The Variational Problem in Lagrange Spaces Endowed with a Special Type of \((\alpha, \beta)\)-Metrics

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Abstract. In this paper, we will continue our investigation on the new recently introduced \((\alpha, \beta)\)-metric

\[ F = \beta + \frac{a \alpha^2 + \beta^2}{\alpha}, \tag{1} \]

where \(\alpha\) is a Riemannian metric; \(\beta\) is an 1-form and \(a \in \left(\frac{1}{4}, +\infty\right)\) is a real positive scalar. We will investigate the variational problem in Lagrange spaces endowed with this type of metrics. Also, we will study the dually local flatness for this type of metric and we will proof that this kind of metric can be reduced to a locally Minkowskian metric. Finally, we will introduce the 2-Killing equation in Finsler spaces.

1. Introduction

The purpose of this paper is twofold. On the one hand, we will investigate the locally dually flatness; the variational problem in Lagrange spaces endowed with the \((\alpha, \beta)\)-metric

\[ F = \beta + \frac{a \alpha^2 + \beta^2}{\alpha}, \]

where \(\alpha\) is a Riemannian metric; \(\beta\) is an 1-form and \(a \in \left(\frac{1}{4}, +\infty\right)\) is a real positive scalar; and on the other hand, we will investigate the 2-Killing equation in Finsler geometry. We introduced this class of \((\alpha, \beta)\)-metrics in [12] and we have analyzed the S-curvature and other important properties of this class of metrics in [13].

The variational problem of Lagrange spaces endowed with \((\alpha, \beta)\)-metrics is very important and worth to be studied not only in Finsler geometry, but also in physics. Some papers in which the variational problem is presented, are ([7], [8], [2]).

Another important topic investigated in this paper, is the dually locally flatness for the \((\alpha, \beta)\)-metric (1). This notion was introduced in Finsler geometry by Z. Shen in [14] where he extend the previous work of...
S.I. Amari and H. Nagaoka from Riemannian geometry (see [1]). We will investigate for our metric (1) the dually locally flatness because this notion play an important role to the study of flat Finsler structures. This we will give us information about the locally flatness of a Finsler spaces endowed with this kind of metric. In some previous works the dually locally flatness was investigated for some \((\alpha, \beta)\)-metrics (see [3], [4]) and this encouraged us to study this notion for our metric (1). Moreover, we will study the 2-Killing equation in Finsler geometry, which is a new topic and worth to be study. The Finsler spaces endowed with \((\alpha, \beta)\)-metrics were investigated in a lot of papers (see [7], [8], [2]).

2. Preliminaries

Let \(M\) be a \(n\)-dimensional \(C^\infty\) manifold. Denote by \(T_x M\) the tangent space at \(x \in M\), by \(TM = \bigcup_{x \in M} T_x M\) the tangent bundle of \(M\), and by \(TM_0 = TM \setminus \{0\}\) the slit tangent bundle on \(M\). A Finsler metric on \(M\) is a function \(F : TM \to [0, \infty)\) which has the following properties:

(i) \(F\) is \(C^\infty\) on \(TM_0\);

(ii) \(F\) is positively 1-homogeneous on the fibers of tangent bundle \(TM\);

(iii) for each \(y \in T_x M\), the following quadratic form \(g_y\) on \(T_y M\) is positive definite,

\[
g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \quad u, v \in T_y M.
\]

The following notion can be found in [7]:

**Definition 2.1.** A Lagrange space is a pair \(L^n = (M, L(x, y))\) formed by a smooth real, \(n\)-dimensional manifold \(M\) and a regular differentiable Lagrangian \(L(x, y)\), for which the \(d\)-tensor field \(g_{ij}\) has constant signature over the manifold \(TM\).

As we know from [17] and [9], Finsler spaces endowed with \((\alpha, \beta)\)-metrics were applied successfully to the study of gravitational magnetic fields. Other important results from [7] are presented as follows:

