Sufficient Conditions for Carathéodory Functions

Mamoru Nunokawa, Oh Sang Kwon, Young Jae Sim, Nak Eun Cho

Abstract. In the present paper, we obtain several sufficient conditions for Carathéodory functions in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). We also obtain sufficient conditions for \( p \)-valent or starlike functions. Moreover, we improve some results due to Nunokawa [Tsukuba J. Math. 13 (1989), 453–455] as some special cases of main results.

1. Introduction

Let \( \mathcal{A}(p) \) denote the class of functions \( f \) of the form
\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,
\]
which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{A} \equiv \mathcal{A}(1) \). A function \( f \in \mathcal{A}(p) \) is called \( p \)-valent in \( U \) if \( f \) satisfies the following two conditions:

(i) for \( w \in \mathbb{C} \), the equation \( f(z) = w \) has at most \( p \) roots in \( U \);

(ii) there exists a \( w_0 \in \mathbb{C} \) such that the equation \( f(z) = w_0 \) has exactly \( p \) roots in \( U \).

A function \( f \in \mathcal{A}(p) \) is said to be \( p \)-valent starlike if
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U).
\]
If a function \( f \in \mathcal{A} \) is 1-valent starlike, then it is called starlike. It is known that that \( p \)-valent starlike function in \( \mathcal{A}(p) \) is \( p \)-valent.

Let \( \mathcal{P} \) be the class of functions \( p \) which are analytic in the unit disk \( U \), with \( p(0) = 1 \) and \( \Re \{ p(z) \} > 0 \) in \( U \). If \( p \in \mathcal{P} \), then we say that \( p \) is a Carathéodory function. It is well-known that if \( f \in \mathcal{A} \) with \( f' \in \mathcal{P} \), then

2010 Mathematics Subject Classification. Primary 30C45

Keywords. Starlike function, \( p \)-valent functions, Carathéodory functions

Received: 28 December 2016; Revised: 15 February 2017; Accepted: 22 March 2017

Communicated by Allaberen Ashyralyev

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450)

Email addresses: mamoru.nunod@doctor.nifty.jp (Mamoru Nunokawa), oskwon@ks.ac.kr (Oh Sang Kwon), yjsim@ks.ac.kr (Young Jae Sim), necho@pknu.ac.kr (Nak Eun Cho)
the function \( f \) is univalent in \( U \) (cf. [1, 10]). In 1935, Ozaki [9] extended the above result as follows: if \( f \) is analytic in a convex domain \( D \) and
\[
\Re \left\{ \exp(i\alpha) f^{(p)}(z) \right\} > 0 \quad (z \in D),
\]
where \( \alpha \) is a real constant, then \( f \) is at most \( p \)-valent in \( D \). This shows that if \( f \in A(p) \) with
\[
\Re \left\{ f^{(p)}(z) \right\} > 0 \quad (z \in U),
\]
then \( f \) is at most \( p \)-valent in \( U \). Nunokawa [3] (see also [4]) improved the above result to the following.

**Theorem A** ([3, Nunokawa]) Let \( p \geq 2 \). If \( f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \) is analytic in \( U \) and
\[
\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \frac{3}{4} \pi \quad (z \in U),
\]
then \( f \) is \( p \)-valent in \( U \).

Recently, Nunokawa et al. [6] found some sufficient conditions for function to be \( p \)-valent by improving Ozaki’s condition given by (1). Also, in [7] and [8], Nunokawa and Sokół obtained another \( p \)-valent conditions by using geometric properties of functions in \( A(p) \).

The purpose of the present paper is to investigate some sufficient conditions for Carathéodory functions and to find some conditions for \( p \)-valent functions or starlike functions. And we improve Theorem A obtained by Nunokawa [3].

The following lemmas will be required for our results.

