On the \((b, c)\)–Inverse in Rings

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Abstract. We present new characterizations for the existence of the \((b, c)\)–inverse in a ring. The set of all \((b, c)\)–invertible elements is described too. Necessary and sufficient conditions which ensure that the \((b, c)\)–inverse of a given element commutes with that element are investigated. As an application of these results, we obtain new characterizations for the existence of the image-kernel \((p, q)\)–inverse.

1. Introduction

Let \(\mathcal{R}\) be an associative ring with the unit 1. The sets of all idempotents and invertible elements of \(\mathcal{R}\) will be denoted by \(\mathcal{R}^e\) and \(\mathcal{R}^{-1}\), respectively.

An element \(a \in \mathcal{R}\) is called regular if there exists \(x \in \mathcal{R}\) satisfying \(axa = a\). In this case, \(x\) is an inner inverse of \(a\). The set of all inner inverses of \(a\) will be denoted by \(a\{1\}\).

Let \(p, q \in \mathcal{R}^e\), \(p \neq q\). Then \(p\mathcal{R}p\) is a ring with the unit \(p\) and we can talk about invertibility of its elements. Since \(p\mathcal{R}q\) does not have a unit, we will talk about invertibility of its elements only in the following sense: let \(p, q \in \mathcal{R}^e\), an element \(a \in \mathcal{R}\) is \((-p, q)\)–invertible if there exists \(a' \in q\mathcal{R}p\) such that

\[
\begin{align*}
a \in p\mathcal{R}q, &\quad aa' = p \quad \text{and} \quad a'a = q.
\end{align*}
\]

If the \((-p, q)\)–inverse \(a'\) of \(a\) exists, it is unique and denoted by \(a^{(-p,q)}\). By \(\mathcal{R}^{(-p,q)}\) will be denoted the set of all \((-p, q)\)–invertible elements of \(\mathcal{R}\).

Lemma 1.1. Let \(a \in \mathcal{R}\). There exist \(p, q \in \mathcal{R}^e\) such that \(a\) is \((-p, q)\)–invertible if and only if \(a\) is regular.

For \(a \in \mathcal{R}\), if \(xax = x\) holds for some \(x \in \mathcal{R}\{0\}\), then \(x\) is an outer generalized inverse of \(a\). The outer inverse is not unique in general, but it is unique if we fix the corresponding idempotents [3]: let \(a \in \mathcal{R}\), and let \(p, q \in \mathcal{R}^e\). An element \(x \in \mathcal{R}\) satisfying

\[
\begin{align*}
xax = x, &\quad xa = p \quad \text{and} \quad 1 - ax = q.
\end{align*}
\]
will be called \((p, q)\)-outer generalized inverse of \(a\), written \(x = a^{(2)}_{p, q}\). If \(a^{(2)}_{p, q}\) exists, it is unique. Note that, for \(a \in \mathcal{R}\) and \(p, q \in \mathcal{R}^*\), \(a^{(2)}_{p, q}\) exists if and only if \((1 - q)a = (1 - q)ap\) and there exists some \(x \in \mathcal{R}\) such that \(px = x, xq = 0, xap = p\) and \(ax = 1 - q\) [3]. If \(a^{(2)}_{p, q}\) satisfies \(aa^{(2)}_{p, q}a = a\), then \(a^{(2)}_{p, q} = a^{(1, 2)}_{p, q}\) is called a \((p, q)\)-reflexive generalized inverse of \(a\).

Instead of prescribing the idempotents \(ax\) and \(xa\), we may prescribe certain kernel and image ideals related to these idempotents [6]: let \(p, q \in \mathcal{R}^*\), an element \(x \in \mathcal{R}\) is the image-kernel \((p, q)\)-inverse of \(a\) if

\[
xxa = x, \quad xax = yR = pR \quad \text{and} \quad (1 - ax)R = qR.
\]

The image-kernel \((p, q)\)-inverse \(x\) is unique if it exists, and it will be denoted by \(a^{x}_{p, q}\). We use \(\mathcal{R}^{(p, q)}_{p, q}\) to denote the set of all image-kernel \((p, q)\)-invertible elements of \(\mathcal{R}\).

**Theorem 1.2.** [8, Theorem 2.1] Let \(p, q \in \mathcal{R}^*\) and \(a \in \mathcal{R}\). Then the following statements are equivalent:

(i) \(a^{x}_{p, q}\) exists,

(ii) there exists some \(x \in \mathcal{R}\) such that

\[
x = px, \quad xap = p, \quad xq = 0, \quad 1 - q = (1 - q)ax.
\]

Observe that element \(x\) in the part (ii) of Theorem 1.2 satisfies \(x = a^{x}_{p, q}\). The image-kernel \((p, q)\)-inverse of Kantún-Montiel [6] coincides with the \((p, q, l)\)-outer generalized inverse of Cao and Xue [2].

