Asymptotic Method for Solution of Identification Problem of the Nonlinear Dynamic Systems

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Abstract. A dynamic system, when the motion of the object is described by the system of nonlinear ordinary differential equations, is considered. The right part of the system involves the phase coordinates as a unknown constant vector-parameter and a small number. The statistical data are taken from practice: the initial and final values of the object coordinates. Using the method of quasilinearization the given equation is reduced to the system of linear differential equations, where the coefficients of the coordinate and unknown parameter, also of the perturbations depend on a small parameter linearly. Then, by using the least-squares method the unknown constant vector-parameter is searched in the form of power series on a small parameter and for the coefficients of zero and the first orders the analytical formulas are given. The fundamental matrices both in a zero and in the first approach are constructed approximately, by means of the ordinary Euler method. On an example the determination of the coefficient of hydraulic resistance (CHR) in the lift in the oil extraction by gas lift method is illustrated, as the obtained results in the first approaching coincides with well-known results on order of $10^{-2}$.

1. Introduction

The problem of identification [3, 9, 13] of the dynamic systems [7, 8, 11, 16] has a lot of practical applications, one of them is the applying in the oil fields, for example, in the oil extraction [8, 13, 20]. Really, in the serve of oil by means of pipelines, in the extraction of oil by the gas lift method or by the rod-pumping setting and other methods - the determination of CHR [3, 6, 9], in layers - the determination of parameters of formation of gas-liquid mixture [10] and other, requires development of numerical methods for the solving of corresponding problems of identification [18] of the dynamic systems. In [9] the gradient method on the basis of Gramm-Schmidt orthogonalization for determination of CHR is given. In [4, 9] the asymptotic method is given for determination of CHR in the first approaching relatively to small parameter, where a small parameter is accepted by inverse to the value of depth of the well. It is shown, that if the use of ordinary Gramm-Schmidt method requires enough large machine time, then an asymptotic method allows to calculate the solution of CHR in the first approaching analytically and the numerical results coincide
with the sought solution to the $10^{-2}$ order. Coming from these results, the authors of [3, 5] generalized the results obtained in [4] for multidimensional case. Further, using the methods of quazilinearization [14] and least-squares [17] the computational algorithm is suggested for solving of general identification problem of the dynamic systems for determination of the constant vector-parameter, which can be used for finding the CHR in the lift and the parameters for formation of gas-liquid mixture in the layers of the wells in the oil production.

The results of calculations show that in simplest case, when the sought (unknown) constant vector-parameter - the subject to determination is scalar, then the method of Gramm-Schmidt requires 2 hours to the right part of corresponding differential equations, where the small parameter is included to the right part of corresponding differential equations, where on the example of the oil fields it is the inverse value of depth of the wells.

In this work it is assumed, that some series of initial and final values (statistical data) of phase coordinate of nonlinear ordinary differential equation\(^1\) (to the right part are included the small parameter and unknown constant vector) are given. It is required to find the small parameter and unknown constant vector in such way that their solutions in the end point coincided (with certain exactness) with statistical data. Using the methods of least-squares and quazilinearization is given an iterative scheme for the construction of asymptotic solutions in the first approach relatively to the small parameter.

The results are illustrated on the example of the oil extraction by gaslift method for determination of CHR, where the small parameter $\varepsilon$ is accepted inverse to the value of depth of the well. Also for the coefficients of asymptotic expressions on $\varepsilon$ the analytical formulas are given, where the linear differential approximation is changed by the discrete approximation using ordinary Euler method. The given numerical results for the values of CHR, which are differs on order of $10^{-2}$ from the [1, 3, 9] can be used as a good initial approach for iterative schemes [3].

