Anti-Invariant Riemannian Submersions from Nearly-K-Cosymplectic Manifolds

Ju Tan*, Na Xu*

*School of Mathematics and Physics Science and Engineering, Anhui University of Technology, Ma'anshan, 243032, People's Republic of China

Abstract. In this paper, we introduce anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. We study the integrability of horizontal distributions. And we investigate the necessary and sufficient condition for an anti-invariant Riemannian submersion to be totally geodesic and harmonic. Moreover, we give examples of anti-invariant Riemannian submersions such that characteristic vector field $\xi$ is vertical or horizontal.

1. Introduction

Let $\pi$ be a $C^\infty$-submersion from a Riemannian manifold $(M, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then according to the different conditions on the map $\pi : (M, g_M) \rightarrow (N, g_N)$, we have the following submersions: Lorentzian submersion and semi-Riemannian submersion [7], slant submersion ([4, 19]), contact-complex submersion [8], almost h-slant submersion and h-slant submersion [16] quaternionic submersion [9], semi-invariant submersion [18], h-semi-invariant submersion [15], etc. In [17], Sahin introduced anti-invariant Riemannian submersions from almost hermitian manifolds onto Riemannian manifolds. Recently, C. Murathan and I. Küpeli Erken have investigated anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds and from cosymplectic manifolds onto Riemannian manifolds ([11, 12]). Furthermore, anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds have also been studied in [2].

In this paper, we study anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. The paper is organized as follows: In section 2, we present some basic facts about Riemannian submersions. Nearly-K-cosymplectic manifolds are introduced in section 3. In section 4, we give the definition of anti-invariant Riemannian submersions and introduce anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. Moreover, we investigate the geometry of leaves of the distributions. In addition, we give two examples of anti-invariant Riemannian submersions such that characteristic vector field $\xi$ is vertical and horizontal respectively.

2010 Mathematics Subject Classification. Primary 53D10; Secondary 53D15

Keywords. Nearly-K-cosymplectic; Riemannian submersion; Totally geodesic; Harmonic.

Communicated by Mića Stanković
Corresponding author: Na Xu

The first author was supported by the grant of China Scholarship Council (No.201606205048), The second author was supported by the grant of China Scholarship Council (No.201606310118).

Email addresses: tanju29070163.com (Ju Tan), xuna4969163.com (Na Xu)
2. Preliminaries

Let \((M, g_M)\) be an \(m\)-dimensional Riemannian manifold, Let \((N, g_N)\) be an \(n\)-dimensional Riemannian manifold. A smooth surjective mapping \(F: (M, g_M) \rightarrow (N, g_N)\) is called a Riemannian submersion if the following conditions are satisfied:

- \(F\) has maximal rank,
- The differential \(F_x\) preserves the lengths of horizontal vectors.

In ([13, 14]), O’Neil have defined the fundamental tensors of a submersion, which are \((1, 2)\)-tensors on \(M\) and are given by the following formulas:

\[
\begin{align*}
T(E, F) &= \mathcal{T}E = \mathcal{H}V E + \mathcal{V}V E H F, \\
A(E, F) &= \mathcal{A}E = \mathcal{V}V E H F + \mathcal{H}V E V F, 
\end{align*}
\]

for any vector field \(E\) and \(F\) on \(M\). Here \(\nabla\) denotes the Levi-Civita connection of \((M, g_M)\). Note that we denote the projection morphism on the distributions \(\ker F\), and \((\ker F)^\perp\) by \(\mathcal{V}\) and \(\mathcal{H}\), respectively. And we have the following lemma ([13, 14]).

**Lemma 2.1.** For any \(U, W\) vertical and \(X, Y\) horizontal vector fields, the tensor fields \(\mathcal{T}, \mathcal{A}\) satisfy:

\[
\begin{align*}
\mathcal{T}(U, W) &= \mathcal{T}(W, U), \\
\mathcal{A}(X, Y) &= -\mathcal{A}(Y, X) = \frac{1}{2} \nabla [X, Y],
\end{align*}
\]

Obviously, \(\mathcal{T}\) is vertical, i.e. \((\mathcal{T}_E = \mathcal{T}_V E)\) And \(\mathcal{A}\) is horizontal, i.e. \((\mathcal{A}_E = \mathcal{A}_H E)\).

For each \(q \in N, F^{-1}(q)\) is a submanifold of \(M\) of dimension \(\dim M - \dim N\). The submanifolds \(F^{-1}(q), q \in N\) are called fibers, and a vector field on \(M\) is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers. A vector field \(X\) on \(M\) is called basic if \(X\) is horizontal and \(F\)-related to a vector field \(\tilde{X}\) on \(N\), i.e. \((\forall p \in M, F, X_p = \tilde{X}_F(p)\))

From (2.1) and (2.2) we have the following basic equations:

\[
\begin{align*}
\nabla_Y W &= \mathcal{T}_V Y + \mathcal{V}V Y W, \\
\nabla_Y X &= \mathcal{H}V V X + \mathcal{T}_V X, \\
\nabla_X Y &= \mathcal{A}_X V + \mathcal{V}V X Y, \\
\nabla_X Y &= \mathcal{H}V (\mathcal{V}V X Y + \mathcal{A}_X Y).
\end{align*}
\]

where \(X, Y\) are horizontal vector fields and \(V, W\) are vertical vector fields.

From (2.1) and (2.2), we can also deduce the following formulas:

\[
\begin{align*}
g(T_E F, G) + g(T_E G, F) &= 0, \\
g(A_E F, G) + g(A_E G, F) &= 0,
\end{align*}
\]

for any \(E, F, G \in \Gamma(TM)\). Moreover, \(T_E, A_E\) reverse the horizontal and the vertical distributions.

It is well-known that a Riemannian submersion has totally geodesic fiber if and only if \(\mathcal{T} = 0\); Horizontal distribution \(\mathcal{H}\) is totally geodesic if and only if \(\mathcal{A} = 0\) (see [10]). Suppose \(e_1, \ldots, e_{m-n}\) be an orthogonal frame of \(\Gamma(\ker F)\), then the horizontal vector field \(H = \frac{1}{n-n} \sum_{i=1}^{m-n} \Gamma_{e_i} e_i\) is called the mean curvature vector field of the fiber. If \(H = 0\) then the Riemannian submersion is called minimal.