Drazin [4] introduced the following class of outer generalized inverses: let \(b, c \in \mathcal{R}\), an element \(a \in \mathcal{R}\) is \((b, c)\)-invertible if there exists \(y \in \mathcal{R}\) such that

\[
y \in (bRy) \cap (yRc), \quad yab = b \quad \text{and} \quad cay = c.
\]

The \((b, c)\)-inverse \(y\) of \(a\) satisfies \(yay = y\), it is unique (if exists) and denoted by \(a^{y}_{b, c}\) [4]. We will use \(\mathcal{R}^{(b, c)}\) to denote the set of all \((b, c)\)-invertible elements of \(\mathcal{R}\).

**Lemma 1.3.** [9] Let \(a, b, c \in \mathcal{R}\). If \(a\) has a \((b, c)\)-inverse, then \(b, c\) and \(cab\) are regular.

The special type of outer inverse is a group inverse. An element \(a \in \mathcal{R}\) is group invertible if there is \(a^g \in \mathcal{R}\) such that

\[
ba^g a = a, \quad a^g a^g = a^g \quad \text{and} \quad aa^g = a^g a.
\]

The group inverse \(a^g\) of \(a\) is uniquely determined by these equations. Denote by \(\mathcal{R}^g\) the set of all group invertible elements of \(\mathcal{R}\). The spectral idempotent of \(a \in \mathcal{R}^g\) is the element \(a^s = 1 - aa^g\).

In this paper, we investigate some properties of the \((b, c)\)-inverse in a ring. Precisely, some new equivalent conditions for the existence of the \((b, c)\)-inverse are presented. We fully characterize the set of all \((b, c)\)-invertible elements. Also, several characterizations for the \((b, c)\)-inverse of a given element to commute with that element are given. We consider too the \((b, c)\)-inverse of a given element which is an inner inverse of that element. As an application of our results, we get new characterizations for the existence of the \(\mathcal{R}^{(p, q)}\)-inverse in a ring.

2. The \((b, c)\)-inverse in rings

In this section, we give new characterizations of the existence of the \((b, c)\)-inverse in a ring.

**Theorem 2.1.** Let \(a, b, c \in \mathcal{R}\). Then

(a) \(a\) is \((b, c)\)-invertible if and only if \(b, c\) are regular and, for \(b^{-} \in b[1]\) and \(c^{-} \in c[1]\), one of the following equivalent statements holds:

\[
\text{(i) \(cabb^{-}\) is \((bb^{-}, 1 - cc^{-})\)-reflexive generalized invertible,}
\]

The image-kernel \((p, q)\)-inverse \(x\) is unique if it exists, and it will be denoted by \(a^{x}_{p, q}\). We use \(\mathcal{R}^{(p, q)}_{p, q}\) to denote the set of all image-kernel \((p, q)\)-invertible elements of \(\mathcal{R}\).

**Theorem 1.2.** [8, Theorem 2.1] Let \(p, q \in \mathcal{R}^*\) and \(a \in \mathcal{R}\). Then the following statements are equivalent:

(i) \(a^{x}_{p, q}\) exists,

(ii) there exists some \(x \in \mathcal{R}\) such that

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x = px, \quad xap = p, \quad xq = 0, \quad 1 - q = (1 - q)ax.
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Observe that element \(x\) in the part (ii) of Theorem 1.2 satisfies \(x = a^{x}_{p, q}\). The image-kernel \((p, q)\)-inverse of Kantún-Montiel [6] coincides with the \((p, q, l)\)-outer generalized inverse of Cao and Xue [2].

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The group inverse \(a^g\) of \(a\) is uniquely determined by these equations. Denote by \(\mathcal{R}^g\) the set of all group invertible elements of \(\mathcal{R}\). The spectral idempotent of \(a \in \mathcal{R}^g\) is the element \(a^s = 1 - aa^g\).

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**Theorem 2.1.** Let \(a, b, c \in \mathcal{R}\). Then

(a) \(a\) is \((b, c)\)-invertible if and only if \(b, c\) are regular and, for \(b^{-} \in b[1]\) and \(c^{-} \in c[1]\), one of the following equivalent statements holds:

\[
\text{(i) \(cabb^{-}\) is \((bb^{-}, 1 - cc^{-})\)-reflexive generalized invertible,}
\]
(ii) \(cabb^-\) is \((-\cdot cc^-, bb^-)\)-invertible.

(b) \(a\) is \((b,c)\)-invertible if and only if \(b, c\) are regular and, for \(b^- \in b[1]\) and \(c^- \in c[1]\), one of the following equivalent statements holds:

(i) \(c^- cab\) is \((b^-b, 1 - c^-c)\)-reflexive generalized invertible,

(ii) \(c^- cab\) is \((-\cdot c^-c, b^-b)\)-invertible.

