{U}d})$ is a principal stratified $\mathbb{H}$-quasi-uniform convergence space. Now it remains to be shown that $(X, \cdot, \lim_{\mathcal{U}d})$ is a principal stratified $\mathbb{H}$-quasi-uniform convergence semigroup, i.e., we show that for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_H^2(X \times X)$: $\mathcal{U}_{\mathcal{U}d}(\mathcal{F}) \wedge \mathcal{U}_{\mathcal{U}d}(\mathcal{G}) \leq \mathcal{U}_{\mathcal{U}d}((m \times m)^{\mathcal{U}}(\mathcal{F} \times \mathcal{G})).$

To prove this, let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_H^2(X \times X)$. Then

$$\mathcal{U}_{\mathcal{U}d}((m \times m)^{\mathcal{U}}(\mathcal{F} \times \mathcal{G})) = \bigwedge_{d \in \mathbb{H}^{X \times X}} (\mathcal{U}(d) \rightarrow (m \times m)^{\mathcal{U}}(\mathcal{F} \times \mathcal{G}(d))$$

$$\geq \bigwedge_{d \in \mathbb{H}^{X \times X}} \left(\frac{1}{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2)} d_1, d_2 \in \mathbb{H}^{X \times X}, d_1 \circ d_2 \leq d\right) \rightarrow (m \times m)^{\mathcal{U}}(\mathcal{F} \times \mathcal{G}(d)) (\Omega)$$

Upon using the stated property, for any $(x, y), (x', y') \in X \times X$, $d_1 \circ d_2 ((x, y), (x', y')) = d_1(x, x') \wedge d_2(y, y') \leq d_1 \circ d_2(xy, x'y') \leq d(xy, x'y') = d(m(x, y), m(x', y')) = (m \times m)^{\mathcal{U}}((x, y), (x', y'))$; note that $x'y' \in X$ as $(X, \cdot)$ is a semigroup. We have
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References