Now, we recall the notion of harmonic maps between Riemannian manifolds. If \(F: M \rightarrow N\) is a smooth map between Riemannian manifolds. Then the differential \(F_\ast\) of \(F\) can be viewed a section of the bundle \(\operatorname{Hom}(TM, F^{-1}TN) \rightarrow M\), where \(F^{-1}TN\) is the pullback bundle which has fibres \((F^{-1}TN)_p = T_{F(p)} N, p \in M\).
Suppose \( M \) have an almost contact metric structure if and only if \( h \) is a Hermitian metric on \( M \). Then the second fundamental form of \( F \) is given by

\[
(VF_*)(X, Y) = \nabla^F_X F_* (Y) - F_* (\nabla^M_X Y),
\]

for any \( X, Y \in \Gamma(TM) \), where \( \nabla^F \) is the pullback connection. It is known that the second fundamental form is symmetric. For a Riemannian submersion \( F \), one can easily obtain:

\[
(VF_*)(X, Y) = 0,
\]

for any \( X, Y \in \Gamma((\ker F)^+) \). A smooth map \( F : M \rightarrow N \) is said to be harmonic if \( \text{trace}(VF_*) = 0 \). On the other hand, the tension field of \( F \) is the section \( \tau(F) \) of \( \Gamma(F^{-1}TN) \) defined by

\[
\tau(F) = \text{div} F_* = \sum_{i=1}^{m} (VF_*)(e_i, e_i),
\]

where \( \{e_1, \ldots, e_m\} \) is the orthonormal frame on \( M \). Then it follows that \( F \) is harmonic if and only if \( \tau(F) = 0 \), (for details, see [1]).

3. Nearly-K-cosymplectic manifolds

A \((2n+1)\)-dimensional \( C^\infty \) differential manifold \( M \) is said to have an almost contact structure or \((\phi, \xi, \eta)\)-structure if there exist on \( M \) a tensor field \( \phi \) of type \((1, 1)\), a vector field \( \xi \) and 1-form \( \eta \) satifying:

\[
\eta(\xi) = 1, \phi^2 = -I + \eta \otimes \xi,
\]

here \( I \) denote the identity tensor, \( \xi \) is called characteristic vector field. And we have the following proposition [3].

**Proposition 3.1.** Suppose \( M^{2n+1} \) has a \((\phi, \xi, \eta)\)-structure. Then \( \phi \cdot \xi = 0 \) and \( \eta \cdot \phi = 0 \). Furthermore, the endomorphism \( \phi \) has rank \( 2n \).

\( M \) is said to have a \((\phi, \xi, \eta, g)\)-structure or an almost contact metric structure if the manifold \( M \) with a \((\phi, \xi, \eta)\)-structure admits a Riemannian metric \( g \) such that

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

here \( X, Y \) are vector fields on \( M \). Obviously, set \( Y = \xi \), We get \( \eta(X) = g(X, \xi) \).

We define an almost complex structure \( f \) on \( M \times R \):

\[
f(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt}),
\]

here \( M \times R \) is considered as the product manifold, And \( M \) have an almost contact structure \((\phi, \xi, \eta)\), \( f \) denotes the \( C^\infty \)-function on \( M \times R \), \( X \) is tangent to \( M \). Now we define a Riemannian metric on \( M \times R \) by

\[
h((X, f \frac{d}{dt}), (Y, g \frac{d}{dt})) = g(X, Y) + fg.
\]

From [7], We have the following proposition:

**Proposition 3.2.** \( M \) have an almost contact metric structure if and only if \( h \) is a Hermitian metric on \( (M \times R, J) \); An \((\phi, \xi, \eta, g)\)-structure is called cosymplectic structure if and only if the structure \((J, h)\) in \( M \times R \) is Kählerian; An \((\phi, \xi, \eta, g)\)-structure is called a nearly-K-cosymplectic structure if \((J, h)\) is nearly Kählerian.
A manifold $M$ endowed with a nearly-K-cosymplectic structure is called nearly-K-cosymplectic manifold. And from [7], $M$ is nearly-K-cosymplectic manifold if and only if it satisfies the following formula:

\[
(V_X\phi)X = 0, \\
\nV_X\xi = 0, \\
\]

(17) (18)

here $X$ is tangent to $M$. Obviously, the first equation is equivalent to

\[
(V_X\phi)Y + (V_Y\phi)X = 0. \\
\]

(19)

It is obvious that a cosymplectic manifold is nearly-K-cosymplectic manifold. The canonical example of nearly-K-cosymplectic manifolds is given by the product $S^n \times R$ nearly Kähler manifold $S^n(j, g)$ with real $R$ line [5]. Now we introduce a nearly-K-cosymplectic manifold example.

**Example 3.3.** Let $L$ be a $(2n + 1)$ dimensional Lie algebra, and choose a basis $\{e_0, e_1, \ldots, e_{2n}\}$ of $L$. The non-vanishing Lie bracket relations are following:

\[
\begin{align*}
[e_0, e_i] &= -a_i e_{n+i}, \\
[e_0, e_{n+i}] &= a_i e_i,
\end{align*}
\]

for $i = 1, \ldots, n, a_i^2 + \ldots + a_n^2 > 0$.

Consider a connected Lie subgroup $G$ of general linear group $GL(k, R)$, for certain $k$, such that the Lie algebra $g$ of $G$ is isomorphic with $L$. Let $\sigma : L \rightarrow g$ be the isomorphism. Let $\{E_0, E_1, \ldots, E_{2n}\}$ be the basis of $G$ formed by left invariant vector fields on $G$ such that $E_j = \sigma(e_j)$ for $j = 0, 1, \ldots, 2n$. Then, the non-vanishing Lie bracket relations on Lie algebra $g$ are following:

\[
\begin{align*}
[E_0, E_i] &= -a_i E_{n+i}, \\
[E_0, E_{n+i}] &= a_i E_i.
\end{align*}
\]

Define a left invariant Riemannian metric $g$ on $G$ by $g(E_j, E_k) = \delta_{jk}$, $j, k = 0, 1, \ldots, 2n$. Then the Levi-Civita connection on $G$ with respect to $g$ is:

\[
\begin{align*}
\nabla_{E_0} E_i &= -a_i E_{n+i}, \\
\nabla_{E_0} E_{n+i} &= a_i E_i.
\end{align*}
\]

Define a 1-form $\eta$ and $(1,1)$-tensor field $\phi$ on $G$ by $\eta(E_j) = \delta_{0j}$, for $j = 0, 1, \ldots, 2n$, and $\phi E_0 = 0, \phi E_i = E_i, \phi E_{n+i} = -E_{n+i}$, for $i = 1, \ldots, n$. Set $\xi = E_0$. Then $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $G$. Notice $\nabla \xi = 0$ and $\nabla (E_0, \phi)E_i = 0$, for $i = 0, 1, \ldots, 2n$. So $(\phi, \xi, \eta, g)$ is a nearly-K-cosymplectic structure. And

\[
\begin{align*}
(V_{E_0}\phi)E_i &= \nabla_{E_0}(\phi E_i) - \phi(\nabla_{E_0}E_i) \\
&= \nabla_{E_0}E_i + \phi(\phi E_{n+i}) \\
&= -2a_i E_{n+i}.
\end{align*}
\]