In addition, if one of the previous statements holds, then

\[
d^{(b,c)} = (cabb^-)^{(1,2)}_{bb^-,1-cc^-} = b (c^- cab)^{(1,2)}_{b^-b,1-c^-c},
\]

\[
(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = d^{(b,c)} c^- = (cabb^-)^{-cc^-bb^-},
\]

\[
(c^- cab)^{(1,2)}_{b^-b,1-c^-c} = b^- d^{(b,c)} = (c^- cab)^{-c^-cb^-b}.
\]

Proof. (a) Suppose that \(a\) is \((b,c)\)-invertible and \(y\) is the \((b,c)\)-inverse of \(a\). Then \(y = bt y = ysc\), for some \(t, s \in \mathbb{R}, yab = b, cay = c\) and, by Lemma 1.3, \(b, c\) are regular. For \(bb^- \in b[1]\) and \(cc^- \in c[1]\), notice that \(cabb^-\) is \((bb^-, 1 - cc^-)\)-reflexive generalized invertible and \(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = yc^-:

\[
yc^-cabb^- = ysc^-cabb^- = yab^- = bb^-cabb^- = yc^-cabb^- = cbb^- yc^- = cayc^- = cc^-cabb^- = yc^-cabb^- = cbb^-.
\]

So, the condition (i) is satisfied. Since \(cabb^- = cc^- cabb^- \in cc^- Rbb^-\) and \(yc^- = bb^- btscc^- \in bb^- Rcc^-\), we deduce that (ii) holds and \((cabb^-)^{-cc^-bb^-} = yc^-\).

Let \(b, c\) be regular, \(b^- \in b[1]\) and \(c^- \in c[1]\). If the statement (i) holds, that is, \(cabb^-\) is \((bb^-, 1 - cc^-)\)-reflexive generalized invertible and \(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = x\), then we verify that \(y = xc\) is the \((b,c)\)-inverse of \(a\):

\[
y = xc = bb^- xc = bb^- y \in bRy
\]

\[
y = xc = xcc^- c = yc^- c \in yRc,
\]

\[
yab = xcb = xcab = bb^-b = b,
\]

\[
cay = caxc = cabb^- xc = cc^-c = c.
\]

In the same way, by condition (ii), we conclude that \(a\) is \((b,c)\)-invertible.

Similarly, we check that (b) is satisfied. \(\square\)

As a consequence of Theorem 2.1, we obtain the next results. The first of them recovers [1, Theorem 4.1].

**Corollary 2.2.** Let \(a, b, c \in \mathbb{R}\). Suppose that \(b, c\) are regular, \(b^- \in b[1]\) and \(c^- \in c[1]\).

(a) If \(bb^- = cc^-\), then the following statements are equivalent:

(i) \(a\) is \((b,c)\)-invertible,

(ii) \(cabb^- \in \mathbb{R}\) and \((cabb^-)^n = 1 - bb^-\),

(iii) \(cabb^- \in (bb^- Rbb^-)^{-1}\).

(b) If \(c^-c = b^-b\), then the following statements are equivalent:

(i) \(a\) is \((b,c)\)-invertible,
Proof. The part (ii) follows similarly. Using (i) and (ii), we get that (iii) holds.

One of the following equivalent statements holds:

(i) \( a b b^* \) is \((b,c)\)-invertible,
(ii) \(c^-ca\) is \((b,c)\)-invertible,
(iii) \(c^-cab\) is \((b,c)\)-invertible.

In addition, if one of the previous statements holds, then

\[
a^{(b,c)} = (a b b^*)^{(b,c)} = (c^-ca)^{(b,c)} = (c^-cab)^{c^-c}.\]

Applying Corollary 2.3, we prove the following result.

**Corollary 2.4.** Let \( a, b, c \in \mathcal{R} \). If \( a \) is \((b,c)\)-invertible and \( x, y \in \mathcal{R} \), then the following statements hold for \( b^- \in b[1] \) and \( c^- \in c[1] \):

(i) \( a + x(1 - bb^-) \) is \((b,c)\)-invertible,
(ii) \( a + (1 - c^-c)y \) is \((b,c)\)-invertible,
(iii) \( a + x(1 - bb^-) + (1 - c^-c)y \) is \((b,c)\)-invertible.

In addition,

\[
a^{(b,c)} = (a + x(1 - bb^-))^{(b,c)} = (a + (1 - c^-c)y)^{(b,c)} = (a + x(1 - bb^-) + (1 - c^-c)y)^{(b,c)}.\]

Proof. Since \( a \) is \((b,c)\)-invertible, by Corollary 2.3, we deduce that \( a b b^* = (a + x(1 - bb^-))b b^- \) is \((b,c)\)-invertible. The part (ii) follows similarly. Using (i) and (ii), we get that (iii) holds. \( \square \)

More characterizations for the existence of the \((b,c)\)-inverse are presented in the next result.

**Theorem 2.5.** Let \( a, b, c \in \mathcal{R} \). Then \( a \) is \((b,c)\)-invertible if and only if \( b, c \) are regular and, for \( b^- \in b[1] \) and \( c^- \in c[1] \), one of the following equivalent statements holds:

(i) \( a \) is \((bb^*, c^-c)\)-invertible,
(ii) \( a \) is image-kernel \((bb^*, 1 - c^-c)\)-invertible.