2. Problem Statement

Let a nonlinear ordinary differential equation

$$\dot{y}(x) = f(y(x), \alpha, \varepsilon)$$

and some sets of initial and final values of the $n$-dimensional phase vector $y(x)$ are given

$$y_k(0) = y_{0k}, \quad y_k(l) = y_{lk}, \quad k = 0, ..., N - 1.$$  \(2\)

Here $\alpha$ is a constant unknown vector, $\varepsilon$ is a small parameter, $f$ is the $n$-dimensional function is differentiable on $y$, $\alpha$, $\varepsilon$. It is required to find such vector-parameter $\alpha = \alpha'$, that the end value $\dot{y}'(l, \alpha', \varepsilon)(the$ solution of equation (1) with an initial value $y_{0k}$ exactly enough coincided with $y_{lk}$ $(k = 0, 1, ..., N - 1)$. Such problem further we will call the identification of the dynamic systems (1).

Let the initial approach $\dot{y}'(x), \alpha'$ be given. Using the method of quazilinearization [3, 17] we present the equation (1) in the linear form relatively $\dot{y}'(x), \alpha'$ and $\varepsilon$, in the following form

$$\dot{y}'(x) = (A_0(y^{-1}(x), \alpha^{-1}) + \varepsilon A_1(y^{-1}(x), \alpha^{-1})) \dot{y}'(x) +$$

$$+ (B_0(y^{-1}(x), \alpha^{-1}) + \varepsilon B_1(y^{-1}(x), \alpha^{-1})) \alpha' +$$

$$+ (C_0(y^{-1}(x), \alpha^{-1}) + \varepsilon C_1(y^{-1}(x), \alpha^{-1})),$$  \(3\)

where $A_i(y^{-1}(x), \alpha^{-1}), B_i(y^{-1}(x), \alpha^{-1}), C_i(y^{-1}(x), \alpha^{-1})$, are the results of decomposition of Taylor in the

\(^1\)In the spatial case [2] the motion of the equation describing by the hyperbolic equations the problem is reduced the solution of the corresponding problem, where the discrete model is considered.
first approach and determined in the following form

\[ A_0 \left( y^{i-1} (x), \alpha^{i-1} \right) = f_y \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right), \quad A_1 \left( y^{i-1} (x), \alpha^{i-1} \right) = f_{y^x} \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right), \]
\[ B_0 \left( y^{i-1} (x), \alpha^{i-1} \right) = f_y \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right), \quad B_1 \left( y^{i-1} (x), \alpha^{i-1} \right) = f_{y^x} \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right), \]
\[ C_0 \left( y^{i-1} (x), \alpha^{i-1} \right) = f \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) - f_y \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) y^{i-1}_0 - f_a \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) \alpha^{i-1}_0, \]
\[ C_1 \left( y^{i-1} (x), \alpha^{i-1} \right) = f_c \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) - f_{y^x} \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) y^{i-1}_0 - f_{ax} \left( y^{i-1}_0, \alpha^{i-1}_0, 0 \right) \alpha^{i-1}_0. \]

(4)

Now we present the solution of linear equation [19] in the following form

\[ y'(t) = \left( \Phi_0^{0j-1} (t) + \epsilon \Phi_1^{0j-1} \right) y'(0) + \left( \Phi_0^{0j} (t) + \epsilon \Phi_1^{0j} \right) \alpha^0 + \left( \Phi_2^{0j-1} (t) + \epsilon \Phi_2^{0j} \right), \]

(5)

where \( \Phi_0^{0j-1} (t, 0), \Phi_0^{0j} (t, 0) \) are determined from the next linear differential equations

\[ \Phi_0^{0j-1} (t, 0) = A_0 \left( y^{i-1} (x), \alpha^{i-1} \right) \Phi_0^{0j-1} (t, 0), \quad \Phi_0^{0j-1} (0, 0) = E, \]
\[ \Phi_0^{0j} (t, 0) = A_0 \left( y^{i-1} (x), \alpha^{i-1} \right) \Phi_0^{0j} (t, 0) + A_1 \left( y^{i-1} (x), \alpha^{i-1} \right) \Phi_0^{0j-1} (t, 0), \]
\[ \Phi_0^{0j-1} (0, 0) = 0 \]