Thus $G$ is not a non-trivial nearly-K-cosymplectic manifold. Moreover, there is a global system of coordinates $(x_i, y_i, z_i), 1 \leq i \leq n$ on nearly-K-cosymplectic manifold $G$ such that

\[
\begin{align*}
E_i &= \frac{\partial}{\partial x_i}, \\
E_{n+i} &= \frac{\partial}{\partial y_i}, \\
E_0 &= \frac{\partial}{\partial z} + \sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial y_j} - \sum_{j=1}^{n} a_j y_j \frac{\partial}{\partial x_j}.
\end{align*}
\]
4. Anti-invariant Riemannian Submersions

Definition 4.1. Let $F$ be a Riemannian submersion from nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to Riemannian manifold $(N, g_N)$. We say $F$ is an anti-invariant Riemannian submersion if the following condition is satisfied:

$$\phi(\ker F) \subseteq (\ker F)^\perp$$

We denote the complementary orthogonal distribution to $\phi(\ker F)$ in $(\ker F)^\perp$ by $\mu$. Then it is easy to prove that $\mu$ is an invariant distribution of $(\ker F)^\perp$, under the action of endomorphism $\phi$.

Now we will give two examples.

Example 4.2. Let $G$ be a nearly-$K$-cosymplectic manifold with dimension seven as in Example 3.3. And set $a_1 = 1, a_2 = 0$, then $\xi = E_0 = \frac{\partial}{\partial u} + x_1 E_3 - y_1 E_1$. Let $N = \{(u, v, w)|u, v, w \in \mathbb{R}, u > 0\}$. The Riemannian metric tensor field $g_N$ is defined by $g_N = \frac{1}{2} du^2 + dv^2 + dw^2$ on $N$.

Let $F: G \to N$ be a map defined by $F(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_1 + y_1}{\sqrt{2}}, \frac{x_2 + y_2}{\sqrt{2}}, \frac{x_3 + y_3}{\sqrt{2}}, z)$. Then by direct calculation, we have

$$\ker F = \text{span}\{V_1 = \frac{1}{\sqrt{2}}(E_2 - E_3), V_2 = \frac{1}{\sqrt{2}}(E_3 - E_6), V_3 = E_0 = \xi\}$$

and

$$(\ker F)^\perp = \text{span}\{H_i = \frac{1}{\sqrt{2}}(E_{i+1} + E_{i+4}), i = 1, 2, H_3 = \frac{1}{\sqrt{2}}(E_1 + E_4), H_4 = \frac{1}{\sqrt{2}}(E_1 - E_4)\}$$

Obviously, $F$ is a Riemannian submersion. Furthermore, $\phi V_1 = H_1, \phi V_2 = H_2$ imply that $(\ker F)^\perp = \phi(\ker F) \oplus \text{span}\{H_3, H_4\}$. Thus $F$ is an anti-invariant Riemannian submersion such that $\xi$ is vertical.

Example 4.3. Let $G$ be a nearly-$K$-cosymplectic manifold with dimension seven as in Example 3.3. And set $a_1 = a_2 = 0, a_3 = 1$, then $\xi = E_0 = \frac{\partial}{\partial u} + x_3 E_6 - y_3 E_3$. Let $N = \{(u_1, u_2, u_3, u_4, u_5)|u_i^2 < 1, u_i \in \mathbb{R}, i = 1, 2, 3, 4, 5\}$. The Riemannian metric tensor field $g_N$ is defined by $g_N = \sum_{i=1}^{4} du_i^2 + (1 - u_3^2 - u_4^2) du_5^2$ on $N$.

Let $F: G \to N$ be a map defined by $F(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_1 + y_1}{\sqrt{2}}, \frac{x_2 + y_2}{\sqrt{2}}, \frac{x_3 + y_3}{\sqrt{2}}, z)$. Then by direct calculation, we have

$$\ker F = \text{span}\{V_1 = \frac{1}{\sqrt{2}}(E_1 - E_4), V_2 = \frac{1}{\sqrt{2}}(E_2 - E_3)\}$$

and

$$(\ker F)^\perp = \text{span}\{H_i = \frac{1}{\sqrt{2}}(E_i + E_{i+4}), i = 1, 2, 3, H_4 = \frac{1}{\sqrt{2}}(E_3 - E_6), H_5 = \xi\}$$

Obviously, $F$ is a Riemannian submersion. Furthermore, $\phi V_1 = H_1, \phi V_2 = H_2$ imply that $\phi(\ker F) \subseteq (\ker F)^\perp$. And $F$ is an anti-invariant Riemannian submersion such that $\xi$ is horizontal.

4.1. Anti-invariant submersions admitting vertical characteristic vector field

In this subsection, we will discuss anti-invariant submersions from a nearly-$K$-cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field $\xi$ is vertical.

On the one hand, because of the invariance of $\mu$ under the action of $\phi$, we can get

$$\phi X = BX + CX,$$

(20)

here $X \in \Gamma((\ker F)^\perp), BX \in \Gamma(\ker F), CX \in \Gamma(\mu)$. On the other hand, since $F$ is a Riemannian submersion and $F_\ast((\ker F)^\perp) = TN$, we get $g_N(F_\ast, \phi V, F_\ast CX) = 0$, for $X \in \Gamma((\ker F)^\perp), V \in \Gamma(\ker F)$. And, we have

$$TN = F_\ast(\phi(\ker F)) \oplus F_\ast(\mu).$$

(21)

By (3.14) and (4.20), it is easy to obtain the following proposition.
Proposition 4.4. Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then we have

\[ BCX = 0, \quad \eta(BX) = 0, \quad C^2X = -X - \phi(BX), \]
\[ C\phi V = 0, \quad C^3X + CX = 0, \quad B\phi V = -V + \eta(V)\xi, \]

where $X \in \Gamma((\ker F)^\perp)$ and $V \in \Gamma(\ker F)$. 

Lemma 4.5. Let $\nabla$ be the connection of a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$. Then we have

\[ \nabla_X Y = -\phi \nabla_X \phi Y + \phi((\nabla_X \phi)Y), \quad (22) \]
\[ \nabla_X Y + \nabla_Y X = -\phi \nabla_X \phi Y - \phi \nabla_Y \phi X, \quad (23) \]

here $X, Y \in \Gamma((\ker F)^\perp)$.