In addition, if one of the previous statements holds, then

\[
a^{(b,c)} = a^{(bb^*, c^-c)} = a_{bb^*}^{(b,c)}.\]

Proof. Let \( a \) be \((b,c)\)-invertible and \( y = a^{(b,c)} \). Since \( y = bty = ysc \), for some \( t, s \in \mathcal{R} \), \( yab = b, cay = c \) and \( b, c \) are regular, for \( b^- \in b[1] \) and \( c^- \in c[1] \), we obtain

\[
y = b b^-b t y = y s c c, \quad y a b b^- = b b^-, \quad c^-c a y = c^-c, \quad \quad \text{(1)}
\]

i.e. \( a \) is \((bb^*, c^-c)\)-invertible and \( y = a^{(bb^*, c^-c)} \). Hence, the statement (i) is satisfied.

By part (i), we have that \( y = a^{(bb^*, c^-c)} \) satisfies (1). Thus,

\[
bb^-y = y, \quad yab b^- = b b^-, \quad y(1 - c^-c) = 0, \quad c^-c a y = c^-c. \quad \text{(2)}
\]

So, by Theorem 1.2(ii), we observe that (ii) holds, that is, \( a \) is image-kernel \((bb^*, 1 - c^-c)\)-invertible and \( a_{bb^*}^{(b,c)} = y \).

Suppose that \( b, c \) are regular and (ii) holds, for \( b^- \in b[1] \) and \( c^- \in c[1] \). Set \( y = a_{bb^*}^{(b,c)} \). Using (2), we have that \( a \) is \((b,c)\)-invertible and \( y = a^{(b,c)} \). \( \square \)
Now, we fully describe the set \( R^{(b,c)} \). The following result recovers [1, Theorem 5.1].

**Theorem 2.6.** Let \( b, c \in \mathcal{R} \) be regular, \( b^- \in b[1] \) and \( c^- \in c[1] \).

(i) Then
\[
R^{(b,c)} = c^- R^{(c^- b^-)} + (1 - c^- c)Rbb^- + R(1 - bb^-).
\]
In addition, for \( x, y \in \mathcal{R} \) and \( u \in \mathcal{R}^{(c^- b^-)} \),
\[
(c^- u)^{[(b,c)]} = (c^- u + (1 - c^- c)xb^- + y(1 - bb^-))^R = u^{(c^- b^-) c}.
\]

(ii) Also,
\[
R^{(b,c)} = R^{(-c^- b^-)} b^- + c^- cR(1 - b^- b) + (1 - c^- c)R.
\]
In addition, for \( x, y \in \mathcal{R} \) and \( v \in \mathcal{R}^{(-c^- b^-)} \),
\[
(vb^-)^{[(b,c)]} = (vb^- + c^- cx(1 - bb^-) + (1 - c^- c)y)^R = b^{(-c^- b^-)} b.
\]

**Proof.** (i) If \( a \in R^{(b,c)} \), then
\[
a = c^- cabb^- + (1 - c^- c)abbb^- + a(1 - bb^-).
\]
By Theorem 2.1, we have that \( cabb^- \in R^{(c^- b^-)} \) and so \( a \in c^- R^{(c^- b^-)} + (1 - c^- c)Rbb^- + R(1 - bb^-) \).

Conversely, assume that \( u \in \mathcal{R}^{(-c^- b^-)} \) and \( a = c^- u \). Since \( cabb^- = c^- ubbb^- = u \in \mathcal{R}^{(-c^- b^-)} \), by Theorem 2.1, we conclude that \( a \in R^{(b,c)} \) and \( a^{[(b,c)]} = u^{(-c^- b^-)} R \). Using Corollary 2.4, notice that \( a + (1 - c^- c)xbbb^- + y(1 - bb^-) \) \( R^{(b,c)} \) and \( a^{[(b,c)]} = (a + (1 - c^- c)xbbb^- + y(1 - bb^-))^R \).

(ii) In the same manner as (i), we verify this part. \( \square \)

Necessary and sufficient conditions which involve the corresponding outer inverses of products \( ab, ca \) or \( cab \), for the existence and representation of \( a^{[(b,c)]} \) are given too.

**Theorem 2.7.** Let \( a, b, c \in \mathcal{R} \). Then

(i) \( a \) is \( (b, c) \)–invertible if and only if \( b \) is regular and, for \( b^- \in b[1] \), \( (ab) \) is \( (b^- b, c) \)–invertible. Moreover,
\[
(ab)^{[(b^- b,c)]} = b^{-1} a^{[(b,c)]} \quad \text{and} \quad a^{[(b,c)]} = b(ab)^{[(b^- b,c)]}.
\]

(ii) \( a \) is \( (b, c) \)–invertible if and only if \( c \) is regular and, for \( c^- \in c[1] \), \( (ca) \) is \( (b, cc^-) \)–invertible. Moreover,
\[
(ca)^{[(b,cc^-)]} = a^{[(b,c)]} c^- \quad \text{and} \quad a^{[(b,c)]} = (ca)^{[(b,cc^-)]} c.
\]