(6)

and

\[ \Phi_0^{0j} (t, 0) = \Phi_0^{0j-1} (t, 0) + \epsilon \Phi_0^{0j} (t, 0) \]

is the fundamental matrix of the homogeneous equation (5) in the first approach relatively small parameter \( \epsilon \), and \( \Phi_0^{0j-1} (t, 0), \Phi_0^{0j} (t, 0), \quad (n = 0, 1) \) from (5) are determined in the following form [5]

\[ \Phi_0^{0j-1} (t, 0) = \int_0^t \Phi_0^{0j-1} (\tau, 0) B_0 (y^{i-1}, \alpha^{i-1}) d \tau, \]
\[ \Phi_0^{0j-1} (t, 0) = \int_0^t \left( \Phi_0^{0j-1} (\tau, 0) B_1 (y^{i-1}, \alpha^{i-1}) + \Phi_0^{0j-1} (\tau, 0) B_0 (y^{i-1}, \alpha^{i-1}) \right) d \tau, \]
\[ \Phi_0^{0j} (t, 0) = \int_0^t \Phi_0^{0j-1} (\tau, 0) C_0 (y^{i-1}, \alpha^{i-1}) d \tau, \]
\[ \Phi_0^{0j} (t, 0) = \int_0^t \left( \Phi_0^{0j-1} (\tau, 0) C_1 (y^{i-1}, \alpha^{i-1}) + \Phi_0^{0j-1} (\tau, 0) C_0 (y^{i-1}, \alpha^{i-1}) \right) d \tau. \]

(7)

Further for the solutions (5) in the point \( x = l \) and for the final values from (2) we construct the functional

\[ I = \sum_{k=0}^{N-1} \left( y_k^f (l) - y_k^f \right) A \left( y_k^f (l) - y_k^f \right), \]

(8)

where \( A \) is the well-known weight matrix, \( y_k^f (l) \) is the solution of the equation (5) at statistical data \( y_k (0) = y_{0k} \) from (2). Thus, if we can choose such \( \alpha^i \), that the functional (8) get the minimum value, then we in fact provide the closeness of the solution \( y(x, \alpha, \epsilon) \) in the point \( x = l \) with \( y_k^f (l) = y_k^f \) from (2).
3. Calculation of the Gradient of the Functional (8)

To obtain the formula for the gradient of the functional, we first put $y_i^j(l)$ from (5) into (8):

$$
F = \sum_{k=0}^{N-1} \left[ y_i^j(\Phi_{0j}^{(1)} + \epsilon\Phi_{1j}^{(1)}) + \alpha''(\Phi_{2j}^{(1)} + \epsilon\Phi_{2j}^{(1)}) - y_i^j \right] \Delta x
$$

$$
\times \left[ \left( \Phi_{0j}^{(1)} + \epsilon\Phi_{1j}^{(1)} \right) y_i^j + \left( \Phi_{1j}^{(1)} + \epsilon\Phi_{1j}^{(1)} \right) \alpha' + \left( \Phi_{2j}^{(1)} + \epsilon\Phi_{2j}^{(1)} \right) - y_i^j \right] =
$$

$$
= \sum_{k=0}^{N-1} \left[ \left( y_i^j, \Phi_{0j}^{(1)} A_2^{(1)} + y_i^j, \Phi_{0j}^{(1)} A_1^{(1)} \alpha' + y_i^j, \Phi_{0j}^{(1)} A_2^{(1)} \right) - y_i^j \left( \Phi_{0j}^{(1)} + \epsilon\Phi_{1j}^{(1)} \right) y_i^j + \left( \Phi_{2j}^{(1)} + \epsilon\Phi_{2j}^{(1)} \right) - y_i^j \right] =
$$

$$
- y_i^j \Phi_{0j}^{(1)} A_2^{(1)} + y_i^j \Phi_{1j}^{(1)} A_2^{(1)} + \alpha'' \Phi_{2j}^{(1)} A_2^{(1)} - y_i^j \Phi_{2j}^{(1)} A_2^{(1)} + \epsilon \left[ y_i^j \Phi_{0j}^{(1)} A_2^{(1)} y_i^j + y_i^j \Phi_{1j}^{(1)} A_2^{(1)} y_i^j + \alpha'' \Phi_{2j}^{(1)} A_2^{(1)} y_i^j - y_i^j \Phi_{2j}^{(1)} A_2^{(1)} y_i^j \right]
$$

