14-Point Difference Operator for the Approximation of the First Derivatives of a Solution of Laplace’s Equation in a Rectangular Parallelepiped

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Abstract. A 14-point difference operator is used to construct finite difference problems for the approximation of the solution, and the first order derivatives of the Dirichlet problem for Laplace’s equations in a rectangular parallelepiped. The boundary functions $\phi_j$ on the faces $\Gamma_j, j = 1, 2, \ldots, 6$ of the parallelepiped are supposed to have $p$th order derivatives satisfying the Hölder condition, i.e., $\phi_j \in C^p(\Gamma_j), 0 < \lambda < 1$, where $p \in \{4, 5\}$. On the edges, the boundary functions as a whole are continuous, and their second and fourth order derivatives satisfy the compatibility conditions which result from the Laplace equation. For the error $u_h - u$ of the approximate solution $u_h$ at each grid point $(x_1, x_2, x_3)$, $|u_h - u| \leq c\rho^{p-4}(x_1, x_2, x_3)h^4$ is obtained, where $u$ is the exact solution, $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point to the boundary of the parallelepiped, $h$ is the grid step, and $c$ is a constant independent of $\rho$ and $h$. It is proved that when $\phi_j \in C^p, 0 < \lambda < 1$, the proposed difference scheme for the approximation of the first derivative converges uniformly with order $O(h^{p-1}), p \in \{4, 5\}$.

1. Introduction

It is well known that the use of difference operators with a low number of pattern and with the highest order of accuracy for the approximate solution of differential equations reduces the effective realization of the obtained system of finite-difference equations. Moreover, to enlarge the class of applied problems the convergence of the difference solutions are preferred to be investigated under the weakened assumptions on the smoothness of the boundary conditions. All of these become more valuable in 3D problems, especially the derivatives of the unknown solution are sought.

The application of derivatives arise in many applied problems such as problems in electrophysics in which the first derivatives of the potential function define the electrostatic field [7], and in the fracture problems where the first derivatives of the stress function define the components of the tangential stress [8].

The investigation of approximate derivatives started in [9], where it was proved that the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the 2D Laplace equation in any strictly interior subdomain, with the same order $h$ with which the difference solution converges on the given domain. The uniform convergence of the difference derivatives over the
whole grid domain to the corresponding derivatives of the solution for the 2D Laplace equation with the order $O(h^2)$ was proved in [14]. In [5], for the first and pure second derivatives of the solution of the 2D Laplace equation special finite difference problems were investigated. It was proved that the solution of these problems converge to the exact derivatives with the order $O(h^4)$.

In [17] for the 3D Laplace equation the convergence of order $O(h^2)$ of the difference derivatives to the corresponding first and pure second derivatives of the exact solution is proved. It was assumed that the boundary functions have third derivatives satisfying the Hölder condition. Furthermore, they are continuous on the edges, and their second derivatives satisfy the compatibility condition that is implied by the Laplace equation. Whereas in [16] when the boundary values on the faces of a parallelepiped are assumed to have the fourth derivatives satisfying the Hölder condition, the constructed difference schemes converge with order $O(h^2)$ to the first and pure second derivatives of the exact solution. In [5] it is assumed that the boundary functions on the faces have sixth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Different difference schemes with the use of the 26-point difference operator are constructed on a cubic grid with mesh size $h$, to approximate the first and pure second derivatives of the solution of the Dirichlet problem with order $O(h^4)$.

In this paper $O(h^{p-1})$, $p = 4, 5$ order of approximation for the first derivatives of the solution of 3D Laplace’s equation is obtained under weaker assumptions on the smoothness of the boundary functions on the faces of the parallelepiped than those used in [6]. Moreover, to construct the finite difference problems the difference operator with a lower number of pattern is used.

Finally, the obtained theoretical results are supported by the illustration of numerical results.

2. Some Properties of a Solution of the Dirichlet Problem on a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < a_i, i = 1, 2, 3\}$ be an open rectangular parallelepiped; $\Gamma_j, j = 1, 2, ..., 6$ be its faces including the edges; $\Gamma_j$ for $j = 1, 2, 3$ ($j = 4, 5, 6$) belongs to the plane $x_j = 0$ ($x_{j-3} = a_{j-3}$), and let $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$ be the boundary of $R$; $\gamma_{\mu\nu} = \Gamma_\mu \cap \Gamma_\nu$ be the edges of the parallelepiped $R$. $C^{p,\lambda}(E)$ is the class of functions that have continuous $k$th derivatives satisfying the Hölder condition with an exponent $\lambda \in (0, 1)$.