(iii) \( a \) is \( (b, c) \)–invertible if and only if \( b, c \) are regular and, for \( b^- \in b[1] \) and \( c^- \in c[1] \), \( (cab) \) is \( (b^- b, cc^-) \)–invertible. Moreover,
\[
(cab)^{[(b^- b,cc^-)]} = b^{(-1)} a^{[(b,c)]} c^- \quad \text{and} \quad a^{[(b,c)]} = b(cab)^{[(b^- b,cc^-)]} c.
\]

**Proof.** (i) \( \Rightarrow \): Because \( a^{[(b,c)]} = bta^{[(b,c)]} = d^{[(b,c)]} s, \) for some \( t, s \in \mathcal{R} \), then
\[
b^- a^{[(b,c)]} = b^- bta^{[(b,c)]} = b^- bbb^- a^{[(b,c)]} \in b^- bRb^- a^{[(b,c)]},
\]
\[
b^- a^{[(b,c)]} = b^- a^{[(b,c)]} sc \in b^- a^{[(b,c)]} R c,
\]
\[
b^- a^{[(b,c)]} = b^- a^{[(b,c)]} c b = b^- a^{[(b,c)]} c = b^- b,
\]
\[
cabb^- a^{[(b,c)]} = c = c,
\]
that is, \( ab^{[(b^- b,c)]} = b^- a^{[(b,c)]} \).

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\[\text{Theorem 2.9}.
\]
\[\tag{ii}
\]
Proof. Hence, for some \(t \in \mathcal{R}\), \(b^{-1}b = (ab)(b^{-1}b)\) and \(cab\)(\(b^{-1}b\)) = \(c\), we get
\[b(ab)(b^{-1}b) = bb^{-1}b(ab)(b^{-1}b) = bt1b^{-1}b(ab)(b^{-1}b) \in \mathcal{R}b(ab)(b^{-1}b),
\]
\[b(ab)(b^{-1}b) = b(ab)(b^{-1}b)b1c \in b(ab)(b^{-1}b)\mathcal{R}c,
\]
\[b(ab)(b^{-1}b)ab = bb^{-1}b = b,
\]
\[cab(ab)(b^{-1}b) = c.
\]
Hence, \(a(\cdot b^{-1}b) = b(ab)(b^{-1}b)\).

Similarly as (i), we prove parts (ii) and (iii). \(\square\)

Now, we will see that \(a\) is \((b, c)\)-invertible if and only if \(au^{-1}\) is \((ub, uc)\)-invertible (or \(u^{-1}a\) is \((bu, cu)\)-invertible).

\[\text{Theorem 2.8.}
\]
\[\text{Let } a, b, c \in \mathcal{R} \text{ and } u \in \mathcal{R}^{-1}. \text{ Then the following statement are equivalent:}
\]
\[\begin{align*}
\text{(i) } & a \text{ is } (b, c)\text{-invertible,} \\
\text{(ii) } & au^{-1} \text{ is } (ub, uc)\text{-invertible,} \\
\text{(iii) } & u^{-1}a \text{ is } (bu, cu)\text{-invertible.}
\end{align*}
\]

In addition, if any of statements (i)–(iii) holds, then
\[a(\cdot b^{-1}b) = u^{-1}(au^{-1})(\cdot b^{-1}b) = (u^{-1}a)(\cdot b^{-1}b)u^{-1},
\]
\[(au^{-1})(\cdot b^{-1}b) = au(\cdot b^{-1}b) \text{ and } (u^{-1}a)(\cdot b^{-1}b) = a(\cdot b^{-1}b)u.
\]

Proof. (i) \(\iff\) (ii): Observe that \(a\) is \((b, c)\)-invertible if and only if there exists \(y \in \mathcal{R}\) such that \(y = bt1y = ysc\), for some \(t, s \in \mathcal{R}\), \(yab = b\) and \(cay = c\) if and only if there exists \(y \in \mathcal{R}\) such that \(uy = (ub)(tu^{-1}uy) = (uy)(tu^{-1}uc)\), for some \(t, s \in \mathcal{R}\), \(uyac^{-1}ub = ub\) and \((uc)(ac^{-1})uy = uc\) which is equivalent to \(au^{-1}\) is \((ub, uc)\)-invertible.