**Lemma 1.1.** ([5, Nunokawa]) Let \( p \) be analytic in \( U \), \( p(z) \neq 0 \) in \( U \), \( p(0) = 1 \) and suppose that there exists a \( z_0 \in U \) such that
\[
\left| \arg p(z) \right| < \frac{\pi}{2} \alpha \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\left| \arg p(z_0) \right| = \frac{\pi}{2} \alpha, \quad \alpha > 0.
\]
Then
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i k \alpha,
\]
where
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right), \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \alpha
\]
and
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right), \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \alpha,
\]
with
\[
p(z_0)^{1/\alpha} = \pm i a.
\]

**Lemma 1.2.** ([2, Nunokawa]) Let \( f \in A(p) \). If there exists a \( (p - k + 1) \)-valent starlike function \( g(z) = z^{p-k+1} + \sum_{n=p-k+2}^{\infty} b_n z^n \) that satisfies
\[
\Re \left\{ \frac{zf^{(k)}(z)}{g(z)} \right\} > 0 \quad (z \in U),
\]
then \( f \) is \( p \)-valent in \( U \).
2. Main Results

**Theorem 2.1.** Let \( p \) be analytic in \( U \), \( p(z) \neq 0 \) in \( U \), \( p(0) = 1 \) and suppose that
\[
\left| \arg \{ p(z) + zp'(z) - \alpha \} \right| < \frac{\pi}{2} + \arctan(\sqrt{1 + 2\alpha}) \quad (z \in U),
\]
where \( 0 \leq \alpha < 1 \). Then, we have
\[
\left| \arg \{ p(z) \} \right| < \frac{\pi}{2} \quad (z \in U),
\]
or
\[
\Re \{ p(z) \} > 0 \quad (z \in U).
\]

**Proof.** If there exists a point \( z_0 \) (\( |z_0| < 1 \)) such that
\[
\left| \arg \{ p(z_0) \} \right| < \frac{\pi}{2} \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\left| \arg \{ p(z_0) \} \right| = \frac{\pi}{2},
\]
then, by Lemma 1.1 with \( \alpha = 1 \), we have
\[
\frac{z_0p'(z_0)}{p(z_0)} = i k.
\]
For the case \( \arg \{ p(z_0) \} = \pi/2 \), \( p(z_0) = ia \) and \( a > 0 \), we have
\[
\arg \{ p(z_0) + z_0p'(z_0) - \alpha \}
\]
\[
= \arg \{ p(z_0) \} + \arg \left\{ \frac{z_0p'(z_0)}{p(z_0)} - \frac{\alpha}{p(z_0)} \right\}
\]
\[
= \frac{\pi}{2} + \arg \left\{ 1 + ik + i\frac{\alpha}{a} \right\}
\]
\[
\geq \frac{\pi}{2} + \arg \left\{ 1 + i \left( a + \frac{1 + 2\alpha}{a} \right) \right\}
\]
\[
\geq \frac{\pi}{2} + \arctan(\sqrt{1 + 2\alpha}),
\]
which contradicts the hypothesis (2).

For the case \( \arg \{ p(z_0) \} = -\pi/2 \), applying the same method as the above, we have
\[
\arg \{ p(z_0) + z_0p'(z_0) - \alpha \} \leq -\left( \frac{\pi}{2} + \arctan(\sqrt{1 + 2\alpha}) \right).
\]
This also contradicts the hypothesis (2) and therefore, it completes the proof of Theorem 2.1.

**Example 2.2.** Consider a function \( p_1 : U \to \mathbb{C} \) defined by
\[
p_1(z) = -\frac{1}{z} \log(1 - z) = \sum_{n=0}^{\infty} \frac{z^n}{n + 1}.
\]

Then we have
\[
p_1(z) + zp'_1(z) = -\frac{1}{z} = \frac{1 + z}{2(1 - z)}.
\]
Hence \( p_1 \) satisfies the condition (2) with \( \alpha = 1/2 \). Therefore, by Theorem 2.1, we have \( \Re \{ p_1(z) \} > 0 \) in \( U \).