Let \(F^n = (M, F(x, y))\) be a Finsler space. It has an \((\alpha, \beta)\)-metric if the fundamental function can be expressed in the following form:

\[ F(x, y) = \tilde{F}(\alpha(x, y), \beta(x, y)) \]

where \(\alpha = a(x)dx^i dx^j\) is a pseudo-Riemannian metric on the base manifold \(M\) and \(b_i(x)dx^i\) is the electromagnetic 1-form on \(M\). As we know from [7], if we denote by \(L^n = (M, L)\) a Lagrange space; the fundamental tensor \(g_{ij}(x, y)\) of \(L^n\) is:

\[ g_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \]

and this tensor can be written as follows for \((\alpha, \beta)\)-Lagrangians:

\[ g_{ij} = \rho_{\alpha ij} + \rho_{\beta ij} + \rho_{-1} (b_i Y_j + b_j Y_i) + \rho_{-2} Y_i Y_j \]

where \(b_i = \frac{\partial \Phi}{\partial y^i}; Y_i = a_{ij} y^j = a \frac{\partial a}{\partial y^i}\);

\(\rho; \rho_{\alpha}; \rho_{-1}; \rho_{-2}\) are invariants of the space \(L^n\).

Here, \(\rho; \rho_{\alpha}; \rho_{-1}; \rho_{-2}\) are given by (see [7]):

\[
\rho = \frac{1}{2} L_{\alpha ij} + \rho_{\alpha} = \frac{1}{2} L_{\beta ij};
\]

\[
\rho_{-1} = \frac{1}{2} L_{\alpha ij} \rho_{-1} = \frac{1}{2} \frac{\partial}{\partial y^i} \left( L_{\alpha ij} - \frac{\alpha}{\alpha} L_{\alpha i} \right).
\]

(2)

where \(L_\alpha = \frac{\partial L}{\partial y^i}; L_\beta = \frac{\partial L}{\partial x^i}; L_{\alpha ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}; L_{\beta ij} = \frac{\partial^2 L}{\partial x^i \partial x^j} \) and \(L_{\alpha ij} = \frac{\partial^2 L}{\partial x^i \partial y^j}\).

Shimada and Sabau in [16], have proved that the system of covectors \([b_i, Y_i]\) is independent. The following formulae holds (see [7]):

\[
y_i = \frac{1}{2} \frac{\partial L}{\partial y^i} = \rho_1 b_i + \rho_1 Y_i; \rho_1 = \frac{1}{2} L_{\beta i};
\]
Some interesting results in the theory of Lagrange spaces are also presented in [5]. Let's recall now some notions regarding 2-Killing vector fields on Riemannian manifolds from paper [11].

The function $F$ is a Finsler function if and only if three conditions are satisfied:

$$
\sigma_{(i,j,k)} \left( \rho_{1}a_{i}b_{k} + \rho_{2}a_{j}b_{k} + \frac{1}{3} \rho_{1}b_{i}b_{j}b_{k} + \rho_{3}b_{i}b_{j}b_{k} + \frac{1}{3} \rho_{4}b_{i}b_{j}b_{k} \right) = 0.
$$

Remark 2.2. From paper [6], we know the following:

The variational problem for Finsler spaces endowed with $(\alpha, \beta)$-metrics is an important topic in Finsler geometry. For such spaces, the Euler-Lagrange equations $E_{i}(L) = \frac{d}{dt} \left( \frac{\partial L}{\partial y^{i}} \right) = 0$, can be give in the following way:

$$
E_{i}(L) = E_{i}(\alpha^{2}) + \frac{2}{\rho} \rho_{1} E_{i}(\beta) + \frac{2}{d} \frac{dx^{i}}{dt} \frac{d}{dy^{j}}
$$

Remark 2.4. If we use the following equations $E_{i}(\beta) = F_{i}(x) \frac{dx^{i}}{ds}$;

$$
F_{ij} = \frac{\partial b_{i}}{\partial x^{j}} - \frac{\partial b_{j}}{\partial x^{i}} = b_{ij} - \rho_{ij},
$$

then (5) can be rewritten in the following way:

$$
E_{i}(\alpha^{2}) + \frac{2}{\rho} \rho_{1} \left( b_{ij} - \rho_{ij} \right) = 0; \quad y^{i} = \frac{dx^{i}}{ds}
$$

Some interesting results in the theory of Lagrange spaces are also presented in [5]. Let’s recall now some notions regarding 2-Killing vector fields on Riemannian manifolds from paper [11].
\textbf{Definition 2.5.} Let \((M, g)\) a Riemannian manifold. A vector field \(X \in \chi(M)\) is called 2-Killing, if \(L_X L_X g = 0\), where \(L\) is the Lie derivative.