(i) \(\iff\) (iii): It follows as (ii) \(\iff\) (ii). \(\square\)

In the cases that \(d\) is \((b, b)\)-invertible and/or \(e\) is \((c, c)\)-invertible, we characterize \((b, c)\)-invertible of \(a\) by \((b, c)\)-invertible of \(abd\), \(eca\) or \(ecab\).

\[\text{Theorem 2.9.}
\]
\[\text{Let } a, b, c, d, e \in \mathcal{R}.
\]
\[\begin{align*}
\text{(i) } & \text{If } d \text{ is } (b, b)\text{-invertible, then } a \text{ is } (b, c)\text{-invertible if and only if } aabd \text{ is } (b, c)\text{-invertible. Moreover, for } b^{-1} \in b[1],
\end{align*}
\]
\[\begin{align*}
(abd)(\cdot b^{-1}b) = a(\cdot b^{-1}b)b^{-1}a(\cdot b^{-1}b) \text{ and } a(\cdot b^{-1}b) = bd(abd)(\cdot b^{-1}b).
\end{align*}
\]
\[\begin{align*}
\text{(ii) } & \text{If } e \text{ is } (c, c)\text{-invertible, } a \text{ is } (b, c)\text{-invertible if and only if } aeca \text{ is } (b, c)\text{-invertible. Moreover, for } c^{-1} \in c[1],
\end{align*}
\]
\[\begin{align*}
(eca)(\cdot b^{-1}b) = a(\cdot b^{-1}b)c^{-1}e(\cdot c^{-1}c) \text{ and } a(\cdot b^{-1}b) = (eca)(\cdot b^{-1}b)e.
\end{align*}
\]
\[\begin{align*}
\text{(iii) } & \text{If } d \text{ is } (b, b)\text{-invertible and } e \text{ is } (c, c)\text{-invertible, then } a \text{ is } (b, c)\text{-invertible if and only if } aecab \text{ is } (b, c)\text{-invertible. Moreover, for } b^{-1} \in b[1] \text{ and } c^{-1} \in c[1],
\end{align*}
\]
\[\begin{align*}
(eca)(\cdot b^{-1}b) = a(\cdot b^{-1}b)b^{-1}d(\cdot b^{-1}b)c^{-1}e(\cdot c^{-1}c) \text{ and } a(\cdot b^{-1}b) = bd(ecab)(\cdot b^{-1}b)e.
\end{align*}
\]
Proof. (i) Assume that \( d \) is \((b, b)\)-invertible and \( a \) is \((b, c)\)-invertible. For \( b^- \in b[1] \) and \( c^- \in c[1] \), by
\[
d^{(b,b)} b^- a^{(b,c)} = bb^- d^{(b,b)} b^- a^{(b,c)} \in bRd^{(b,b)} b^- a^{(b,c)},
\]
we deduce that \( abd \) is \((b, c)\)-invertible and \((ab)^{(b,c)} = d^{(b,b)} b^- a^{(b,c)}\).

Conversely, let \( d \) be \((b, b)\)-invertible and \( abd \) be \((b, c)\)-invertible. Since, for \( b^- \in b[1] \) and \( c^- \in c[1] \),
\[
bd(abd)^{(b,c)} b^- b^{-}bd(abd)^{(b,c)} b^- \in bRbd(abd)^{(b,c)},
\]
we can prove parts (ii) and (iii) in the same manner. \( \square \)

Remark that the condition \( d \) is \((b, b)\)-invertible in Theorem 2.9 can be replaced with \( d \) is Mary invertible along \( b \). For details about the Mary inverse, see [7]. Notice that Theorem 2.9 recovers [10, Theorem 3.7].

In the following theorem, we investigate when the equality \( aa^{(b,c)} = a^{(b,c)} a \) is satisfied. If \( a^{(b,c)} \) satisfies \( a^{(b,c)} = a^{(b,c)} a \), then \( a^{(b,c)} \in R^a \) and \( (a^{(b,c)})^* = a^2 a^{(b,c)} \).

**Theorem 2.10.** Let \( a, b, c \in R \). If \( a \) is \((b, c)\)-invertible, then the following statements are equivalent:

(i) \( a^{(b,c)} = a^{(b,c)} a \),

(ii) there exist \( c^{-}(c^{-}, a^{(b,c)}) \) and \( b^{-}(b^{-}, a, b^{-}) \) such that \( c^{-}(c^{-}, a^{(b,c)}) = a^{(b,c)} a^{-} \) and \( b^{-}(b^{-}, a, b^{-}) = b^{-} a^{(b,c)} \), for \( b^- \in b[1] \) and \( c^- \in c[1] \),

(iii) there exist \( c^{(1,2),1} \) and \( b^{(1,2),1} \) such that \( c^{(1,2),1} = a^{(b,c)} a^{-} \) and \( b^{(1,2),1} = b^{-} a^{(b,c)} \), for \( b^- \in b[1] \) and \( c^- \in c[1] \).

**Proof.** (i) \( \Rightarrow \) (ii): Set \( x = a^{(b,c)} a^{-} \), for \( c^- \in c[1] \). The equality \( aa^{(b,c)} = a^{(b,c)} a \) implies
\[
c = cc^- c = cc^- c a^{(b,c)} \in cc^- Raa^{(b,c)},
\]
\[
x = a^{(b,c)} a^{-} = aa^{(b,c)} c c^- \in aa^{(b,c)} Rcc^-,
\]
\[
xc = ca^{(b,c)} a^{-} = ca^{(b,c)} c c^- = cc^-,
\]
\[
xc = a^{(b,c)} a^{-} c c^- = aa^{(b,c)} c c^- = aa^{(b,c)}.
\]

Thus, there exists \( c^{-}(c^{-}, a^{(b,c)}) = x \). Similarly, we check that \( b^{-}(b^{-}, a, b^{-}) \) exists and \( b^{-}(b^{-}, a, b^{-}) = b^{-} a^{(b,c)} \), for \( b^- \in b[1] \).