Consider the boundary value problem

\[ \Delta u = 0 \text{ on } R, \ u = \varphi_j \text{ on } \Gamma_j, \ j = 1, 2, ..., 6 \]  

(1)

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \varphi_j$ are given functions.

Assume that

\[ \varphi_j \in C^{p,\lambda}(\Gamma_j), \ 0 < \lambda < 1, \ j = 1, 2, ..., 6, \ p \in \{4, 5\} \]  

(2)

\[ \varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu}, \]  

(3)

\[ \frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_\nu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \]  

(4)

\[ \frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_{\mu\nu}^2 \partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \]  

(5)

where $1 \leq \mu < \nu \leq 6, \nu - \mu \neq 3, \nu_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$ and $t_\mu$ and $t_\nu$ is an element of the normal to $\gamma_{\mu\nu}$ on the face $\Gamma_\mu$ and $\Gamma_\nu$, respectively.

The following Lemma follows from Theorem 2.1 in [12].

**Lemma 2.1.** Under conditions (2)–(5), the solution $u$ of the Dirichlet problem (1) belong to the Hölder class $C^{p,\lambda}(R)$, $0 < \lambda < 1, p \in \{4, 5\}$. 
Lemma 2.2. Let \( \rho (x_1, x_2, x_3) \) be the distance from the current point of the open parallelepiped \( R \) to its boundary and let 
\[
\frac{\rho}{h} = a_1 \frac{x_1}{h} + a_2 \frac{x_2}{h} + a_3 \frac{x_3}{h},
\]
where \( a_1^2 + a_2^2 + a_3^2 = 1 \).

Then the next inequality holds
\[
\left| \frac{\partial^p u (x_1, x_2, x_3)}{\partial p^6} \right| \leq c p^{\rho - 6} (x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R \text{ and } p \in \{4, 5\}
\]
where \( u \) is the solution of the problem (1), \( c \) is a constant independent of the direction of derivative \( \frac{\partial}{\partial \rho} \).

Proof. Since \( u \in C^{\rho, 1}(\bar{R}), p \in \{4, 5\} \) (Lemma 2.1) the proof Lemma 2.2 follows with the use of Lemma 3 in [10] (Chap. 4, Sec. 3).

3. Finite Difference Problem

We introduce a cubic grid with a step \( h > 0 \) defined by the planes \( x_i = 0, h, 2h, \ldots, i = 1, 2, 3 \). It is assumed that the edge lengths of \( R \) and \( h \) are such that \( \frac{h}{\rho} \geq 4 \) \( (i = 1, 2, 3) \) are integers.

Let \( D_h \) be the set of nodes of the grid constructed, \( \bar{R}_h = \bar{R} \cap D_h, R_h = R \cap D_h, R_h^c \subset R_h \) be the set of nodes of \( R_h \) lying at a distance of \( kh \) away from the boundary \( \Gamma \) of \( R \), and \( \Gamma_h = \Gamma \cap D_h \).

The 14-point difference operator \( S \) on the grid is defined as (see [19])
\[
S u(x_1, x_2, x_3) = \frac{1}{56} \left( 8 \sum_{p=10}^6 u_p + \sum_{q=70}^{14} u_q \right), \quad (x_1, x_2, x_3) \in R_h,
\]
where \( \sum_{(m)} \) is the sum extending over the nodes lying at a distance of \( m^{1/2}h \) away from the point \( (x_1, x_2, x_3) \) and \( u_p \) and \( u_q \) are the values of \( u \) at the corresponding nodes.