In this paper we will extend this notion for the case of Finsler spaces. First we will recall the notion of Killing vectors in Finsler spaces. We will follow the results from [15]. We will consider the coordinate transformation:

\[ \begin{align*}
\bar{x}^i &= x^i + \epsilon V_i^j; \\
\bar{y}^j &= y^j + \epsilon \frac{\partial V_i^j}{\partial x^i} y^i
\end{align*} \]

Under this change of coordinates, a Finsler structure became (see [15]):

\[ \bar{F}(\bar{x}, \bar{y}) = \bar{F}(x, y) + \epsilon V^i \frac{\partial F}{\partial x^i} + \epsilon y^j \frac{\partial F}{\partial y^j}. \]

where \(\bar{F}(\bar{x}, \bar{y})\) must be equal with \(F(x, y)\). Using this remark, the authors of paper [15], concluded that the Killing equation in Finsler space is:

\[ K_V(F) = V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^i} \frac{\partial F}{\partial y^j} = 0 \]  \hspace{1cm} (8)

Also, for an \((\alpha, \beta)\)-metric, the authors of paper [15], remarked that the Killing equation for the Finsler spaces endowed with this kind of metrics, is given as follows:

\[ 0 = K_V(\alpha)\phi(s) + aK_V(\phi(s)) \]

\[ \Rightarrow 0 = \left( \phi(s) - s \frac{\partial \phi(s)}{\partial s} \right) K_V(\alpha) + \frac{\partial \phi(s)}{\partial s} K_V(\beta) \]  \hspace{1cm} (9)

where \(\partial /\partial s\) denotes the covariant derivative with respect to Riemannian metric \(\alpha\). As we presented in previous section, in Introduction, the dually locally flatness on Finsler spaces is an important topic and worth to be studied because give us important informations about the flatness of the space. An important result obtained in [18], is the following one:

\textbf{Theorem 2.6.} ([18]) Let \(F = \alpha \phi(s), s = \frac{\ell}{\alpha}\) be an \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold \(M^n, (n \geq 3)\), where \(\alpha = \sqrt{a_{ij}y_j^i}\) is a Riemannian metric and \(\beta = b_i(x)y^i \neq 0\) is an 1-form on \(M\). Suppose that \(F\) is not Riemannian and \(\phi'(s) \neq 0; \phi''(0) \neq 0; \beta \neq 0\). Then \(F\) is a locally dually flat on \(M\) if and only if \(\alpha, \beta\) and \(\phi = \phi(s)\), satisfy:

1. \(s_0 = \frac{1}{2}(\beta \theta_1 - \theta b_1)\),
2. \(r_0 = \frac{3}{2} \theta \beta + \left[ \theta + \frac{3}{2}(b^2 \theta - \theta b_1) \right] a^2 + \frac{3}{2} (a k_2 - 2 - 3 k_2 b_2) \theta b_2\),
3. \(G_3 = \frac{1}{2} [2 \theta + (3 k_1 - 2) \theta b_1] y^j + \frac{1}{2} (\theta - \tau b_2) a^2 + \frac{1}{2} k_2 \beta \beta_1 \),
4. \(4 \tau [s(k_2 - k_3 \phi')^2 (\phi \phi' - s \phi' - s \phi') - (\phi^2 + \phi \phi'') + k_1 \phi(\phi - s \phi')] = 0,\)

where \(\tau = \tau(x)\) is a scalar function; \(\theta = \theta(x)\) is an 1-form on \(M\), \(\theta_1 = d \theta_0 \theta_0\),

\[ \begin{align*}
k_1 &= \Pi(0); k_2 &= \frac{\Pi'(0)}{Q(0)}; k_3 &= \frac{1}{6 \Pi(0)} \left[ 3 \Pi''(0) \Pi'(0) - 6 \Pi(0)^2 - Q(0) \Pi'''(0) \right], \hspace{1cm} (10)
\end{align*} \]

and \(Q = \frac{\phi'}{\phi''} - \frac{\phi'}{\phi''}, \quad \Pi = \frac{\phi'' + \phi \phi''}{\phi'(0 - s \phi')}\).
3. Main Results