Actually, the function \( p_1 \) satisfies that \( \Re \{ p_1(z) \} > \log 2 = 0.693147 \cdots \) in \( U \) (See Figure 1 below).
Applying Theorem 2.1, we have the following corollary.

**Corollary 2.3.** Let \( p \geq 2 \). If \( f \in A(p) \) satisfies \( f^{(p-1)} \neq 0 \) in \( U \) and

\[
\left| \arg \left( f^{(p)}(z) - \alpha \cdot p! \right) \right| < \frac{\pi}{2} + \arctan(\sqrt{1 + 2\alpha}) \quad (z \in U),
\]

where \( 0 \leq \alpha < 1 \), then \( f \) is \( p \)-valent in \( U \).

**Proof.** Let us put

\[
p(z) = \frac{f^{(p-1)}(z)}{p!z}, \quad p(0) = 1.
\]

Then it follows that

\[
\left| \arg \left( p(z) + zp'(z) - \alpha \right) \right| = \left| \arg \left( \frac{f^{(p)}(z)}{p!} - \alpha \right) \right| = \left| \arg \left( f^{(p)}(z) - \alpha \cdot p! \right) \right| < \frac{\pi}{2} + \arctan(\sqrt{1 + 2\alpha}).
\]

From Theorem 2.1, we have \( \Re \{p(z)\} > 0 \) in \( U \), or equivalently,

\[
\Re \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > 0 \quad (z \in U).
\]

This shows that \( f \) is \( p \)-valent in \( U \). \( \Box \)

**Example 2.4.** Consider a function \( f_{1} : U \to \mathbb{C} \) defined by

\[
f_{1}(z) = 2[z + (1 - z) \log(1 - z)] = z^2 + \frac{1}{3}z^3 + \frac{1}{6}z^4 + \frac{1}{10}z^5 + \cdots.
\]
Then, we have
\[ \left| \arg \left( f_1''(z) - 1 \right) \right| = \left| \arg \left( p_1(z) + z p_1'(z) - \frac{1}{2} \right) \right| < \frac{\pi}{2}, \]

where \( p_1 \) is the function defined by (3). Therefore, by Corollary 2.3 with \( p = 2 \) and \( \alpha = 1/2 \), the function \( f_1 \) is 2-valent in \( U \).

**Remark 2.5.** For the case \( \alpha = 0 \) in Corollary 2.3, we have Theorem A as aforementioned.

**Theorem 2.6.** Let \( p \) be analytic in \( U \), \( p(0) = 1 \), \( p(z) \neq 0 \) in \( U \) and suppose that
\[ \left| \arg \left( p(z) + \frac{z p'(z)}{p(z)} + \alpha \right) \right| < \frac{\pi}{2} - \arctan \left( \frac{\alpha}{\sqrt{3}} \right) \quad (z \in \mathbb{U}), \]

where \( 0 \leq \alpha < \infty \). Then we have
\[ \left| \arg (p(z)) \right| < \frac{\pi}{2} \quad (z \in \mathbb{U}). \]

**Proof.** If there exists a point \( z_0 \) (\( |z_0| < 1 \)) such that
\[ \left| \arg (p(z_0)) \right| < \frac{\pi}{2} \quad \text{for} \quad |z| < |z_0| \]

and
\[ \left| \arg (p(z_0)) \right| = \frac{\pi}{2}, \]
then, by Lemma 1.1 with \( \alpha = 1 \), we have
\[ \frac{z_0 p'(z_0)}{p(z_0)} = i k. \]

For the case \( \arg (p(z_0)) = \pi/2 \), \( p(z_0) = i a \) and \( a > 0 \), we have
\[
\arg \left( p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} + \alpha \right) \\
= \arg (p(z_0)) + \arg \left( 1 + \frac{z_0 p'(z_0)}{p(z_0) \alpha} \right) \\
= \frac{\pi}{2} + \arg \left( 1 + \frac{k}{a} - i \frac{\alpha}{a} \right) \\
= \frac{\pi}{2} - \arctan \left( \frac{\alpha}{a + k} \right) \\
\geq \frac{\pi}{2} - \arctan \left( \frac{\alpha}{\sqrt{3}} \right),
\]

which contradicts the hypothesis (4).