Proof. Denote $g_M(\cdot, \cdot)$ by $\langle \cdot, \cdot \rangle$. Since $\xi$ is vertical and $\nabla_X \xi = 0$, by (2.7), (2.8) and (2.10), we have:

\[ \eta(\nabla_X Y) = \langle H\nabla_X Y + A_X Y, \xi \rangle = \langle A_X Y, \xi \rangle = -\langle Y, A_X \xi \rangle = -\langle Y, V_X \xi - \nabla X \xi \rangle = 0. \]

And

\[ \nabla_X (\phi Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y), \]

So

\[ \phi(\nabla_X \phi Y) = \phi((\nabla_X \phi)Y) + \phi^2(\nabla_X Y) = \phi((\nabla_X \phi)Y) - \nabla X Y + \eta(\nabla_X Y)\xi, \]

Thus we obtain (4.22). To see (4.23), By (3.19) and (4.22), we have

\[ \nabla_X Y = -\phi \nabla_X \phi Y - \phi((\nabla_Y \phi)X) = -\phi \nabla_X \phi Y - \phi(\nabla_Y \phi X) + \phi^2(\nabla_X Y) = -\phi \nabla_X \phi Y - \phi(\nabla_Y \phi X) - \nabla_Y X + \eta(\nabla_Y X)\xi. \]

Hence, we get

\[ \nabla_X Y + \nabla_Y X = -\phi \nabla_X \phi Y - \phi \nabla_Y \phi X. \]

Lemma 4.6. Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then we have

\[ A_X \xi = 0, \quad (24) \]
\[ T_U \xi = 0, \quad (25) \]
\[ g_M(CX, \phi U) = 0, \quad (26) \]
\[ g_M(V_X CY, \phi U) = g_M(CY, \nabla_U \phi X) - 2g_M(CY, \phi(\nabla_U X)), \quad (27) \]

here $X, Y \in \Gamma((\ker F)^\perp), U \in \Gamma(\ker F)$. 

\[ \square \]
Proof. By (3.18) and (2.7), (2.5), and notice \( \mathcal{A}_X, T_U \) reverse the distributions, we get (4.24) and (4.25).

By (3.15) and (4.20), we have
\[
\begin{align*}
g_M(\mathcal{A}_X, \phi U) &= g_M(\phi X - B X, \phi U) \\
&= g_M(X, U) - \eta(X)\eta(U) + g_M(\phi BX, \phi(\phi U)).
\end{align*}
\]

Since \( \phi BX \in \Gamma((\ker F)^+) \), \( U, \xi \in \Gamma(\ker F) \), we get (4.26).

Since \( [X, U] \in \Gamma(\ker F) \), we have \( g_M(\mathcal{A}_X, \phi U) = 0 \) and \( g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)) = g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)). \) By (4.26) and (3.19), we obtain
\[
\begin{align*}
g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)) &= g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)) - g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)) \\
&= g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)) - g_M(\mathcal{A}_X, \phi(\mathcal{A}_X U)).
\end{align*}
\]

Next, we study the integrability of the horizontal distribution and then we investigate the geometry of leaves of \( \ker F \), and \((\ker F)^+\).

**Theorem 4.7.** Let \( F \) be an anti-invariant Riemannian submersion from a nearly-\( K \)-cosymplectic manifold \( M(\phi, \xi, \eta, g_M) \) to a Riemannian manifold \( (N, g_N) \). Then the following criteria are equivalent:

1. \((\ker F)^+\) is integrable,
2. \[
g_N((VF)(Y, BX), F, \phi V) = g_M((\mathcal{A}_X, \phi(\mathcal{A}_X U)) Y, V) - g_M(\mathcal{A}_X Y, V) - g_M(\mathcal{A}_X V, Y) + g_M(\phi BX, \phi(\phi U)) \]
3. \[
g_M(\mathcal{A}_X Y, \mathcal{A}_X BX, \phi V) = g_M(\mathcal{A}_X Y, \mathcal{A}_X BX, \phi V) + g_M(\phi BX, \phi(\phi U)) - g_M(\phi BX, \phi(\phi U))
\]

here \( X, Y \in \Gamma((\ker F)^+) \), \( V \in \Gamma(\ker F) \).

**Proof.** For \( X, Y \in \Gamma((\ker F)^+) \), \( V \in \Gamma(\ker F) \), we have
\[
\begin{align*}
g_M([X, Y], V) &= g_M(\mathcal{A}_X Y, V) - g_M(\mathcal{A}_X V, Y) \\
&= g_M(\phi BX, \phi V).
\end{align*}
\]

Then from (4.20), we have
\[
\begin{align*}
g_M([X, Y], V) &= g_M([\phi BX, \phi V), \phi(\phi U)] V - g_M([\phi BX, \phi V), \phi(\phi U)] Y + g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y \\
&= g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y + g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y \\
&= g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y + g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y.
\end{align*}
\]

Since \( F \) is a Riemannian submersion and \( \phi V \in \Gamma((\ker F)^+) \), we get
\[
\begin{align*}
g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y + g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y
\end{align*}
\]

From (2.11) and (4.27), we get
\[
\begin{align*}
g_M([X, Y], V) &= -g_N((VF)(X, BY), F, \phi V) + g_M(\mathcal{A}_X Y, V) - g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y + 2g_M([\phi BX, \phi V), \phi(\phi U)] Y \\
&= -g_N((VF)(X, BY), F, \phi V) + g_M(\mathcal{A}_X Y, V) - g_M([\phi BX, \phi V), \phi(\phi U)] Y - g_M([\phi BX, \phi V), \phi(\phi U)] Y + 2g_M([\phi BX, \phi V), \phi(\phi U)] Y.
\end{align*}
\]
which proves (1) ⇔ (2). On the other hand, by (2.11), we have

\[(\nabla F)(Y, BX) - (\nabla F)(X, BY) = -F_\ast(V_Y BX - V_X BY)\]

Then, according to (2.7), we get

\[(\nabla F)(Y, BX) - (\nabla F)(X, BY) = -F_\ast(\mathcal{A}_Y BX - \mathcal{A}_X BY).\]

Notice \(\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((kerF)^\perp),\) this implies that (2) ⇔ (3). □

If \(\phi(kerF) = (kerF)^\perp,\) then we can get \(C = 0\) and \(TN = F_\ast(\phi(kerF)).\) We have the following corollary.