In the following lines, we will give our first main result. Using our metric (1), in which \( \phi(s) = s^2 + s + a \), with \( s = \frac{2}{a} \), we will compute \( Q(s), \Pi(s) \) and \( k_1, k_2, k_3 \) given in (10) for our metric. After tedious computations, one obtains:

\[
Q(s) = \frac{2s + 1}{a - s^2}; \quad Q(0) = \frac{1}{a}; \quad Q'(0) = \frac{2}{a}.
\]

\[
\Pi(s) = \frac{6s^2 + 6s + 2a + 1}{(s^2 + s + a)(a - s^2)}; \quad \Pi'(0) = \frac{6}{a^2} - \frac{1 + 2a}{a^3}.
\]

\[
k_1 = \Pi(0) = \frac{1 + 2a}{a^2} \quad \text{(11)}
\]

\[
k_2 = \frac{\Pi'(0)}{Q(0)} = \frac{4a - 1}{a^2} \quad \text{(12)}
\]

\[
k_3 = -\frac{1}{6Q(0)^2} \left[ 3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0) \right] = \frac{2(1 - 4a)}{a^3} \quad \text{(13)}
\]

Now, using Theorem 2.6 and the above relations (11)-(13), we will formulate the following:

**Theorem 3.1.** Let \( M^n, (n \geq 3) \) an \( n \)-dimensional manifold and let \( F = \beta + \frac{\alpha x^2 + \beta y}{a} \); where \( \alpha \) is a Riemannian metric; \( \beta \) is an 1-form and \( a \in \left( \frac{1}{2}, +\infty \right) \) is a real positive scalar; given in (1). Supposing that \( F \) is not Riemannian, then \( F \) is locally dually flat on \( M \), if and only if \( \alpha, \beta \) and \( \phi(s) = s^2 + s + a \), satisfy:

- \( 1. s_{i0} = \frac{1}{3}(\beta \theta_i - \theta \beta_i) \),
- \( 2. r_{00} = \frac{2}{3}\theta \beta + \left[ \theta + \frac{1}{2}(\theta b^2 - \theta \beta b) \right] \alpha^2 + \left[ \left( \frac{4\alpha - 1}{a^2} \right) \left( 1 + \frac{2\beta}{a} \right) \right] \tau \beta^2 \),
- \( 3. G_a = \frac{1}{3} \left[ 2\theta + \left( \frac{-2\alpha^2 + 4\alpha + 1}{a^2} \right) \tau \beta^2 \right] \theta^2 + \frac{1}{2}(\theta^2 - \tau \beta^2) \alpha^2 - \frac{4\alpha - 1}{a^2} \tau \beta^2 \beta \),
- \( 4. \tau \left[ \frac{4\alpha - 1}{a^2} \left( 1 + \frac{2\beta}{a} \right) (a - 3s^3 - 3s^2 - (6s^2 + 6s + 2a + 1) + \frac{4\beta}{a} (s^2 + s + a)(a - s^2) \right] = 0 \),

where \( \tau = \tau(x) \) is a scalar function; \( \theta = \theta_i(x) \) is an 1-form on \( M \), \( \theta^i = a^m \theta_m \).

**Proof.** Using the above relations (11)-(13) computed for our metric (1) and replacing them in Theorem 2.6, we get easily the assertion of the theorem. \( \Box \)

**Remark 3.2.** Using Theorem 2.6, in paper [19], is presented the following corollary:

**Corollary 3.3.** ([19]) Let \( F = \alpha \phi(s), s = \frac{2}{a} \), be an \( (\alpha, \beta) \)-metric on a manifold \( M \) of dimension \( n \geq 3 \) with the same assumptions as in Theorem 2.6. Let \( \phi \), satisfy:

\[
s(k_2 - k_3 s^2)(\phi \phi' - s \phi' - s \phi''') - (\phi^2 + \phi \phi'') + k_1 \phi (\phi - s \phi') \neq 0.
\]

Then, \( F \) is locally dually flat on \( M \) if and only if:

\[
s_{i0} = \frac{1}{3}(\beta \theta_i - \theta \beta_i),
\]

\[
r_{00} = \frac{2}{3} \left[ \theta \beta - (\theta \beta) \alpha^2 \right],
\]

\[
G_a = \frac{1}{3} \left[ 2\theta \beta^2 + \theta \beta \alpha^2 \right]
\]

where \( k_i, (1 \leq i \leq 3) \) are the same with those of Theorem 2.6.
Another important result from paper [19], is the following lemma:

**Lemma 3.4.** ([19]) Let $F = \alpha \phi(s)$, $s = \frac{\beta}{a}$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold and $b = \|\beta\|_a$. Suppose that $\phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3$ for any constant $c_1 > 0$; $c_2$ and $c_3$. Then $F$ is of isotropic $S$-curvature, $S = (n + 1) c F$, if and only if one of the following holds:

(a) $\beta$ satisfies:
$$ r_{ij} = e \left[ b^2 a_{ij} - b_i b_j \right], s_j = 0, $$
where $e = e(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies:
$$ \Phi = -2(n + 1)k \frac{\phi \Delta^2}{\beta^2 - s^2}, $$
where $k$ is a constant. In this case, $S = (n + 1) c F$, with $c = kc$.
(b) $\beta$ satisfies:
$$ r_{ij} = 0; s_j = 0 $$
In this case, $S = 0$, regardless of choices of a particular $\phi$.