On the boundary \( \Gamma \) of \( R \), we define continuous on the entire boundary including the edges of \( R \), the function \( \varphi \) as follows
\[
\varphi = \left\{ \begin{array}{ll}
\varphi_1 & \text{on } \Gamma_1 \\
\varphi_j & \text{on } \Gamma_j \setminus \bigcup_{i=1}^{j-1} \Gamma_i, \quad j = 2, 3, \ldots, 6.
\end{array} \right.
\]

Obviously,
\[
\varphi = \varphi_j \text{ on } \Gamma_j, \quad j = 1, 2, \ldots, 6.
\]

We consider the finite difference problem approximating Dirichlet problem (1):
\[
u_h = S u_h \text{ on } R_h, \quad u_h = \varphi \text{ on } \Gamma_h,
\]
where \( S \) is the difference operator given by (7) and \( \varphi \) is the function defined by (8). By maximum principle, the system (9) has a unique solution (see [11], Chap. 4).

In what follows and for simplicity, we denote by \( c, c_1, c_2, \ldots \) constants, which are independent of \( h \) and the nearest factors, the identical notation will be used for various constants.

Consider two systems of grid equations
\[
\begin{align*}
v_h &= S v_h + g_h \text{ on } R_h, \quad v_h = 0 \text{ on } \Gamma_h, \\
\bar{v}_h &= S \bar{v}_h + \bar{g}_h \text{ on } R_h, \quad \bar{v}_h = 0 \text{ on } \Gamma_h
\end{align*}
\]
where \( g_h \) and \( \bar{g}_h \) are given functions and \( |\bar{g}_h| \leq g_h \) on \( R_h \).

Lemma 3.1. The solutions \( v_h \) and \( \bar{v}_h \) of systems (10) and (11) satisfy the inequality
\[
|\bar{v}_h| \leq v_h \text{ on } R_h.
\]
The proof of Lemma 3.1 is similar to that of the comparison theorem in [11] (Chap.4, Sec.3). Define

$$N(h) = \left\lfloor \frac{\min\{a_1, a_2, a_3\}}{2h} \right\rfloor,$$

(12)

where \([a]\) is the integer part of \(a\).

Consider for a fixed \(k\), \(1 \leq k \leq N(h)\) the systems of grid equations

$$v^k_h = Su^k_h + g^k_h \quad \text{on} \quad R^k_h, \quad v^k_h = 0 \quad \text{on} \quad \Gamma_h,$$

(13)

where

$$g^k_h = \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh. \end{cases}$$

**Lemma 3.2.** The solution \(v^k_h\) of the system (13) satisfies the inequality

$$v^k_h(x_1, x_2, x_3) \leq T_h^k, \quad 1 \leq k \leq N(h),$$

(14)

where \(T_h^k\) is defined as

$$T_h^k = T_h^k(x_1, x_2, x_3) = \begin{cases} \frac{5^p}{5k}, & 0 \leq \rho(x_1, x_2, x_3) \leq kh, \\ \rho(x_1, x_2, x_3) > kh. \end{cases}$$

(15)

**Proof.** By the direct calculation of the expression \(ST_h^k\), we obtain

$$T_h^k - ST_h^k \geq \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh, \end{cases}$$

(16)

on \(R_h\). On the basis of (13), inequalities (16) and taking the boundary condition \(T_h^k = 0\) on \(\Gamma_h\) into account, by Lemma 3.1, we get (14). \(\square\)

Let \(x_0 = (x_{10}, x_{20}, x_{30})\), be some point in \(R_h\). By Taylor’s formula for the solution \(u\) of the problem (1) around the point \(x_0\), we have

$$u(x_1, x_2, x_3) = p_5(x_1, x_2, x_3; x_0) + r_5(x_1, x_2, x_3; x_0),$$

(17)

where \(p_5\) is fifth-degree Taylor polynomial and \(r_5\) is remainder.

Since \(u\) is a harmonic function and \(S\) is linear, by taking into account that \(Sp_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30})\) from (17) follows

$$Su(x_{10}, x_{20}, x_{30}) = u(x_{10}, x_{20}, x_{30}) + Sr_5(x_{10}, x_{20}, x_{30}; x_0).$$

(18)

**Lemma 3.3.** The following estimation holds

$$\max_{(x_1, x_2, x_3) \in R_h} |Su - u| \leq c_4 \frac{h^{p+\lambda}}{k^{k-p-\lambda}}, \quad k = 1, 2, \ldots, N(h), \quad p \in \{4, 5\},$$