Now we take the derivative on an unknown constant vector $\alpha'$ from (9) where $I''_{\alpha'}$ is determined in the following form:

$$
I''_{\alpha'} = \sum_{k=0}^{N-1} \left( \Phi_{0j}^{(1)} A_0^{(1)} y_i^j - \Phi_{0j}^{(1)} A_1^{(1)} \alpha' + \Phi_{0j}^{(1)} A_2^{(1)} \alpha' - \Phi_{0j}^{(1)} A_0^{(1)} y_i^j + 2 \Phi_{0j}^{(1)} A_0^{(1)} y_i^j \right) +
$$

$$
+ \left( \Phi_{0j}^{(1)} A_0^{(1)} y_i^j + \alpha'' \Phi_{1j}^{(1)} A_2^{(1)} y_i^j + 2 \Phi_{0j}^{(1)} A_2^{(1)} y_i^j + \Phi_{0j}^{(1)} A_2^{(1)} \alpha' \right) +
$$

$$
+ \left( \Phi_{0j}^{(1)} A_0^{(1)} y_i^j + \Phi_{0j}^{(1)} A_1^{(1)} \Phi_{2j}^{(1)} y_i^j + \Phi_{0j}^{(1)} A_2^{(1)} y_i^j \right) +
$$

$$
- \Phi_{0j}^{(1)} A_2^{(1)} \alpha' + \Phi_{0j}^{(1)} A_0^{(1)} \Phi_{2j}^{(1)} y_i^j \right] \alpha' +
$$

$$
+ \left( \Phi_{0j}^{(1)} A_0^{(1)} + \epsilon \Phi_{1j}^{(1)} A_2^{(1)} y_i^j \right) \alpha' \right].
$$
We equate to zero the first derivative $l_{\alpha}^r$ and search $\alpha^i$ in the form

\[
y^0(t) = 1.
\]

For determination $\alpha_0^i$ and $\alpha_1^i$ we have the next algebraic equations:

\[
\sum_{k=0}^{N-1} \left( \Phi_0^{0,1-j} A \Phi_1^{0,1-j} y_0^i - \Phi_1^{0,1-j} A y_0^i + 2 \Phi_1^{0,1-j} A \Phi_0^{0,1-j} \alpha_0^i \right) = 0,
\]

\[
\sum_{k=0}^{N-1} \left( \Phi_0^{0,1-j} A \Phi_1^{1,1-j} y_0^i + \Phi_1^{1,1-j} A \Phi_0^{0,1-j} y_0^i + \Phi_1^{1,1-j} A \Phi_2^{0,1-j} - \Phi_1^{1,1-j} A y_0^i + 2 \Phi_1^{1,1-j} A \Phi_1^{0,1-j} \alpha_1^i \right) = 0.
\]

After solving the equations (12) relatively to $\alpha_1^i$, correspondingly, we have:

\[
\alpha_0^i = -\frac{1}{2} \sum_{k=0}^{N-1} \left( \Phi_1^{0,1-j} A \Phi_1^{0,1-j} - \Phi_1^{0,1-j} A y_0^i + 2 \Phi_1^{0,1-j} A \Phi_2^{0,1-j} \right),
\]

\[
\alpha_1^i = -\frac{1}{2} \sum_{k=0}^{N-1} \left( \Phi_1^{1,1-j} A \Phi_1^{1,1-j} - \Phi_1^{1,1-j} A y_0^i + 2 \Phi_1^{1,1-j} A \Phi_2^{1,1-j} \right)
\times \left( \Phi_1^{0,1-j} A \Phi_1^{0,1-j} - \Phi_1^{0,1-j} A y_0^i + 2 \Phi_1^{0,1-j} A \Phi_2^{0,1-j} \right).
\]

Thus calculating $\alpha$ in the first approach on $\xi$ we obtain: $\alpha^i = \alpha_0^i + \varepsilon \alpha_1^i$.