Next, using the above results, we are ready to proof the following theorem:

**Theorem 3.5.** Let $F = \beta + \frac{a_2 + b^2}{a}$ our metric (1) with $\phi(s) = s^2 + s + \alpha$; where $\alpha$ is a Riemannian metric; $\beta$ is an 1-form and $a \in \left( \frac{1}{4}, +\infty \right)$ is a real positive scalar, on the manifold $M$ with $n \geq 3$, on the same assumptions as in Theorem 2.6. Let $\phi$, satisfy:
$$ s \left( \frac{4a - 1}{a^2} \left( 1 + \frac{2s^2}{a} \right) \right) (a - s^2) + (s^2 + s + \alpha) \left( \frac{1 + 2a}{a^2} - 6 \right) + 4a - 1 \neq 0 $$
Then $F$ is locally dually flat on $M$ and of isotropic $S$-curvature $S = (n + 1) c F$, if and only if:

$$ s_{00} = \frac{1}{3} \left( \beta \theta_0 - \theta b \right), $$
$$ r_{00} = \frac{2}{3} \theta \beta + \left[ \theta + \frac{2}{3} (b^2 \theta - \theta b') \right] a^2, $$
$$ G_{0'}^l = \frac{1}{3} \left[ 2 \theta + \left( \frac{-2a^2 + 6a + 3}{a^2} \right) \tau \beta \right] a' $$
where $\theta = \theta_i(x) y^i$ is an 1-form on $M$.

**Proof.** We will use the same approach as in paper [19], but for our metric (1). In this case, $\phi(s) = s^2 + s + \alpha$; $\phi' = 2s + 1$; $\phi''(0) \neq 0$.

From (14), we know the condition:
$$ s(k_2 - k_3 s^2)(\phi \phi' - s \phi' - s \phi \phi'') - (\phi^2 + \phi \phi'') + k_1 (\phi - s \phi') \neq 0. $$
This condition, for our metric (1), can be written in the following way:
$$ s \left( \frac{4a - 1}{a^2} \left( 1 + \frac{2s^2}{a} \right) \right) (a - s^2) + (s^2 + s + \alpha) \left( \frac{1 + 2a}{a^2} - 6 \right) + 4a - 1 \neq 0. $$
Now, from Theorem 3.1, we know:
$$ r_{00} = \frac{2}{3} \theta \beta + \left[ \theta + \frac{2}{3} (b^2 \theta - \theta b') \right] a^2 + \left[ \left( \frac{4a - 1}{a^2} \left( 1 + \frac{2b^2}{a} \right) - \frac{2}{3} \right) \tau \beta \right] a'. $$

**Lemma 3.6.** ([19]) Let $F = \alpha \phi(s)$, $s = \frac{\beta}{a}$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold and $b = \|\beta\|_a$.
By assumption,\( L = \frac{1}{3} [2\mu + \left( -\frac{2\alpha^2 + 6\alpha + 3}{a^2} \right) \tau \beta] y' + \frac{1}{3} (\tau' - \tau \beta') a^2 - \frac{4 \alpha - 1}{a^3} \tau \beta^2 \beta', \) 

(16)

By assumption, \( \frac{\beta''(\theta)}{\beta'} = \beta. \) Replacing this expression in (15), we get:

\[
r_{00} = \frac{2b^l \partial_l \theta}{b^2} \beta^2 + \left[ \tau + \frac{2}{3} (b^2 \tau - \partial_l \theta^l) \right] \alpha^2 - \left[ \frac{4 \alpha - 1}{a^2} \left( 1 + \frac{2b^2}{a} \right) - \frac{2}{3} \right] \tau \beta^2 = 0.
\]