(19)

where \(u\) is the solution of the Dirichlet problem (1), \(S\) is the difference operator defined by (7), and \(N(h)\) is given by (12).
Proof. Let \( x_0 = (x_{10}, x_{20}, x_{30}) \) be some point in \( \mathbb{R}^3 \), and let \( P_{mnq} = (x_{10} + mh, x_{20} + nh, x_{30} + qh) \), where \( m, n, q = 0, \pm 1, m^2 + n^2 + q^2 \neq 0 \), be any point in the pattern of operator \( S \). Then by using the integral form of the remainder term of Taylor’s formula for each point \( P_{mnq} \) by virtue of Lemma 2.1 and Lemma 2.2, we obtain

\[
|r_5 (x_{10} + mh, x_{20} + nh, x_{30} + qh; x_0)| \leq ch^{p+\lambda}, \quad p \in \{4, 5\}.
\]

(20)

From the structure (7) of the operator \( S \) follows that its norm in the uniform metric is equal to one, then using by (20), we have

\[
|S_{r_5} (x_{10}, x_{20}, x_{30}; x_0)| \leq ch^{p+\lambda}, \quad p \in \{4, 5\}.
\]

(21)

On the basis of (18) and (21) follows the inequality (19), for \( k = 1 \). Let \( x_0 \in R^k_h \) be an arbitrary point for \( 2 \leq k \leq N(h) \), and let \( r_5 (x_1, x_2, x_3; x_0) \) be remainder term of the Taylor formula (17) in the Lagrange form. Then \( S_{r_5} (x_{10}, x_{20}, x_{30}; x_0) \) can be expressed linearly in terms of the 14 number of sixth derivatives of \( u \) at some points on the open intervals connecting the points of pattern of the operator \( S \) with the point \( x_0 \). The sum of the absolute values of the coefficients multiplying the sixth derivatives does not exceed \( ch^6 \) which is independent \( k_0 \) \((2 \leq k_0 \leq N(h))\) or the point \( x_0 \in R^k_h \). Using the estimation of the sixth derivatives by Lemma 2.2, for all \( 2 \leq k \leq N(h) \), we obtain

\[
|S_{r_5} (x_{10}, x_{20}, x_{30}; x_0)| \leq c_1 \frac{h^6}{(kh)^{6-p-\lambda}} = c_1 \frac{h^{p+\lambda}}{k^{6-p-\lambda}}.
\]

(22)

By virtue of (18) and (22) follows the estimation (19). □

Theorem 3.4. Assume that the boundary functions \( \varphi \) satisfy conditions (2)–(5). Then at each point \( (x_1, x_2, x_3) \in R_h \)

\[
|u_h - u| \leq c_0 h^4 p^{n-4}, \quad p \in \{4, 5\},
\]

(23)

where \( u_h \) is the solution of the finite difference problem (9), \( u \) is the exact solution of problem (1), and \( \rho = \rho (x_1, x_2, x_3) \) is the distance from the current point \( (x_1, x_2, x_3) \in R_h \) to the boundary of the rectangular parallelepiped \( R \).

Proof. Let \( \varepsilon_h^k \) \( 1 \leq k \leq N(h) \), be a solution of the system

\[
\varepsilon_h^k = S \varepsilon_h^k + \mu_h^k \text{ on } R_h, \quad \varepsilon_h^k = 0 \text{ on } \Gamma_h,
\]

(24)

where

\[
\mu_h^k = \begin{cases} S u - u \text{ on } R_h^k \text{,} \\ 0 \text{ on } R_h \setminus R_h^k. \end{cases}
\]

(25)

Let

\[
\varepsilon_h = u_h - u \text{ on } \overline{R}_h.
\]

(26)

By (9) and (26) the error function \( \varepsilon_h \) satisfies the system of equations

\[
\varepsilon_h = S \varepsilon_h + (S u - u) \text{ on } R_h, \quad \varepsilon_h = 0 \text{ on } \Gamma_h.
\]

(27)

We represent a solution of the system (27) as follows

\[
\varepsilon_h = \sum_{k=1}^{N(h)} \varepsilon_h^k
\]

(28)

where \( N(h) \) defined by (12), \( \varepsilon_h^k \), \( 1 \leq k \leq N(h) \), is a solution of the system

\[
\varepsilon_h^k = S \varepsilon_h^k + \alpha_h^k \text{ on } R_h, \quad \varepsilon_h^k = 0 \text{ on } \Gamma_h,
\]