(ii) \( \Rightarrow \) (i): If \( c^{-}(c^{-}, a^{(b,c)}) = a^{(b,c)} a^{-} \) and \( b^{-}(b^{-}, a, b^{-}) = b^{-} a^{(b,c)} \), for \( b^- \in b[1] \) and \( c^- \in c[1] \), then
\[
aa^{(b,c)} = a^{(b,c)} a^{-} c = bb^- a^{(b,c)} a^{-} c = bb^- a^{(b,c)} = a^{(b,c)} a.
\]

(i) \( \Leftrightarrow \) (iii): In the similar way as (i) \( \Leftrightarrow \) (ii). \( \square \)

By Theorem 2.10, we obtain the next result.
Corollary 2.11. Let \( a, b, c \in \mathcal{R} \). If \( a \) is \((b, c)\)-invertible, \( cc^- = aa^{(b,c)} \) and \( b^-b = a^{(b,c)}a \), for \( b^-b \in b(1) \) and \( c^-c \in c(1) \), then the following statements are equivalent:

(i) \( a^{(b,c)} = a^{(b,c)}a \),

(ii) there exist \( c^b \) and \( b^c \) such that \( c^b = a^{(b,c)}ac^- \), \( c^b = 1 - cc^- \), \( b^c = b^-a^{(b,c)}a \) and \( b^c = 1 - b^-b \).

Now, we study equivalent conditions for the \((b, c)\)-inverse \( a^{(b,c)} \) to be an inner inverse of \( a \).

Theorem 2.12. Let \( a, b, c \in \mathcal{R} \). If \( a \) is \((b, c)\)-invertible, then the following statements are equivalent:

(i) \( a^{(b,c)}a = a \),

(ii) \( \mathcal{R} = b\mathcal{R} \oplus a^* \),

(iii) \( \mathcal{R} = \mathcal{R}c \oplus a^*a \).

Proof. Recall that \( a^{(b,c)}a = a \Leftrightarrow \mathcal{R} = a^{(b,c)}\mathcal{R} \oplus a^* \Leftrightarrow \mathcal{R} = \mathcal{R}a^{(b,c)} \oplus a^*a \). The rest follows by \( a^{(b,c)}\mathcal{R} = b\mathcal{R} \) and \( \mathcal{R}a^{(b,c)} = \mathcal{R}c \). \( \square \)

Theorem 2.13. Let \( a, b, c \in \mathcal{R} \). Then the following statements are equivalent:

(i) \( a \) is \((b, c)\)-invertible, and \( a^{(b,c)}a = a \),

(ii) \( a \in ab\mathcal{R} \), \( b \in ab\mathcal{R} \) and \( c \in ca\mathcal{R} \).

Proof. (i) \( \Rightarrow \) (ii): This follows by the definition of \((b, c)\)-inverse.

(ii) \( \Rightarrow \) (i): From the hypotheses, we have that

\[ a = a^1b_1 = a^2c_1, b = c_1a_1 \text{ and } c = c_1a_1. \]

Then \( b = t_3t_2cab \in Rca \) and \( c = cabb_1t_4 \in cabR \), which imply \( a \) is \((b, c)\)-invertible by [4, Theorem 2.2]. Also, \( a^{(b,c)}a = a^{(b,c)}a^1 = a^1b_1 = a \). \( \square \)

Theorem 2.14. Let \( a, b, c \in \mathcal{R} \). If \( a \) is \((b, c)\)-invertible, \( a^{(b,c)}a = a \), \( b^-b \in b(1) \) and \( c^-c \in c(1) \), then \( a^{(b,c)} = (c^-cabb^-)^{(1,2)} \). In addition, if \( bb^- = c^-c \), then \( c^-cabb^- \in \mathcal{R}^2 \) and \( a^{(b,c)} = (c^-cabb^-)^b \).

Proof. Since

\[ a^{(b,c)}c^-cabb^-a^{(b,c)} = a^{(b,c)}aa^{(b,c)} = a^{(b,c)}, \]
\[ c^-cabb^-a^{(b,c)}c^-cabb^- = c^-caaa^{(b,c)}abb^- = c^-cabb^-, \]
\[ a^{(b,c)}c^-cabb^- = a^{(b,c)}abb^- = bb^-, \]
\[ c^-cabb^-a^{(b,c)} = c^-caaa^{(b,c)} = c^-c, \]

we deduce that \( (c^-cabb^-)^{1,2}b_{b-,1-c-c} = a^{(b,c)} \). \( \square \)

One new representation for \( a^{(b,c)} \) is given now.