Differentiating this relation with respect to \( y^m \) and taking into account that \( r_{00} = 0 \) by Lemma 3.4, one obtains:

\[
\frac{4b^l \partial_l \tau}{b^2} \beta b_m + 2 \left[ \tau + \frac{2}{3} (b^2 \tau - \partial_l \theta^l) \right] \alpha^2 - \left[ \frac{4 \alpha - 1}{a^2} \left( 1 + \frac{2b^2}{a} \right) - \frac{2}{3} \right] \tau \beta b_m = 0.
\]

Multiplying with \( b^m, \) one obtains:

\[
2 \left[ \frac{4 \alpha - 1}{a^2} \left( 1 + \frac{2b^2}{a} \right) - \frac{2}{3} \right] \tau \beta = 0.
\]

By assumption,

\[
\left[ \frac{4 \alpha - 1}{a^2} \left( 1 + \frac{2b^2}{a} \right) - \frac{2}{3} \right] \tau \beta \neq 0.
\]

So, we can deduce that \( \tau = 0. \) Taking into above relations (15) and (16), we get the desired result. \( \square \)

### 3.1. The variational problem for Finsler spaces endowed with the \((\alpha, \beta)\)-metric (1)

Starting from (1), in this section, we will consider a Finsler space endowed with the fundamental function \( L = F^2 = \left( \beta + \frac{aa^2 + \beta^2}{a} \right)^2, \) where \( \alpha \) is a Riemannian metric; \( \beta \) is an 1-form and \( \alpha \in \left( \frac{1}{a}, +\infty \right) \) is a real positive scalar. We will investigate the variational problem for this Finsler space endowed with this \((\alpha, \beta)\)-metric. First, after tedious computations, one obtains:

\[
L_\alpha = \frac{2(\beta \alpha + aa^2 + \beta^2)(aa^2 - \beta^2)}{a^3}; \quad L_\beta = \frac{2(\beta \alpha + aa^2 + \beta^2)(\alpha + 2\beta)}{a^2}
\]

\[
L_{\alpha \beta} = \frac{2(-4\beta^3 - 3\beta^2 \alpha + aa^3)}{a^3}; \quad L_{\beta \beta} = \frac{2(\alpha^2 + 6\beta \alpha + 6\beta^2 + 2aa^2)}{a^2}; \quad L_{\alpha \alpha} = \frac{2(a^2 \alpha^4 + 3\beta^4 + 2\beta^3 \alpha)}{a^4}
\]

\[
L_{\alpha \alpha \alpha} = \frac{-12\beta^3 (\alpha + 2\beta)}{a^5}; \quad L_{\alpha \beta \beta} = \frac{-12(\alpha + 2\beta) \beta^2}{a^3}; \quad L_{\beta \beta \beta} = \frac{12(\alpha + 2\beta) \beta^2}{a^2}; \quad L_{\alpha \alpha \beta} = \frac{12(\alpha + 2\beta) \beta^2}{a^4}
\]

(17)

Next, we will compute for our metric (1):

\[
\rho = \frac{1}{2a} L_\alpha = \frac{\beta (aa^2 + 2\beta^2)(aa^2 - \beta^2)}{a^4}; \quad \rho_0 = \frac{1}{2} L_{\beta \beta} = \frac{\alpha^2 + 6\beta \alpha + 6\beta^2 + 2aa^2}{a^2};
\]

\[
\rho_{-1} = \frac{1}{2a} L_{\alpha \beta} = \frac{aa^3 - 3aa^2 - 4\beta^3}{a^4}; \quad \rho_1 = \frac{1}{2a} L_\beta = \frac{(aa^2 + 2\beta^2)(\alpha + 2\beta)}{a^2}
\]

\[
\rho_{-2} = \frac{1}{2a^2} \left( L_{\alpha \alpha} - \frac{1}{a} L_\alpha \right) = \frac{\beta(4\beta^3 + 3aa^2 - aa^3)}{a^6}
\]

(18)
Next, we compute:
\[ r_1 = \frac{1}{2} L_{\alpha\beta\gamma} = \frac{6(\alpha + 2\beta)}{a^2}; \quad r_2 = \frac{1}{2a} L_{\alpha\beta\gamma} = -\frac{6(\alpha + 2\beta)}{a^4}; \]
\[ r_3 = \frac{1}{2a^2} \left( L_{\alpha\alpha\beta} - \frac{1}{a} L_{\alpha\beta} \right); \quad r_4 = \frac{1}{2a^2} \left( L_{\alpha\alpha\alpha} - \frac{3}{a^2} L_{\alpha} \right) = \frac{3\beta(-8\beta^3 - 5\alpha\beta^2 + a\alpha^3)}{a^6} \]
(19)