(29)
where $u$ is the solution of the problem

\begin{equation}
\phi = \frac{\partial \phi}{\partial x}, \quad \text{on } R_h^k, \quad \text{when}
\end{equation}

Then on the basis of (28), (29), (30), Lemma 3.2 and Lemma 3.3, for the solution of (27), we have

\begin{align}
|\epsilon_h| &\leq \sum_{k=1}^{N(h)} |\phi^e_k| \leq \sum_{k=1}^{N(h)} \frac{p}{(kh)^{p-\lambda}} |Su - u| \\
&\leq 5c_1h^{p+\lambda} \sum_{k=1}^{p/h} k + 5c_1h^6 \sum_{k=p/h}^{N(h)} \frac{p}{(kh)^{p-\lambda}} \\
&\leq 5c_1h^{p+\lambda} \sum_{k=1}^{p/h} k - 5p+\lambda \quad + 5c_1h^{p-1+\lambda} \sum_{k=p/h}^{N(h)} k^{p-\lambda} \\
&\leq c_2h^p \rho^{p+\lambda} + c_3h^4 \rho \leq c_4h^4 \rho^{p-4}, \quad p \in \{4, 5\}. 
\end{align}

From (26) and (31), for any point $(x_1, x_2, x_3) \in R_h$, we obtain

\begin{equation}
|u_h - u| = |\epsilon_h| \leq c_0h^4 \rho^{p-4}(x_1, x_2, x_3), \quad p \in \{4, 5\}. 
\end{equation}

4. Approximation of the First Derivative

4.1. Boundary Function is from $C^{5, \lambda}$

Let the boundary functions $\phi_j, j = 1, 2, ..., 6$, in problem (1) on the faces $\Gamma_j$ be satisfied the conditions

\begin{equation}
\phi_j \in C^{5, \lambda}(\Gamma_j), \quad 0 < \lambda < 1, \quad j = 1, 2, ..., 6, 
\end{equation}

i.e., $p = 5$ in (2). Let $u$ be a solution of the problem (1) with the conditions (32) and (3) – (5).

We put $v = \frac{\partial \phi}{\partial x}$ and $\Phi_j = \frac{\partial \phi}{\partial x}$ on $\Gamma_j, \quad j = 1, 2, ..., 6$. It is obvious that the function $v$ is a solution of the following boundary value problem

\begin{equation}
\Delta v = 0 \text{ on } R, \quad v = \Phi_j \text{ on } \Gamma_j, \quad j = 1, 2, ..., 6, 
\end{equation}

where $u$ is a solution of the problem (1) for $p = 5$.

We define an approximate solution of problem (33) as a solution of the following finite difference problem

\begin{equation}
v_h = S v_h \text{ on } R_h, \quad v_h = \Phi_j(u_h) \text{ on } \Gamma_j^k, \quad j = 1, 2, ..., 6, 
\end{equation}

where $u_h$ is the solution of the problem (9), $\Phi_h$ is the fourth order forward (backward) numerical differentiation operator (see [1], [2]) used in [6] with the 26-point difference operator. On the nodes $\Gamma_j^k$, the boundary values are defined as $\Phi_j(u_h) = \frac{\partial v}{\partial x^j}, \quad p = 2, 3, 5, 6$.

\textbf{Theorem 4.1.} The estimation is true

\begin{equation}
\max_{(x_1, x_2, x_3) \in R_h} \left| \frac{\partial u}{\partial x_1} \right| \leq c h^4, 
\end{equation}