Summarizing the above it is possible to present the computational algorithm [12] for solving of the problem of identification (1), (2), (8)

**Algorithm 1.**

1. The nonlinear function $\delta_2$ from (1), the initial $y_0(0)$ and final $y_k(l)$, the given weight matrices $A_i$, the initial approaches $y^r(x), \alpha^r$ and small number $\alpha^r$ are forming.
2. $f_y(y_0, a_0, 0), f_y(y_0, a_0, 0), f_y(y_0, a_0, 0), f_y(y_0, a_0, 0), f_y(y_0, a_0, 0), f_y(y_0, a_0, 0)$ are calculating.
3. $A_0(y^r(x), \alpha^r), A_1(y^r(x), \alpha^r), B_0(y^r(x), \alpha^r), B_1(y^r(x), \alpha^r), C_0(y^r(x), \alpha^r), \alpha_0^r$ are forming from (3).
4. The fundamental matrices $\Phi_0^{0,1-j}(t, 0), y_0$ are calculating [19] at the initial $\Phi_0^{0,1-j}(0, 0) = E, \alpha_0^r$ from (6).
5. By using of the fundamental matrices $\Delta, \alpha^r = \alpha_0^r + \varepsilon \alpha_1^r$, $y_{-1}, \Phi_1^{0,1-j}(t, 0)$ are calculating the integrals.
6. $\Delta^r$ and $\alpha^r$ are calculating from (11).
7. $\Phi_1^{1,1-j}(t, 0)$ we accept $y^r(x)$ as the initial iteration. If it is satisfied, we go to the step 2. Otherwise the calculation process is stopped.

\[\text{[15]}\]
4. Ordinary Algorithm of Euler for Solving (1) and the Approximate Formulas for $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{1}^{\, j-1}(t, 0)$, $\Phi_{2}^{\, j-1}(t, 0)$, $\Phi_{1}^{\, i-1}(t, 0)$, $\Phi_{1}^{\, j-1}(t, 0)$, $\Phi_{1}^{\, j-1}(t, 0)$, $\Phi_{2}^{\, j-1}(t, 0)$, $\Phi_{2}^{\, j-1}(t, 0)$, $\Phi_{2}^{\, j-1}(t, 0)$

We note that at calculating $\alpha^i$ over the algorithm 1 one of the difficulties is the procedure of finding of $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$and $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$, $\Phi_{0}^{\, j-1}(t, 0)$ However by help of approximate methods it is possible to restore them.

Now discretizing on the step $\Delta$ the equation (3) on the first Euler method [8, 20], we obtain:

\[
y_{i+1} = \left( E + \Delta (A_0 (y_{i-1}, \alpha^{i-1}) + \varepsilon A_1 (y_{i-1}, \alpha^{i-1})) \right) y_i + \\
+ \Delta \left( B_0 (y_{i-1}, \alpha^{i-1}) + \varepsilon B_1 (y_{i-1}, \alpha^{i-1}) \right) a_i + \Delta \left( C_0 (y_{i-1}, \alpha^{i-1}) + \varepsilon C_1 (y_{i-1}, \alpha^{i-1}) \right). \tag{15}
\]

It is now possible to express $y_i$ through an initial condition $y_0$. First we show these expressions for $y_1, y_2$

\[
y_1 = ((E + \Delta A_0) + \Delta A \varepsilon) y_0 + (B_0 + \varepsilon B_1) a^1 + (C_0 + \varepsilon C_1),
\]

\[
y_2 = ((E + \Delta A_0)^2 + 2 \varepsilon \Delta (E + \Delta A_0) A_1) y_0 + \\
+ [(E + \Delta A_0) B_0 + B_1 + \varepsilon ((E + \Delta A_0) B_1 + B_1 + \Delta A_1 B_0)] a^1 + \\
+ [(E + \Delta A_0)] C_0 + C_0 + \varepsilon ((E + \Delta A_0) + 2 \varepsilon \Delta (E + \Delta A_0) C_1 + C_1 + \Delta A_1 C_0] \tag{16}.
\]