**Corollary 4.8.** Let \(F\) be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold \(M(\phi, \xi, \eta, g_M)\) to a Riemannian manifold \((N, g_N),\) and \(\phi(kerF) = (kerF)^\perp.\) Then the following assertions are equivalent to each other:

1. \((kerF)^\perp\) is integrable,
2. \((\nabla F)(X, \phi Y) - (\nabla F)(Y, \phi X) = 2F_\ast((\nabla_Y \phi)X),\)
3. \(\mathcal{A}_X \phi Y - \mathcal{A}_Y \phi X = -2H((\nabla_Y \phi)X)\) for \(X, Y \in \Gamma((kerF)^\perp).\)

**Theorem 4.9.** Let \(F\) be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold \(M(\phi, \xi, \eta, g_M)\) to a Riemannian manifold \((N, g_N),\) then the following criteria are equivalent:

1. \((kerF)^\perp\) defines a totally geodesic foliation on \(M,
2. \(g_M(\mathcal{A}_X BY, \phi V) = g_M((\nabla_X \phi)Y, \phi V) - g_M(CY, (\nabla_Y \phi)X)) + 2g_M(CY, \phi(\nabla_Y V)),\)
3. \(g_N(\nabla F_\ast(X, \phi Y), F_\ast \phi V) = -g_M((\nabla_X \phi)Y, \phi V) + g_M(CY, (\nabla_Y \phi)X)) - 2g_M(CY, \phi(\nabla_Y V)),\)

for \(X, Y \in \Gamma((kerF)^\perp), V \in \Gamma(kerF).\)

**Proof.** For \(X, Y \in \Gamma((kerF)^\perp), V \in \Gamma(kerF),\) by (3.15), we get

\[g_M(\nabla_X Y, V) = g_M((\nabla_X \phi)Y, \phi V) - g_M((\nabla_X \phi)Y, \phi V).\]

And using (2.7), (4.20) and (4.27), we have

\[g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY, \phi V) + g_M((\nabla_Y \phi)X, \phi V) - g_M((\nabla_X \phi)Y, \phi V)\]

\[= g_M(\mathcal{A}_X BY, \phi V) + g_M(CY, (\nabla_Y \phi)X) - g_M((\nabla_X \phi)Y, \phi V) - 2g_M(CY, \phi(\nabla_Y V)).\]

The above equation shows (1) ⇔ (2).

Since \(F\) is a Riemannian submersion and \(\phi V \in \Gamma((kerF)^\perp),\) we have

\[g_M(\mathcal{A}_X BY, \phi V) = g_M(\nabla_X BY, \phi V) = g_M(F_\ast \mathcal{A}_X BY, F_\ast \phi V).\]

Using (2.11) and (2.12), we get

\[g_M(\mathcal{A}_X BY, \phi V) = -g_N((\nabla F_\ast)(X, BY), F_\ast \phi V)\]

\[= -g_N((\nabla F_\ast)(X, \phi Y), F_\ast \phi V),\]

which shows that (2) ⇔ (3). □

**Corollary 4.10.** Let \(F\) be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold \(M(\phi, \xi, \eta, g_M)\) to a Riemannian manifold \((N, g_N)\) such that \(\phi(kerF) = (kerF)^\perp.\) Then the following assertions are equivalent to each other:

1. \((kerF)^\perp\) defines a totally geodesic foliation on \(M,
2. \(\mathcal{A}_X \phi Y = H((\nabla_X \phi)Y),\)
3. \((\nabla F)(X, \phi Y) = -F_\ast((\nabla_X \phi)Y)\) for \(X, Y \in \Gamma((kerF)^\perp).\)
Theorem 4.11. Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then the following criteria are equivalent:

1. $(\ker F)\perp$ defines a totally geodesic foliation on $M$.
2. $g_M((\nabla F)(V, \phi X), F, \phi W) + g_M(\phi W, (\nabla V\phi)X) = 0,$
3. $\mathcal{H}((\nabla V\phi)X) = \nabla V\mathcal{A}XV \in \Gamma(g_\mu),$

Here $X \in \Gamma((\ker F)^\perp), V, W \in \Gamma(\ker F).$

Proof. For $X \in \Gamma((\ker F)^\perp), V, W \in \Gamma(\ker F)$, since $\xi \in \Gamma(\ker F)$, by (2.6) and (4.25), it is easy to obtain $
 g_M(\nabla VX, \xi) = 0.$ Then by (3.15) and (2.6), we have

$$
g_M(\nabla VX, \xi) = -g_M(W, \nabla VX)$$

Since $[V, \phi X] \in \Gamma(\ker F), \phi W \in \Gamma(\ker F)^\perp,$ then $g_M([V, \phi X], \phi W) = 0.$ By (2.11), we have

$$
g_M(\nabla VX, \xi) = -g_N(F_\phi W, [\nabla V\phi X] + g_M(\phi W, (\nabla V\phi)X)$$

which shows (1) $\Leftrightarrow$ (2). Next, by some calculation, we get

$$
g_M((\nabla F)(V, \phi X), F, \phi W) = -g_M(\phi W, \nabla V\phi X).$$

Using (4.20), we have

$$
g_N((\nabla F)(V, \phi X), F, \phi W) = -g_M(\phi W, \nabla VBX + \nabla VX).$$

Hence, we have

$$
g_N((\nabla F)(V, \phi X), F, \phi W) = -g_M(\phi W, \nabla VBX + [V, CX] + \nabla VX).$$

Since $[V, CX] \in \Gamma(\ker F)$, using (2.5) and (2.7), we get

$$
g_N((\nabla F)(V, \phi X), F, \phi W) = -g_M(\phi W, \nabla VBX + \nabla VX).$$

This shows (2) $\Leftrightarrow$ (3). □

Corollary 4.12. Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $\phi(\ker F) = (\ker F)^\perp$. Then the following assertions are equivalent to each other:

1. $(\ker F)\perp$ defines a totally geodesic foliation on $M$.
2. $(\nabla F)(V, \phi X) + F_\phi((\nabla V\phi)X) = 0,$
3. $\mathcal{H}((\nabla V\phi)X) = \nabla V\phi X, \forall X \in \Gamma((\ker F)^\perp), V \in \Gamma(\ker F).$

We recall that a $C^\infty$ map $F$ between two Riemannian manifolds is called totally geodesic if $\nabla F = 0$. For an anti-invariant Riemannian submersion such that $\phi(\ker F) = (\ker F)^\perp$, we have the following theorem.