where $u$ is the solution of the problem (1), $v_h$ is the solution of the finite difference problem (34).
Proof. Let
\[
e_h = \nu_h - \nu \text{ on } \overline{R},
\]
where \( \nu = \frac{\partial u}{\partial x_1} \). From (34) and (36), we have
\[
e_h = S\epsilon + (S\nu - \nu) \text{ on } R_{hr},
\]
\[
e_h = \Phi_{kh}(u_h) - \nu \text{ on } \Gamma^{kh}_1, \ k = 1, 4, \ e_h = 0 \text{ on } \Gamma^{kh}_p, \ p = 2, 3, 5, 6.
\]
We put
\[
e_h = \epsilon^1_h + \epsilon^2_h,
\]
where
\[
e^1_h = S\epsilon^1 + \text{ on } R_{hr},
\]
\[
e^1_h = \Phi_{kh}(u_h) - \nu \text{ on } \Gamma^{kh}_1, \ k = 1, 4, \ e^1_h = 0 \text{ on } \Gamma^{kh}_q, \ q = 2, 3, 5, 6;
\]
\[
e^2_h = S\epsilon^2 + (S\nu - \nu) \text{ on } R_{hr}, \ e^2_h = 0 \text{ on } \Gamma^{kh}_j, \ j = 1, 2, ..., 6.
\]
First, we estimate the difference \( \Phi_{kh}(u_h) - \nu \) on \( \Gamma^{kh}_1, \ k = 1, 4 \) using the representation
\[
\Phi_{kh}(u_h) - \nu = (\Phi_{kh}(u_h) - \Phi_{kh}(u)) + (\Phi_{kh}(u) - \nu)
\]
Since \( \Phi_{kh}(u), \ k = 1, 4 \) are the fourth order approximation of \( \partial u / \partial x_1 \) on \( \Gamma_1 \) and by Lemma 2.1 the fifth order partial derivatives of the solution \( u \) are bounded in \( \overline{R} \), the difference \( \Phi_{kh}(u) - \nu \) has estimation (see [1], [2])
\[
\max_{k=1,4} \max_{(x_1,x_2,x_3)\in \Gamma^{kh}_1} |\Phi_{kh}(u) - \nu| \leq c_1 h^4.
\]
To estimate \( \Phi_{kh}(u_h) - \Phi_{kh}(u) \), we take the fourth order forward formula \( (k = 1) \),
\[
\Phi_{kh}(u_h) = \frac{1}{12h^4}[-25\phi_1(x_2,x_3) + 48u_h(h,x_2,x_3) - 36u_h(2h,x_2,x_3) + 16\phi_1(3h,x_2,x_3) - 3u_h(4h,x_2,x_3)] \text{ on } \Gamma^{kh}_1.
\]
Using the pointwise estimation (24) in Theorem 3.4, when \( p = 5 \), and taking into account the values of the distance function \( \rho(x_1,x_2,x_3) \) in the formula (43), we have
\[
|\Phi_{kh}(u_h) - \Phi_{kh}(u)| \leq c_2 h^4.
\]
The estimation (44) is true for the backward formula \( (k = 4) \), also. On the basis of (41), (42), (44), by using the maximum principle, for the solution of system (38), (39), we have
\[
\max_{(x_1,x_2,x_3)\in \overline{R}} |\epsilon^1_h| \leq c_3 h^4.
\]
The solution \( \epsilon^2_h \) of system (40) is the error function of the finite difference solution for problem (33), when the boundary functions \( \Phi_j = \partial u / \partial x_j, \ j = 1, 2, ..., 6 \), as follows from (2) – (5) satisfy the conditions
\[
\Phi_j \in C^{4,1}(\overline{\Gamma_j}), \ 0 < \lambda < 1, \ j = 1, 2, ..., 6, \Phi_j = \Phi_0 \text{ on } \gamma_{j},
\]
\[
\frac{\partial^2 \Phi_j}{\partial x_j^2} + \frac{\partial^2 \Phi_j}{\partial x_k^2} + \frac{\partial^2 \Phi_j}{\partial x_j \partial x_k} = 0 \text{ on } \gamma_{j},
\]
Then, on the basis of Theorem 4 in [19] for the error \( \epsilon^2_h \), we have
\[
\max_{(x_1,x_2,x_3)\in \overline{R}} |\epsilon^2_h| \leq c_4 h^4.
\]
By virtue of (37), (45), and (46) follows the inequality (35). \( \square \)
4.2. Boundary Function From $C^4$.

Let the boundary functions $q_j \in C^4 \Gamma_j$, $0 < \lambda < 1$, $j = 1, 2, ..., 6$, in (1) - (5), i.e., $p = 4$ in (2), and let $v = \frac{\partial u}{\partial x_1}$ and let $\Phi_j = \frac{\partial u}{\partial x_1}$ on $\Gamma_j$, $j = 1, 2, ..., 6$, and consider the boundary value problem:

$$\nabla v = 0 \text{ on } R, \quad v = \Phi_j \text{ on } \Gamma_j, \quad j = 1, 2, ..., 6, \quad (47)$$

where $u$ is a solution of the boundary value problem (1).