Let for $y_{i-1}$ be true the relations

\[
y_{i-1} = \left( (E + \Delta A_0) N^{-1} + N \Delta (E + \Delta A_0) N^{-2} \varepsilon \right) y_0 + \\
+ \left( (E + \Delta A_0) N^{-2} B_0 + \varepsilon ((E + \Delta A_0) N^{-2} B_1 + \\
+ N (E + \Delta A_0) N^{-2} \Delta A_1 B_0 + \varepsilon N (E + \Delta A_0) N^{-2} \Delta A_1 C_0 + \\
+ \varepsilon (E + \Delta A_0) N^{-1} C_1 + N (E + \Delta A_0) N^{-1} \Delta A_1 C_0 \right). 
\]

By mathematical induction we will prove easily, that

\[
y_i = \left( (E + \Delta A_0) N + N \Delta (E + \Delta A_0) N^{-1} \varepsilon \right) y_0 + \\
+ \left( (E + \Delta A_0) N^{-1} B_0 + \varepsilon ((E + \Delta A_0) N^{-1} B_1 + \\
+ N (E + \Delta A_0) N^{-1} \Delta A_1 B_0 + \varepsilon N (E + \Delta A_0) N^{-1} C_0 + \\
+ \varepsilon (E + \Delta A_0) N^{-1} C_1 + N (E + \Delta A_0) N^{-1} \Delta A_1 C_0 \right). \tag{16}
\]

Now from (16) we define the approximate formulas for $\Phi_j(0, l)$.

Note that the finding of $\alpha_0^i, \alpha_1^i$ from (14) makes difficulty from calculations of fundamental matrices $\Phi_0^j(0, l), \Phi_1^j(0, l), \Phi_2^j(0, l)$($n = 0, 1$) from the system of linear differential equations (5). Therefore in the next point by means of ordinary method of Euler we discretize the equation (5) and restore approximately the fundamental matrices $\Phi_0^1(0, l), (n = 0, 1)(i = 0, 1, 2)$ in the first approach. As is shown from (13) - (14) for renewal of $\alpha_0^i, \alpha_1^i$ it is necessary to take into account the higher $\Phi_j(0, l)$. The relation (16) allows to find them. Therefore comparing (16) with (5) we have:

\[
\begin{align*}
\Phi_0^{\, j-1} &= (E + \Delta A_0) N, \\
\Phi_1^{\, j-1} &= N \Delta (E + \Delta A_0) N^{-1}, \\
\Phi_2^{\, j-1} &= (E + \Delta A_0) N^{-1} B_0, \\
\Phi_1^{\, j-1} &= (E + \Delta A_0) N^{-1} B_1 + N (E + \Delta A_0) N^{-1} \Delta A_1 B_0, \\
\Phi_2^{\, j-1} &= (E + \Delta A_0) N^{-1} C_0, \\
\Phi_1^{\, j-1} &= (E + \Delta A_0) N^{-1} C_1 + N (E + \Delta A_0) N^{-1} \Delta A_1 C_0. 
\end{align*} \tag{17}
\]

Thus it is simpler to calculate $\alpha_0^i, \alpha_1^i$ from (13), (14) by means of approximate formulas.
We note that with the help of (17) we can easily find for \( \alpha_{i-1} \), \( \alpha_{i} \) from (13) and (14) the next expressions in an obvious form

\[
\alpha_{i} = -\frac{1}{2} \sum_{k=0}^{N-1} \left( B_{i}^{k} (E + \Delta A_{0})^{N-1} A (E + \Delta A_{0})^{N-1} B_{i} \right)^{-1} \times \times \left( y_{i}^{0} B_{i}^{0} (E + \Delta A_{0})^{N-1} A (E + \Delta A_{0})^{N-1} B_{i} + B_{i}^{0} (E + \Delta A_{0})^{N-1} + \right.
\]