Theorem 4.13. Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $\phi(\ker F) = (\ker F)^\perp$. Then $F$ is a totally geodesic map if and only if

$$
\phi T_W\phi V + \mathcal{H}((\nabla W\phi)\phi V) = 0,$$

and

$$
\phi \mathcal{A}_X\phi W + \mathcal{H}((\nabla X\phi)\phi W) = 0,$$

for $V, W \in \Gamma(\ker F), X \in \Gamma((\ker F)^\perp).$
Proof. For \( V, W \in \Gamma(\ker F) \), \( X \in \Gamma((\ker F)^\perp) \), since \( \phi(\ker F) = (\ker F)^\perp \) and \( \xi \) is vertical, by (2.6) and (3.18), it is easy to obtain
\[
(\nabla F^*)(W, V) = F_*((\nabla_w \phi)^T W \phi V) + F_*((\nabla_w \phi)^T \phi V).
\]
(30)

One the other hand, by (3.14) and (2.11), we have
\[
F_*((\phi \nabla X \phi W) \phi V) = (\nabla F^*)(X, W) - F_*((\nabla X \phi)^T \phi W).
\]
Then, by (2.8), we get
\[
(\nabla F^*)(X, W) = F_*((\nabla X \phi)^T \phi W) + F_*((\phi A X \phi W)^T \phi W).
\]
(31)

Hence, proof comes from (2.12) (4.30) and (4.31).

Finally, we study the necessary and sufficient condition for an anti-invariant Riemannian submersion such that \( \phi(\ker F) = (\ker F)^\perp \) to be harmonic.

**Theorem 4.14.** Let \( F \) be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold \( (\phi, \xi, \eta, \tilde{M}) \) to a Riemannian manifold \( (N, g_N) \) such that \( \phi(\ker F) = (\ker F)^\perp \). Then \( F \) is harmonic if and only if trace \( (\phi^T V) \phi W \) = 0, for \( V \in \Gamma(\ker F) \).

Proof. From [6] we know that \( F \) is harmonic if and only if \( F \) has minimal fibres. Thus \( F \) is harmonic if and only if \( \sum_{i=1}^{k} T_{e_i} e_i = 0 \), where \( k \) denotes the dimension of \( \ker F \). On the other hand, by (3.15), we get
\[
\mathcal{H}(\phi V W) = \phi(V V W),
\]
(32)
for \( V, W \in \Gamma(\ker F) \). By (4.32) and some calculations, we obtain
\[
T_{e_i} W - \phi T_{e_i} W = V((\phi V) W).
\]
Then, by (3.17), we have
\[
\sum_{i=1}^{k} g_M(T_{e_i} e_i, V) = \sum_{i=1}^{k} g_M(\phi T_{e_i} e_i, V) + \sum_{i=1}^{k} g_M((\nabla e_i \phi) e_i, V) = - \sum_{i=1}^{k} g_M(T_{e_i} e_i, \phi V)
\]
for any \( V \in \Gamma(\ker F) \). And by (2.9), we get
\[
\sum_{i=1}^{k} g_M(\phi e_i, T_{e_i} V) = \sum_{i=1}^{k} g_M(T_{e_i} e_i, \phi V).
\]
By (2.3) and (3.14), we have
\[
\sum_{i=1}^{k} g_M(e_i, T_{e_i} V) = - \sum_{i=1}^{k} g_M(T_{e_i} e_i, \phi V).
\]
This completes the proof. \( \square \)
4.2. **Anti-invariant submersions admitting horizontal characteristic vector field.**

In this subsection, we will discuss anti-invariant submersions from a nearly-K-cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field $\xi$ is horizontal. Since $\phi \mu \subseteq \mu$, by (3.14), it is easy to obtain: $\mu = \phi \mu \oplus \{\xi\}$. For any horizontal vector field $X$, we write

$$\phi X = BX + CX,$$

where $BX \in \Gamma(ker F)$, $CX \in \Gamma(\mu)$.

Now we suppose that $X$ is horizontal and $V$ is vertical vector field. From $g_M(\phi V, CX) = 0$, we can obtain $g_N(F, \phi V, F, CX) = 0$, which implies that

$$TN = F_*(\phi(\ker F)) \oplus F_*(\mu).$$

By (3.14) and (4.33), we have the following proposition.

**Proposition 4.15.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then we have

$$BCX = 0, \quad C^2X = \phi^2X - \phi(BX), \quad C\phi V = 0, \quad C^3X + CX = 0, \quad B\phi V = -V,$$

where $X \in \Gamma((ker F)^\perp)$ and $V \in \Gamma(ker F)$.

By (3.14), it is easy to get

$$\nabla_X Y = -\phi(\nabla_X \phi Y) + \phi((\nabla_X \phi)Y) + \eta(\nabla_X Y)\xi, \quad \forall X, Y \in \Gamma((ker F)^\perp)$$

**Lemma 4.16.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then we have

$$\mathcal{A}_X \xi = 0,$$

$$T_U \xi = 0,$$

$$g_M(CX, \phi U) = 0,$$

$$g_M(\nabla_X CY, \phi U) = g_M(CY, (\nabla_U \phi)X) - g_M(CY, \phi \mathcal{A}_X U),$$

where $X, Y \in \Gamma((ker F)^\perp)$, $U \in \Gamma(ker F)$.

**Proof.** Assume that $X, Y \in \Gamma((ker F)^\perp)$, $U \in \Gamma(ker F)$. By (2.8), (2.6) and (3.18), we obtain (4.36) and (4.37).

Using (3.15) and (4.33), $\eta \cdot \phi = 0$, since $\phi BX, \xi \in \Gamma((ker F)^\perp)$, $U \in \Gamma(ker F)$, we have

$$g_M(CX, \phi U) = g_M(\phi X - BX, \phi U) = g_M(\phi X, \phi U) - \eta(\phi X)\eta(U) + g_M(\phi BX, \phi(\phi U)) = g_M(\phi BX, -U + \eta(U)\xi) = 0$$

For $X, Y \in \Gamma((ker F)^\perp)$, $U \in \Gamma(ker F)$, by (3.19) we have

$$g_M(\nabla_X CY, \phi U) = -g_M(CY, \nabla_X (\phi U)) = -g_M(CY, (\nabla_U \phi)X + \phi(\nabla_U U)) = g_M(CY, (\nabla_U \phi)X) - g_M(CY, \phi(\nabla_U U)).$$

Since $\phi(\nabla_U U) \in \phi(ker F)$, by (2.7), we have

$$g_M(\nabla_X CY, \phi U) = g_M(CY, (\nabla_U \phi)X) - g_M(CY, \phi \mathcal{A}_X U) - g_M(CY, \phi(\nabla_U U)) = g_M(CY, (\nabla_U \phi)X) - g_M(CY, \phi \mathcal{A}_X U).$$

$\square$
Next, we study the integrability of the horizontal distribution and then we investigate the geometry of leaves of $\ker F$, and $(\ker F)^\perp$.