For the case \( \arg (p(z_0)) = -\pi/2 \), applying the same method as the above, we have
\[
\arg \left( p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} + \alpha \right) \leq - \left( \frac{\pi}{2} - \arctan \left( \frac{\alpha}{\sqrt{3}} \right) \right).
\]

This also contradicts the hypothesis (4) and therefore, it completes the proof of Theorem 2.6. □
Corollary 2.7. Let \( f \in \mathcal{A} \) and suppose that
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} + \alpha \right) \right| < \frac{\pi}{2} - \operatorname{arctan} \left( \frac{\alpha}{\sqrt{3}} \right) \quad (z \in \mathbb{U}),
\]
where \( 0 \leq \alpha < \infty \). Then \( f \) is starlike in \( \mathbb{U} \).

Proof. Let us put
\[
p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1.
\]
Then it follows that
\[
\left| \arg \left( p(z) + \frac{zp'(z)}{p(z)} + \alpha \right) \right| = \left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} + \alpha \right) \right| < \frac{\pi}{2} - \operatorname{arctan} \left( \frac{\alpha}{\sqrt{3}} \right).
\]

From Theorem 2.6, we have \( \Re \{p(z)\} > 0 \) in \( \mathbb{U} \) and
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}).
\]
This shows that \( f \) is starlike in \( \mathbb{U} \). \( \square \)

Theorem 2.8. Let \( p \) be analytic in \( \mathbb{U} \), \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( \mathbb{U} \) and suppose that
\[
\Re \left\{ \sqrt{p(z)} + zp'(z) \right\} > 0 \quad (z \in \mathbb{U}). \tag{5}
\]
Then we have
\[
\left| \arg \{p(z)\} \right| < \frac{\pi}{2} \alpha_1 \quad (z \in \mathbb{U}),
\]
where \( \alpha_1 \) is the positive root of the equation
\[
\alpha + \frac{2}{\pi} \arctan(\alpha) = 2 \tag{6}
\]
and \( 1.39 < \alpha_1 < 1.40 \).

Proof. If there exists a point \( z_0 \) \((|z_0| < 1)\) such that
\[
\left| \arg \{p(z)\} \right| < \frac{\pi}{2} \alpha_1 \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\left| \arg \{p(z_0)\} \right| = \frac{\pi}{2} \alpha_1,
\]
then, by Lemma 1.1 with \( a = \alpha_1 \), we have
\[
\frac{zp'(z_0)}{p(z_0)} = i\alpha_1 k.
\]
For the case $\arg p(z_0) = \pi\alpha_1/2$, we have

$$
\arg \left\{ \sqrt{p(z_0) + z_0p'(z_0)} \right\}
= \frac{1}{2} \left( \arg \{p(z_0)\} + \arg \left\{ 1 + \frac{z_0p'(z_0)}{p(z_0)} \right\} \right)
\geq \frac{1}{2} \left( \frac{\pi}{2}\alpha_1 + \arctan(\alpha_1) \right)
= \frac{\pi}{2},
$$
which implies that

$$\Re \left\{ \sqrt{p(z_0) + z_0p'(z_0)} \right\} \leq 0.$$

And this contradicts the hypothesis (5).