\[
+ B_{i}^{1} \Delta A \left( E + \Delta A_{0} \right)^{N-1} \Delta A_{1} C_{0} - B_{i}^{1} (E + \Delta A_{0})^{N-1} + \right.
\]

\[
+ B_{i}^{1} \Delta A \left( E + \Delta A_{0} \right)^{N-1} \Delta A_{1} C_{0} - B_{i}^{1} (E + \Delta A_{0})^{N-1} + \left. \right) \times \left( y_{i}^{0} (E + \Delta A_{0})^{N-1} A (E + \Delta A_{0})^{N-1} B_{i} - B_{i}^{0} (E + \Delta A_{0})^{N-1} A y_{i}^{0} + \right.
\]

\[
+ \left. y_{i}^{0} (E + \Delta A_{0})^{N-1} A \left( (E + \Delta A_{0})^{N-1} B_{i} + N (E + \Delta A_{0})^{N-1} \Delta A_{1} B_{i} \right) \right) .
\]

We note that formulas (18) and (19) allow approximately to find \( \alpha_{i} \) from (2) and (3) \((i = 0, 1, ..., N)\) are calculating from (4).

Algorithm 2.

1. The function \( f(y, x, \alpha) \) and the statistical data \( y_{i}^{0}, y_{i}^{1} \) from (2) and (3) \((i = 0, 1, ..., N)\) are given.
2. The derivatives \( f_{y}, f_{y} \) are calculating
3. \( A_{0}, A_{1}, B_{0}, B_{1}, C_{0}, C_{1} \) are calculating from (4).
4. \( \varphi_{0}^{0}, \varphi_{1}^{0}, \varphi_{0}^{1}, \varphi_{1}^{1}, \varphi_{0}^{1}, \varphi_{1}^{1} \) are calculating from (7) \((or \ (17))\).
5. \( \alpha_{i}^{0} \) and \( \alpha_{i}^{1} \) are calculating by the formulas (13) and (14), correspondingly \((or \ (18), (19))\).
6. \( \alpha^{1} \approx \alpha_{i}^{0} + \epsilon \alpha_{i}^{1} \) are calculating
7. The condition \( l_{i} > l_{i+1} \) is checked up. If the condition is not satisfied, we go to the step 2. Otherwise the calculation process is stopped.

We note that formulas (18) and (19) allow approximately to find \( \alpha \).
We consider the following example.

5. Example

Let us consider the gas lift process for the oil production where the motion equation is described by the following nonlinear ordinary differential equation \([3, 6, 8, 10]\)

\[
Q = \frac{2a(\lambda_{c})pFQ^{2}}{\epsilon_{2}2^{2}p^{2}F^{2} - Q^{2}}, \quad Q(0) = u,
\]

where \( c >> \omega_{s} \), except \( Q = \rho Q, F \) all values are constant, \( F \) is the cross-sectional area of pump-compressor pipes, that is constant relatively to axes.
Here it is assumed that the transition from the end of ring pipe through the layer to beginning of the lift \((x = l)\) is executed on the following difference equation:

\[
\begin{align*}
Q(l + 0) &= γ Q(l - 0) + γ_1(Q(l - 0))Q,
γ(Q(l - 0)) &= -β3(Q(l - 0) - β2)^2 + β1,
\end{align*}
\]

where \(γ\), \(β1\), \(β2\), \(β3\) are constant real numbers, which are subject to determination.

For simplicity we suppose that parameters \(γ\), \(β1\), \(β2\), \(β3\) are known and it is required to find the CHR \(λ\), included to (20) through \(α(λ)\).