Next we will follow the same treatment as in [7], but this time for the metric (1). Using equations (3), we will get the following results:
\[ y_i = \frac{(\alpha\beta + aa^2 + \beta^2)(\alpha + 2\beta)}{a^2} b_i + \frac{(\alpha\beta + aa^2 + \beta^2)(aa^2 - \beta^2)}{a^4} \]
\[ \frac{\partial \rho_1}{\partial y^i} = \frac{a^2 + 6\alpha\beta + 6\beta^2 + 2aa^2}{a^2} b_i + \frac{aa^2 - 3\alpha\beta^2 - 4\beta^3}{a^4} Y_i \]
\[ \frac{\partial \rho}{\partial y^i} = \frac{a^3 - 3\alpha\beta^2 - 4\beta^3}{a^4} b_i + \frac{\beta(4\beta^3 + 3\alpha\beta^2 - aa^3)}{a^6} Y_i \]
\[ \frac{\partial \rho_0}{\partial y^i} = \frac{6(\alpha + 2\beta)}{a^2} b_i - \frac{6\beta(\alpha + 2\beta)}{a^4} Y_i \]
\[ \frac{\partial \rho_{-2}}{\partial y^i} = \frac{4\beta^3 - 6\beta a^2 - 9a^2\beta - 9\alpha\beta^2 - aa^3}{a^6} b_i + \frac{3\beta(-8\beta^3 - 5\alpha\beta^2 + a\alpha^3)}{a^8} Y_i \]
\[ \frac{\partial \rho_{-1}}{\partial y^i} = \frac{-6\beta(\alpha + 2\beta)}{a^4} b_i + \frac{4\beta^3 - 6\beta a^2 - 9a^2\beta - aa^3}{a^6} Y_i \]
(20)

Now, we are ready to give the following result:

**Theorem 3.6.** The Cartan tensor for the \((\alpha, \beta)\)-metric (1), has the following form:
\[ 2C_{ij} = \frac{\sigma_{(i,j)}}{\alpha^2}(a_3 - 3\alpha\beta^2 - 4\beta^3) a_i a_j \]
\[ + \frac{\beta(4\beta^3 + 3\alpha\beta^2 - aa^3)}{a^6} a_i Y_j + \frac{2(\alpha + 2\beta)}{a^2} b_i b_j b_k - \frac{6(\alpha + 2\beta)}{a^4} b_i b_j Y_k \]
\[ + \frac{4\beta^3 - 6\beta a^2 - 9a^2\beta - aa^3}{a^6} b_i Y_j Y_k + \frac{\beta(aa^3 - 8\beta^3 - 5\alpha\beta^2)}{a^8} Y_i Y_j Y_k \]
(21)
where \(\sigma_{(i,j,k)}\) is the cyclic sum of the indices.

**Proof.** From (4), we know the general form of the Cartan tensor for an \((\alpha, \beta)\)-metric. Replacing (15) and (16) in (4), we get the Cartan tensor for the metric (1). \(\square\)

**Theorem 3.7.** The Euler-Lagrange equations for the metric (1), can be expressed in the following way:
\[ E_i(\alpha^2) + 2(\alpha + 2\beta)(\alpha^2 - \beta^2) a_i (b_{\beta i} - b_{\beta j}) = 0, \quad y' = \frac{dx^i}{ds} \]

**Proof.** The proof is direct, using (5) and (15). \(\square\)

Now, using (8), (9) and Definition 2.5, we will study the 2-Killing equation for an \((\alpha, \beta)\)-metric. In this regard, we can give now the following:
Theorem 3.8. The 2–Killing equation for an general \((\alpha, \beta)\)-metric, \(F = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\), has the following form:

\[
K_V \left( \frac{\partial \phi(s)}{\partial s} \right) (K_V(\beta) - K_V(\alpha)) + \frac{\partial \phi(s)}{\partial s} (K_V(K_V(\beta)) - sK_V(K_V(\alpha))) + \\
\phi(s)K_V(K_V(\alpha)) + K_V(\alpha)K_V(\phi(s)) - K_V(s)K_V(\alpha) \frac{\partial \phi(s)}{\partial s} = 0
\] 

(22)

with

\[
K_V(\alpha) = \frac{1}{2\alpha} \left( V_{ij} - V_{ji} \right) y^i y^j
\]

\[
K_V(\beta) = \left( V_i \frac{\partial b_j}{\partial x^i} + b_i \frac{\partial V_j}{\partial x^i} \right) y^j
\]

where "\(\nabla\)" denotes the covariant derivative with respect to Riemannian metric \(\alpha\).