For the case $\arg p(z_0) = -\pi\alpha_1/2$, applying the same method as the above, we have

$$
\arg \left\{ \sqrt{p(z_0) + z_0p'(z_0)} \right\} \leq -\frac{1}{2}\pi,
$$
or

$$\Re \left\{ \sqrt{p(z_0) + z_0p'(z_0)} \right\} \leq 0.$$

This also contradicts the hypothesis (5) and therefore it completes the proof of Theorem 2.8. □

**Example 2.9.** Consider a function $p_2 : \mathbb{U} \to \mathbb{C}$ defined by

$$
p_2(z) = \frac{5 - z}{1 - z} + \frac{4}{z} \log(1 - z)
= 1 + 2z + \frac{8}{3}z^2 + 3z^3 + \frac{16}{5}z^4 + \frac{10}{3}z^5 + \cdots.
$$

A simple calculation leads us to the equation

$$p_2(z) +zp_2'(z) = \left( \frac{1 + z}{1 - z} \right)^2.$$

Therefore the function $p_2$ satisfy the inequality (5) and it follows from Theorem 2.8 that

$$\left| \arg \{p_2(z)\} \right| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}).$$

Let us put

$$f(\theta) := \Re \left\{ p_2 \left( e^{i\theta} \right) \right\}
= 3 + 2\cos \theta \log(2 - 2\cos \theta) - 4\sin \theta \arctan\left( \frac{\sin \theta}{1 - \cos \theta} \right) \quad (\theta \in (0, \pi))$$
and

$$g(\theta) := 3 \left\{ p_2 \left( e^{i\theta} \right) \right\}
= \frac{3\sin \theta}{1 - \cos \theta} - 2\sin \theta \log(2 - 2\cos \theta) - 4\cos \theta \arctan\left( \frac{\sin \theta}{1 - \cos \theta} \right) \quad (\theta \in (0, \pi)).$$

Then we have

$$\left| \arg \{p_2 \left( e^{i\theta} \right)\} \right| \leq \left| \arg \left\{ p_2 \left( e^{i\theta} \right) \right\} \right| < 2.022 \quad (\theta \in (0, \pi)).$$
where $\theta_0 (0.804 < \theta_0 < 0.805)$ is the root of the equation $g'(\theta) f(\theta) = f'(\theta) g(\theta)$ (see figure 2 above). Thus, this implies that

$$\left| \arg \{p_2(z)\} \right| < \frac{\pi}{2} \alpha_1 \quad (z \in \mathbb{U}).$$

Applying Theorem 2.8, we have the following corollary.

**Corollary 2.10.** Let $p \geq 4$. Let $f \in \mathcal{A}(p)$ satisfy $f^{(k)} \neq 0$ for $k = p - 1$, $p - 2$ and $p - 3$ in $\mathbb{U}$. If

$$\left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| < \pi \quad (z \in \mathbb{U}),$$

then $f$ is $p$-valent in $\mathbb{U}$.

**Proof.** Let us put

$$q_1(z) = \frac{f^{(p-1)}(z)}{p!z}, \quad q_1(0) = 1.$$

Then it follows that

$$q_1(z) + zq_1'(z) = \frac{f^{(p)}(z)}{p!}.$$

Applying Theorem 2.8, we have

$$\left| \arg \left\{ \frac{f^{(p-1)}}{z} \right\} \right| = \left| \arg \{q_1(z)\} \right| < \frac{\pi}{2} \alpha_1 \quad (z \in \mathbb{U}),$$

where $\alpha_1 (1.39 < \alpha_1 < 1.40)$ is the positive root of the equation given by (6).