Further some nominal trajectory \(Q^0(x)\) and parameter \(α^0\) are searching supposing that \(k\)th iteration is already done. Linearizing the equation (20) near these data we have

\[
Q^k(x) = A \left( Q^{k-1}, α^{k-1} \right) Q^k(x) + B \left( Q^{k-1}, α^{k-1} \right) α^k + C \left( Q^{k-1}, α^{k-1} \right),
\]

where

\[
\begin{align*}
A_0 \left( y^{-1}(x), α^{-1} \right) &= 0, & A_1 \left( y^{-1}(x), α^{-1} \right) &= 4α^{-1} c^2 ρ^3 F^2 Q^{-1},
B_0 \left( y^{-1}(x), α^{-1} \right) &= -2ρF, & B_1 \left( y^{-1}(x), α^{-1} \right) &= 2ρ^3 F^2 c^2
C_0 \left( y^{-1}(x), α^{-1} \right) &= 2ρ Fa^{-1}, & C_1 \left( y^{-1}(x), α^{-1} \right) &= 2α^{-1} c^2 ρ^3 \left( 2(Q^{-1})^2 + 1 \right).
\end{align*}
\]

Note that by help of relations (17), (18) the matrices \(Φ_{1}^{k-1}(x, 0), Φ_{2}^{k-1}(x, 0)\) are calculating in the following form

\[
\begin{align*}
Φ_{1}^{k-1}(x, 0) &= \left( \sum_{i=N+2}^{2N-1} \prod_{j=2N-1}^{i-1} (E + A \left( Q^{k-1}(x_i), α^{k-1} \right)) h \right) B \left( Q^{k-1}(x_{i-1}), α^{k-1} \right) h +
+ B \left( Q^{k-1}(x_{2N-1}), α^{k-1} \right) h,
Φ_{2}^{k-1}(x, 0) &= \left( \sum_{i=N+2}^{2N-1} \prod_{j=2N-1}^{i-1} (E + A \left( Q^{k-1}(x_i), α^{k-1} \right)) h \right) C \left( Q^{k-1}(x_{i-1}), α^{k-1} \right) h +
+ C \left( Q^{k-1}(x_{2N-1}), α^{k-1} \right) h,
\end{align*}
\]

where \(h\) is an enough small number, which is the step of integration.

Let the statistical data, that are the results of measuring of debit \(\bar{Q}_{2m}\) on leaving with the given initial volume of gas are given, i.e. \(\bar{Q}_{0}, \bar{Q}_{2m}, i = 1, 5\) are known.

<table>
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<tr>
<th>(y^{i0})</th>
<th>5.5698</th>
<th>5.5732</th>
<th>5.5761</th>
<th>5.5810</th>
<th>5.5848</th>
<th>5.5852</th>
<th>5.5824</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y^{i})</td>
<td>4.4242</td>
<td>4.4248</td>
<td>4.4254</td>
<td>4.4262</td>
<td>4.4266</td>
<td>4.4263</td>
<td>4.4251</td>
</tr>
</tbody>
</table>

Then the functional from (8) has the following form:

\[
I = \frac{1}{2} \sum_{i=1}^{5} \left| Q^{i}_{0} - Q^{i}_{1} \right|^2,
\]

where \(Q^{i}_{1}\) is the solution of the equation (3) for the initial conditions \(Q^{i}_{0}\). Let the parameters of the equation (20) look like as:

\[
\begin{align*}
\text{at } 0 ≤ x < l = 0: & \quad l = 1485 m, s = 331 m/s, \quad ρ = \frac{0.717 k_{D}}{m}, \quad d = \sqrt{1142^2 - 732^2} 10^{-3} m, \quad λ = 0.01,
\text{at } l + 0 < x ≤ 2l: & \quad s = 850 m/s, \quad ρ = \frac{700 k_{D}}{m}, \quad d = 0.073 m, \quad λ = 0.23.
\end{align*}
\]
Now we pass to implementation of the above algorithm.

The initial value of CHR $\lambda^0$ we accept equal to 1. Accepting $y^0(t) = 1$ and repeating the procedure 1-5 from the algorithm 2 we determine the values of $\lambda, y^1(t)$. After 44 iterations the following result was obtained:

$$\lambda_c \approx 0.29834,$$

that coincides with $\lambda$ from (23) within $10^{-2}$.

Note that such approach can be satisfactory to finding of initial iterations of ordinary gradient method, for finding of CHR [3], linearizing [1] and other.

References