We define the following third order numerical differentiation operators $\Phi_{ph}$, $\nu = 1, 4$:

$$\Phi_{ph}(u_h) = \frac{1}{6h}[-11q_j(x_1, x_3) + 18u_h(h, x_2, x_3) - 9u_h(2h, x_2, x_3)$$

$$+ 2u_h(3h, x_2, x_3)] \text{ on } \Gamma_j^h, \quad (48)$$

$$\Phi_{ph}(u_h) = \frac{1}{6h}[11q_j(x_2, x_3) - 18u_h(a_1 - h, x_2, x_3) + 9u_h(a_1 - 2h, x_2, x_3)$$

$$- 2u_h(a_1 - 3h, x_2, x_3)] \text{ on } \Gamma_j^h, \quad (49)$$

and we put

$$\Phi_{ph}(u_h) = \frac{\partial \Phi_j}{\partial x_1}, \quad \text{on } \Gamma_j^h, \quad p = 2, 3, 5, 6 \quad (50)$$

where $u_h$ is the solution of the finite difference problem (9).

It is obvious that $\Phi_j$, $j = 1, 2, ..., 6$, satisfy the conditions

$$\Phi_j \in C^3 \Gamma_j, \quad 0 < \lambda < 1, \quad j = 1, 2, ..., 6, \quad (51)$$

$$\Phi_p = \Phi_v \text{ on } \gamma_{ph}, \quad (52)$$

$$\frac{\partial^2 \Phi_j}{\partial t^2} + \frac{\partial^2 \Phi_j}{\partial t^2} + \frac{\partial^2 \Phi_j}{\partial t^2} = 0 \text{ on } \gamma_{ph}. \quad (53)$$

Let $v_h$ be the solution of the following finite difference problem

$$v_h = Sv_h \text{ on } R_h, \quad v_h = \Phi_{ph} \text{ on } \Gamma_j^h, \quad j = 1, 2, ..., 6, \quad (54)$$

where $\Phi_{ph}$, $j = 1, 2, ..., 6$, are defined by (48) - (50).

**Theorem 4.2.** Let the boundary function $q_j \in C^4 \Gamma_j, \quad j = 1, ..., 6$. The estimation is true

$$\max_{(x_1, x_2, x_3) \in R_h} \left| v_h - \frac{\partial u}{\partial x_1} \right| \leq ch^3, \quad (55)$$

where $u$ is the solution of the problem (1), $v_h$ is the solution of the finite difference problem (54).

**Proof.** The proof of Theorem 4.2 is similar to that of Theorem 4.1, with the following differences in estimation for the errors $\epsilon_1$ and $\epsilon_2$ in (37): (i) putting $p = 4$ in Lemma 2.1 and Theorem 3.4, and taking into account that the formulae (48) and (49) are the third order, the estimation

$$\max_{(x_1, x_2, x_3) \in R_h} |\epsilon_1| \leq c_0 h^3.$$  

is proved. (ii) on the basis of (51)-(53) and Theorem 2 in [19], we obtain

$$\max_{(x_1, x_2, x_3) \in R_h} |\epsilon_2| \leq c_2 h^3.$$

□

**Remark 4.3.** We have investigated the method of high order approximations of the first derivative $\partial u/\partial x_1$. The same results are obtained for the derivatives $\partial u/\partial x_l$, $l = 2, 3$ analogously, by using the same order forward and backward formulae in appropriate faces of the parallelepiped.
5. Numerical Results

Example 5.1. Let $R = \{(x_1, x_2, x_3) : 0 < x_i < 1, i = 1, 2, 3\}$, and let $\Gamma_j, j = 1, ..., 6$ be its faces. We consider the following problem:

$$\Delta u = 0 \text{ on } R, \quad u = \varphi (x_1, x_2, x_3) \text{ on } \Gamma_j, \ j = 1, ..., 6,$$

where

$$\varphi (x_1, x_2, x_3) = \left( x_3 - \frac{1}{2} \right)^2 - \left( \frac{x_1^2 + x_2^2}{2} \right) + \left( x_1^2 + x_2^2 \right)^{\frac{1}{2}} \cos \left( \left( 5 + \frac{1}{30} \right) \arctan \left( \frac{x_2}{x_1} \right) \right)$$

is the exact solution of this problem, which is in $C^{5,1/30}$.