Next, let us put

$$q_2(z) = \frac{2f^{(p-2)}}{p!z^2}, \quad q_2(0) = 1.$$
Then it follows that
\[ 2q_2(z) + zq'_2(z) = q_2(z) \left( 2 + \frac{zq'_2(z)}{q_2(z)} \right) = \frac{2f^{(p-1)}}{p!z}. \]

Let \( \alpha_2 \) be the positive root of the equation
\[ \alpha + \frac{2}{\pi} \arctan \left( \frac{\alpha}{2} \right) = \alpha_1 \]
and
\[ 1.08 < \alpha_2 < 1.09. \]

If there exists a point \( z_1, |z_1| < 1 \) such that
\[ |\arg(q_2(z))| < \frac{\pi}{2}\alpha_2 \quad \text{for} \quad |z| < |z_1| \]
and
\[ |\arg(q_2(z_1))| = \frac{\pi}{2} \alpha_2, \]
then we have
\[ \frac{z_1q'_2(z_1)}{q_2(z_1)} = ia_2k. \]

For the case \( \arg(q_2(z_1)) = \pi\alpha_2/2 \), we have
\[
\arg \left\{ 2q_2(z_1) + z_1q'_2(z_1) \right\} = \arg \left\{ \frac{f^{(p-1)}(z_1)}{z_1} \right\} = \arg q_2(z_1) + \arg \left\{ 2 + \frac{z_1q'_2(z_1)}{q_2(z_1)} \right\} = \frac{\pi}{2}\alpha_2 + \arg \{ 2 + ia_2k \} \geq \frac{\pi}{2}\alpha_2 + \arctan \frac{\alpha_2}{2} = \frac{\pi}{2} \alpha_1,
\]
which contradicts (7).

For the case \( \arg(q_2(z_1)) = -\pi\alpha_2/2 \), we have
\[
\arg \left\{ 2q_2(z_1) + z_1q'_2(z_1) \right\} = \arg \left\{ \frac{2f^{(p-1)}(z_1)}{p!z_1} \right\} = \arg \left\{ \frac{f^{(p-1)}(z_1)}{z} \right\} \leq -\frac{\pi}{2}\alpha_1.
\]
This also contradicts (7) and therefore, we have
\[ |\arg(q_2(z))| = \left| \arg \left\{ \frac{f^{(p-2)}(z)}{z^2} \right\} \right| < \frac{\pi}{2}\alpha_2 \quad (z \in U), \]
where
\[ \alpha_2 + \frac{2}{\pi} \arctan \frac{\alpha_2}{2} = \alpha_1. \]
and

\[1.08 < \alpha_2 < 1.09.\]

Let

\[q_3(z) = \frac{6f^{(\nu-3)}(z)}{p!z^3}, \quad q_3(0) = 1.\]

Then it follows that

\[3q_3(z) + zq'_3(z) = \frac{6f^{(\nu-2)}(z)}{p!z^2}.\]

Applying the same method as the above, we have

\[\left| \arg \left\{ 3q_3(z) + zq'_3(z) \right\} \right| = \left| \arg \left\{ \frac{6f^{(\nu-2)}(z)}{p!z^2} \right\} \right| = \left| \arg \left\{ \frac{z^2}{2\alpha_3 + 2\pi \arctan \left( \frac{\alpha_3}{3} \right)} \right\} \right| = \frac{\pi}{2} \alpha_2,
\]

where

\[0.903 < \alpha_3 < 0.904.\]

This shows that

\[\left| \arg \left\{ \frac{z^{f^{(\nu-3)}(z)}}{z^3} \right\} \right| = \left| \arg \left\{ \frac{z^{f^{(\nu-3)}(z)}}{z^3} \right\} \right| < \frac{\pi}{2} \alpha_3 < \frac{\pi}{2} \quad (z \in \mathbb{U}),\]

or

\[\Re \left\{ \frac{z^{f^{(\nu-3)}(z)}}{z^4} \right\} > 0 \quad (z \in \mathbb{U}).\]

It is trivial that \(g(z) = z^4\) is 4-valent starlike function in \(\mathbb{U}\). Therefore, from (8) and Lemma 1.2, we see that \(f\) is \(p\)-valent in \(\mathbb{U}\). This completes our proof of Corollary 2.10. \(\square\)

**Remark 2.11.** We remark that Corollary 2.10 improves Theorem A for the case \(p \geq 4\).

References