Proof. We will start with the 2-Killing equation adapted this time for Finsler spaces endowed with \((\alpha, \beta)\)-metrics:

\[K_V(K_V(\alpha \phi(s))) = 0.\]

After computation, we get:

\[K_V(K_V(\alpha \phi(s))) = K_V \left( K_V(\alpha) \left( \phi(s) - s \frac{\partial \phi(s)}{\partial s} \right) + \frac{\partial \phi(s)}{\partial s} K_V(\beta) \right) + \\
K_V(K_V(\alpha)) \left( \phi(s) - s \frac{\partial \phi(s)}{\partial s} \right) + K_V(\alpha) \left( K_V(\phi(s)) - K_V(s \frac{\partial \phi(s)}{\partial s} - sK_V \left( \frac{\partial \phi(s)}{\partial s} \right) \right) + \\
K_V \left( \frac{\partial \phi(s)}{\partial s} K_V(\beta) + \frac{\partial \phi(s)}{\partial s} K_V(\beta) \right)
\]

and after we group the terms, we get the desired result. \(\Box\)

In the following lines, we will recall some notions from [10].

Proposition 3.9. ([10]) We have the relations:

\[
\partial_i \alpha = \frac{1}{\alpha} y_i; \quad \partial_i \partial_i \alpha = \frac{1}{\alpha} \gamma_{ij}(x) - \frac{1}{\alpha^3};
\]

\[
\partial_i \beta = A_i(x); \quad \partial_i \partial_i \beta = 0
\]

The moments of the Lagrangian \(L(\alpha(x, y), \beta(x, y))\) (see [10]), is given by:

\[
p_i = \frac{1}{2} \left( L_\alpha \partial_i \alpha + L_\beta \partial_i \beta \right)
\]

Proposition 3.10. ([10]) The moments of the Lagrangian \(L(x, y)\), are given by: \(p_i = \rho y_i + \rho_1 A_i\), where \(\rho = \frac{1}{L_\alpha} \frac{1}{L_\beta} \rho_1 = \frac{1}{L_\beta}\). The derivatives of the principal invariants of the Lagrange space are given by:

\[
\partial_i \rho = \rho_{-2} y_i + \rho_{-1} A_i; \quad \partial_i \rho_1 = \rho_{-1} y_i + \rho_0 A_i
\]

Now, we are able to give the following result for our metric (1):

Proposition 3.11. The moments of the Lagrangian \(L(x, y)\) for the metric (1), are given by:

\[
p_i = \frac{(\alpha \beta + aa^2 + \beta^2)(aa^2 - \beta^2)}{\alpha^4} y_i + \frac{(\alpha \beta + aa^2 + \beta^2)(a + 2\beta)}{\alpha^2} A_i
\]
Proof. The proof is direct, using (15) and Proposition 3.10.

**Proposition 3.12.** The derivatives of the principal invariants of the Lagrange space endowed with metric (1), are given by:

\[
\partial_i \rho = \frac{\beta(4\beta^3 + 3\alpha \beta^2 - a \alpha^3)}{a^6} y_i + \frac{a \alpha^3 - 3 \alpha \beta^2 - 4 \beta^3}{a^4} A_i
\]

\[
\partial_i \rho_1 = \frac{a \alpha^3 - 3 \alpha \beta^2 - 4 \beta^3}{a^4} y_i + \frac{\alpha^2 + 6 \alpha \beta + 6 \beta^2 + 2 a \alpha^2}{a^4} A_i.
\]

Proof. The proof is obvious and follow easily, using (18) and the above Proposition 3.10.

**Remark 3.13.** In this paper we used the Maple 13 program at computations.

4. Conclusion

In this paper we have continued the investigations on the new introduced \((\alpha, \beta)\)-metric (1) and we succeed to investigate the dually locally flatness and the Cartan tensor for this type of metrics. Also we investigated the 2-Killing equation for Finsler spaces, which represent an important step for the study of Finsler spaces.

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