**Theorem 4.17.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other:

1. $(\ker F)^\perp$ is integrable,
2. $g_N((\nabla F_\ast)(Y, BX), F, \phi V) = g_N((\nabla F_\ast)(X, BY), F, \phi V) - g_M(CY, (\nabla \phi) X) + g_M(CX, (\nabla \phi) Y) - 2g_M((\nabla \phi) X, \phi V) + g_M(CY, \phi, \mathcal{A}_V) - g_M(CX, \phi, \mathcal{A}_V),$ 
3. $g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) = -g_M(CY, (\nabla \phi) X) + g_M(CX, (\nabla \phi) Y) - 2g_M((\nabla \phi) X, \phi V) + g_M(CY, \phi, \mathcal{A}_V) - g_M(CX, \phi, \mathcal{A}_V),$ 

for $X, Y \in \Gamma((\ker F)^\perp), V \in \Gamma(\ker F).$

**Proof.** For $X, Y \in \Gamma((\ker F)^\perp), V \in \Gamma(\ker F)$, we have

$$g_M([X, Y], V) = g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) = g_M(\phi \nabla_X Y, \phi V) - g_M(\phi \nabla_Y X, \phi V).$$

Then from (4.20), we have

$$g_M([X, Y], V) = g_M(\nabla_X \phi Y, \phi V) - g_M((\nabla_X \phi) Y, \phi V) + g_M((\nabla_Y \phi) X, \phi V)$$

$$= g_M(\nabla_X \phi Y, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) - g_M(\nabla_Y CX, \phi V).$$

Since $F$ is a Riemannian submersion and $\phi V \in \Gamma((\ker F)^\perp)$, we get

$$g_M(\nabla_X \phi Y, \phi V) = g_N(F, \nabla_X BY, F, \phi V), g_M(\nabla_Y BX, \phi V) = g_N(F, \nabla_Y BX, F, \phi V).$$

From (2.11) and (4.39), we get

$$g_M([X, Y], V) = -g_M((\nabla F)(BY, X), F, \phi V) + g_M(CY, (\nabla \phi) X)$$

$$- g_M(CX, (\nabla \phi) Y) + 2g_M((\nabla \phi) X, \phi V)$$

$$- g_M(CY, \phi, \mathcal{A}_V) + g_M(CX, \phi, \mathcal{A}_V)$$

$$+ g_M((\nabla F)(BX, Y), F, \phi V)$$

which proves (1) $\Leftrightarrow$ (2). On the other hand, by (2.11), we have

$$(\nabla F)(Y, BX) - (\nabla F)(X, BY) = -F_\ast(\nabla_Y BX - \nabla_X BY).$$

Then, according to (2.7), we get

$$(\nabla F)(Y, BX) - (\nabla F)(X, BY) = -F_\ast(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

Notice $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((\ker F)^\perp)$, this implies that (2) $\Leftrightarrow$ (3). 

**Remark 4.18.** If $(\ker F)^\perp = \phi(\ker F) \oplus \{\xi\}$, then we can get $CX = 0$ for $X \in \Gamma((\ker F)^\perp)$.

**Corollary 4.19.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(\ker F)^\perp = \phi(\ker F) \oplus \{\xi\}$. Then the following assertions are equivalent to each other:
1. $(\ker F)^⊥$ is integrable,
2. $(\nabla F)(Y, \phi X) - (\nabla F)(X, \phi Y) = -2F,((\nabla \phi)X)$,
3. $\mathcal{A}_X\phi Y - \mathcal{A}_Y\phi X = -2\mathcal{H}((\nabla \phi)X)$,

for $X, Y \in \Gamma((\ker F)^⊥)$.

**Theorem 4.20.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then the following criteria are equivalent:

1. $(\ker F)^⊥$ defines a totally geodesic foliation on $M$,
2. $g_M(\mathcal{A}_XBY, \phi V) = g_M((\nabla_X\phi)Y, \phi V) - g_M(CY, (\nabla_V\phi)X) + g_M(CY, \phi \mathcal{A}_XV)$,
3. $g_N(\nabla F, (X, \phi Y), F, \phi V) = -g_M((\nabla_X\phi)Y, \phi V) + g_M(CY, (\nabla_V\phi)X) - g_M(CY, \phi \mathcal{A}_XV)$,

for $X, Y \in \Gamma((\ker F)^⊥); V \in \Gamma(\ker F)$.

**Proof.** For $X, Y \in \Gamma((\ker F)^⊥), V \in \Gamma(\ker F)$, by (3.15), we get

$$g_M(\nabla_XY, V) = g_M((\nabla_X\phi)Y, \phi V) - g_M((\nabla_X\phi)Y, \phi V).$$

And using (2.7), (4.33) and (4.39), we have

$$g_M(\nabla_XY, V) = g_M(\mathcal{A}_XBY, \phi V) + g_M((\nabla_X\phi)Y, \phi V) - g_M((\nabla_X\phi)Y, \phi V) - g_M(CY, (\nabla_V\phi)X) - g_M(CY, \phi \mathcal{A}_XV).$$

The above equation shows (1) $\Leftrightarrow$ (2). Since $F$ is a Riemannian submersion and $\phi V \in \Gamma((\ker F)^⊥)$, we have

$$g_M(\mathcal{A}_XBY, \phi V) = g_M((\nabla_X\phi)Y, \phi V) = g_M(F, \nabla_XBY, F, \phi V).$$

Using (2.11) and (2.12), we get

$$g_M(\mathcal{A}_XBY, \phi V) = -g_M((\nabla_XBY), F, \phi V) = -g_M((\nabla F)(X, \phi Y), F, \phi V),$$

which shows that (2) $\Leftrightarrow$ (3). $\square$

**Corollary 4.21.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(\ker F)^⊥ = \phi(\ker F) \oplus \{\xi\}$. Then the following assertions are equivalent to each other:

1. $(\ker F)^⊥$ defines a totally geodesic foliation on $M$,
2. $\mathcal{A}_X\phi Y = \mathcal{H}((\nabla_X\phi)Y)$,
3. $\nabla F(X, \phi Y) = -F_1((\nabla_X\phi)Y),$

for $X, Y \in \Gamma((\ker F)^⊥)$.