We solve the system (9) and (34) to find the approximate solution $u_h$ for $u$ and approximate first derivative $v_h$ for $\varphi$ respectively.

In Tables 1 and 2 the maximum errors are given. Table 1 shows that the convergence order more than 4 which is corresponds to the product $\rho$ in Theorem 3.4. Table 2 justified estimation (35) in Theorem 4.1, i.e., the fourth order convergence.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u_h - u|_{\mathbb{R}^3}$</th>
<th>$E^n_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>7.517E-09</td>
<td>32.13</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>2.339E-10</td>
<td>32.66</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>7.163E-12</td>
<td>32.24</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>2.188E-13</td>
<td>32.75</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>6.682E-15</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 Errors for the solution in maximum norm

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|v_h - v|_{\mathbb{R}^3}$</th>
<th>$E^n_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>4.5436E-03</td>
<td>13.40</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>3.3909E-04</td>
<td>14.76</td>
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<tr>
<td>$2^{-5}$</td>
<td>2.2975E-05</td>
<td>15.40</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>1.4922E-06</td>
<td>15.70</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>9.5053E-08</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 Errors for the first derivative in maximum norm with the fourth-order formulae

Example 5.2. Let $u$ be a solution of problem (56) when the boundary function $\varphi$ is chosen from $C^{4,\frac{1}{7}}$ as

$$\varphi (x_1, x_2, x_3) = \left( x_3 - \frac{1}{2} \right)^2 - \left( \frac{x_1^2 + x_2^2}{2} \right) + \left( x_1^2 + x_2^2 \right)^{\frac{1}{2}} \cos \left( \left( 4 + \frac{1}{30} \right) \arctan \left( \frac{x_2}{x_1} \right) \right).$$

Table 3 and 4 give the fourth order convergence when boundary function is from $C^{4,\frac{1}{7}}$ for both, solution and first derivative which are the numerical justification of Theorem 3.4 and 4.2 respectively.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u_h - u|_{\mathbb{R}^3}$</th>
<th>$E^n_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>3.4801E-08</td>
<td>16.20</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>2.1486E-09</td>
<td>16.36</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>1.3135E-10</td>
<td>16.37</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>8.0228E-12</td>
<td>16.37</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>4.8998E-13</td>
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</tr>
</tbody>
</table>

Table 3 Errors for the solution in maximum norm

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|v_h - v|_{\mathbb{R}^3}$</th>
<th>$E^n_v$</th>
</tr>
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<tbody>
<tr>
<td>$2^{-3}$</td>
<td>3.4801E-08</td>
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</tr>
<tr>
<td>$2^{-7}$</td>
<td>4.8998E-13</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 Errors for the first derivative in maximum norm with the fourth-order formulae
In Tables 1-4 we have used the following notations: 

\[ ||U_h - U||_{\mathcal{R}_h} = \max_{x \in \mathcal{R}_h} |U_h(x) - U(x)| \quad \text{and} \quad E^m_{U} = \frac{||U - U_{h,w}||_{\mathcal{R}_h}}{||U - U_{h,w+1}||_{\mathcal{R}_h}} \] 

where \( U \) be the exact solution of the continuous problem, and \( U_h \) be its approximate values on \( \mathcal{R}_h \).

6. Conclusion

Three different schemes with the 14-point difference operator are constructed on a cubic grid with mesh size \( h \), whose solutions separately approximate the solution of the Dirichlet problem for 3D Laplace’s equation with the order \( O(h^p\rho^{-\alpha}) \), \( p \in [4, 5] \), where \( \rho = \rho(x_1, x_2, x_3) \) is the distance from the current point \( (x_1, x_2, x_3) \in \mathcal{R}_h \) to the boundary of the rectangular parallelepiped \( \mathcal{R} \) and its first derivatives with the orders \( O(h^{1-\alpha}) \).

The obtained results can be used to highly approximate the derivatives of the solution of 3D Laplace’s boundary value problems on a prism with an arbitrary polygonal base and on polyhedra by developing the combined or composite grid methods [13, 15]. For the 2D case see [3, 4, 18, 20].

References