For the vertical distribution $\ker F$, we have:

**Theorem 4.22.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other:

1. $(\ker F)$ defines a totally geodesic foliation on $M$,
2. $g_N((\nabla F)(V, \phi X), F, \phi W) + g_M(\phi W, (\nabla_V\phi)X) = 0$,
3. $\mathcal{H}((\nabla_V\phi)X) - \mathcal{T}_V^BX - \mathcal{A}_CXV \in \Gamma(\mu),$

for $X \in \Gamma((\ker F)^⊥); V, W \in \Gamma(\ker F)$.
Proof. For $X \in \Gamma((\ker F)^{-})$, $V, W \in \Gamma(\ker F_{*})$, since $\xi \in \Gamma((\ker F_{*})^{-})$, we have $g_{M}(W, \xi) = 0$. Then by $g_{M}(W, X) = 0$, we have $g_{M}(V_{V}W, X) = -g_{M}(W, V_{V}X)$. By (2.6) and (3.15), we obtain
\[ g_{M}(V_{V}W, X) = -g_{M}(W, V_{V}X) = -g_{M}(\phi W, \mathcal{H}V_{V}\phi X) + g_{M}(\phi W, (V_{V}\phi)X). \]
Since $[V, \phi X] \in \Gamma(\ker F_{*})$, $\phi W \in \Gamma((\ker F_{*})^{-})$, then $g_{M}([V, \phi X], \phi W) = 0$. By (2.11), we have
\[ g_{M}(V_{V}W, X) = -g_{N}(F_{*}\phi W, F_{*}\mathcal{H}V_{V}\phi X) + g_{M}(\phi W, (V_{V}\phi)X) = g_{N}([V, \phi X], F_{*}\phi W) + g_{M}(\phi W, (V_{V}\phi)X), \]
which shows (1) $\iff$ (2). Next, by some calculation, we get
\[ g_{N}([V, \phi X], F_{*}\phi W) = -g_{M}(\phi W, V_{V}\phi X). \]

Using (4.33), we have
\[ g_{N}([V, \phi X], F_{*}\phi W) = -g_{M}(\phi W, V_{V}BX + V_{V}CX). \]

Hence, we have
\[ g_{N}([V, \phi X], F_{*}\phi W) = -g_{M}(\phi W, V_{V}BX + [V, CX] + V_{CX}V). \]
Since $[V, CX] \in \Gamma((\ker F_{*})^{-})$, using (2.5) and (2.7), we get
\[ g_{N}([V, \phi X], F_{*}\phi W) = -g_{M}(\phi W, T_{V}BX + \mathcal{A}_{CX}V). \]
This shows (2) $\iff$ (3). \( \square \)

**Corollary 4.23.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_{M})$ to a Riemannian manifold $(N, g_{N})$ such that $(\ker F_{*})^{-} = \phi(\ker F_{*}) \oplus \{\xi\}$. Then the following assertions are equivalent to each other:

1. $(\ker F_{*})^{-}$ defines a totally geodesic foliation on $M$,
2. $(\mathcal{V}F)$($V, \phi X$) + $F_{*}(\mathcal{V}_{V}\phi X) = 0$,
3. $\mathcal{H}((\mathcal{V}_{V}\phi)X) = T_{V}\phi X$,

for $X \in \Gamma((\ker F_{*})^{-})$, $V \in \Gamma(\ker F_{*})$.

**Theorem 4.24.** Let $F$ be an anti-invariant Riemannian submersion from a nearly-$K$-cosymplectic manifold $M(\phi, \xi, \eta, g_{M})$ to a Riemannian manifold $(N, g_{N})$ such that $(\ker F_{*})^{-} = \phi(\ker F_{*}) \oplus \{\xi\}$. Then $F$ is a totally geodesic map if and only if
\[ \phi T_{W}\phi V + \mathcal{H}((\mathcal{V}_{W}\phi)\phi V) = 0, \tag{40} \]
and
\[ \phi \mathcal{A}_{X}\phi W + \mathcal{H}((\mathcal{V}_{X}\phi)\phi W) = 0, \tag{41} \]
for $V, W \in \Gamma(\ker F_{*})$, $X \in \Gamma((\ker F_{*})^{-})$.

Proof. \( \forall X \in \Gamma((\ker F_{*})^{-}) \) put $X = \phi X_{1} + a\xi$, $X_{1} \in \ker F_{*}, a \in R$, then we have
\[ F_{*}\phi(X) = F_{*}\phi(\phi X_{1} + a\xi) = F_{*}(X_{1} - \eta(X_{1})\xi) = 0. \]
Thus
\[ F_{*}\phi(X) = 0, \forall X \in \Gamma((\ker F_{*})^{-}). \]
For $V, W \in \Gamma(\ker F^\ast)$, $X \in \Gamma((\ker F^\ast)^\perp)$, by (2.6) and (3.18), it is easy to obtain
\[(\nabla F^\ast)(W, V) = F_\ast(\phi T_W \phi V) + F_\ast((\nabla_W \phi) \phi V). \tag{43}\]

One the other hand, by (3.14) and (2.11), we have
\[F_\ast(\phi \nabla_X \phi W) = (\nabla F^\ast)(X, W) - F_\ast((\nabla_X \phi) \phi W). \tag{44}\]

Hence, proof comes from (2.12) (4.43) and (4.44). □

Finally, we study the necessary and sufficient condition for anti-invariant Riemannian submersion such that $(\ker F^\ast)^\perp = \phi(\ker F^\ast) \oplus \{\xi\}$ to be harmonic.

**Theorem 4.25.** Let $F$ is an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(\ker F^\ast)^\perp = \phi(\ker F^\ast) \oplus \{\xi\}$. Then $F$ is harmonic if and only if trace$(\phi T_V) = 0$, for $V \in \Gamma(\ker F^\ast)$.

**Proof.** From [6] we know that $F$ is harmonic if and only if $F$ has minimal fibres. Thus $F$ is harmonic if and only if $\sum_{i=1}^k T_e_i e_i = 0$, where $k$ denotes the dimension of $\ker F^\ast$. On the other hand, by (3.15), we get
\[H(\phi \nabla_V W) = \phi(\nabla_V W), \tag{45}\]
for $V, W \in \Gamma(\ker F^\ast)$. By (4.45) and some calculations, we obtain
\[T_V \phi W - \phi T_V W = \nabla((\nabla_V \phi) W). \]

Then, by (3.17), we have
\[\sum_{i=1}^k g_M(T_e_i, e_i, V) = \sum_{i=1}^k g_M(\phi T_e_i, e_i, V) + \sum_{i=1}^k g_M((\nabla_e \phi) e_i, V) = -\sum_{i=1}^k g_M(T_e_i, e_i, \phi V)\]
for any $V \in \Gamma(\ker F^\ast)$. And by (2.9), we get
\[\sum_{i=1}^k g_M(\phi e_i, T_e_i V) = \sum_{i=1}^k g_M(T_e_i e_i, \phi V). \]

By (2.3) and (3.14), we have
\[\sum_{i=1}^k g_M(e_i, \phi T_e_i e_i) = -\sum_{i=1}^k g_M(T_e_i e_i, \phi V). \]

This completes the proof. □
References