Theorem 2.15. Let \( a, b, c \in \mathcal{R} \). If \( a \) is \((b, c)\)-invertible and \( x \in (cab)(1) \), then \( a^{(b,c)} = bxc \).

Proof. By Lemma 1.3, \( b, c \) and \( cab \) are regular. Let \( x \in (cab)(1) \), \( b^-b \in b(1) \), \( c^-c \in c(1) \) and \( y = bxc \). Then \( y = bxc = bb^-bxc = bb^-y \) \( y \in b\mathcal{R} y \) and \( y = bxc = bxc^-c = yc^-c \in c\mathcal{R} y \). Since \( cabxcab = cab \), then \( abxcab - ab \in c^-c = (a^{(b,c)})^-c \). So, \( a^{(b,c)}abxcab = a^{(b,c)}ab \), i.e. \( yab = bxcab = b \). Also, by \( cabxca - ca \in c^-c = (a^{(b,c)})^-c \), we get \( cabxca^{(b,c)} = caa^{(b,c)} \), that is, \( cay = cabx = c \). Therefore, \( y = a^{(b,c)} \). \( \square \)

Next, we consider the reverse order law for the \((b, c)\)-inverse. 
Theorem 2.16. Let \( a, b, c, d \in \mathcal{R} \) be such that \( ab = ba \) and \( ac = ca \). If both \( a \) and \( d \) are \((b, c)\)-invertible, then \( ad \) is \((b, c)\)-invertible and \((ad)^{(b, c)} d = a^{(b, c)} a \).

Proof. Let \( y = d^{(b, c)} a^{(b, c)} \). Then we obtain that

\[
y = bb^{-1} d^{(b, c)} a^{(b, c)} \in b\mathcal{R}y \quad \text{and} \quad y = d^{(b, c)} a^{(b, c)} c^{-1} c \in y\mathcal{R}c.
\]

From the conditions \( ab = ba \) and \( ac = ca \), it follows that \( a^{(b, c)} a = a^{(b, c)} a \) by [5, Corollary 2.4(i)]. Then

\[
y(ad)b = d^{(b, c)} a^{(b, c)} d b = d^{(b, c)} a d^{(b, c)} d b = d^{(b, c)} c^{-1} (ca d^{(b, c)}) d b = d^{(b, c)} d b = b
\]

and

\[
c(ad)y = c a d^{(b, c)} a^{(b, c)} d = ac d^{(b, c)} a^{(b, c)} = a c a^{(b, c)} = c.
\]

This completes the proof of the theorem. \( \square \)

3. The image-kernel \((p, q)\)-invertible in rings

In this section, as an application of results proved in Section 2, we obtain new characterizations for the existence of the image-kernel \((p, q)\)-invertible in rings.

Applying Theorem 2.5, notice that \( a \in \mathcal{R} \) is \((p, q)\)-invertible if and only if \( a \) is image-kernel \((p, 1 - q)\)-invertible in the case that \( p, q \in \mathcal{R}^{*} \).

Corollary 3.1. Let \( a \in \mathcal{R} \) and \( p, q \in \mathcal{R}^{*} \). Then the following statements are equivalent:

(i) \( a \) is \((p, q)\)-invertible,

(ii) \( a \) is image-kernel \((p, 1 - q)\)-invertible.

Moreover, if one of the previous statements holds, then \( a^{(p, q)} = a^{x}_{p, 1 - q} \).

By Corollary 3.1 and Theorem 2.1, we get next equivalent conditions for the existence of the image-kernel \((p, q)\)-invertible.

Corollary 3.2. Let \( a \in \mathcal{R} \) and \( p, q \in \mathcal{R}^{*} \). Then the following statements are equivalent:

(i) \( a \) is image-kernel \((p, q)\)-invertible,

(ii) \((1 - q)ap \) is \((p, q)\)-reflexive generalized invertible,

(iii) \((1 - q)ap \) is \((-1, 1 - q, p)\)-invertible.

In addition, if one of the previous statements holds, then

\[
a^{x}_{p, q} = ((1 - q)ap)^{1, 2}_{p, q} (1 - q) = p ((1 - q)ap)^{1, 2}_{p, q},
\]

\[
((1 - q)ap)^{1, 2}_{p, q} = a^{x}_{p, q} (1 - q) = pa^{x}_{p, q} = ((1 - q)ap)^{-1}_{1 - q, p}.
\]

Using Corollary 3.2, notice that the following results hold.

Corollary 3.3. Let \( a \in \mathcal{R} \) and \( p \in \mathcal{R}^{*} \). Then the following statements are equivalent:

(i) \( a \) is image-kernel \((p, 1 - p)\)-invertible,

(ii) \( pap \in \mathcal{R}^{*} \) and \( (pap)^{n} = 1 - p \),

(iii) \( pap \in (p\mathcal{R}p)^{-1} \).
Corollary 3.4. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^*$. Then the following statements are equivalent:

(i) $a$ is $(p, 1 - q)$-reflexive generalized invertible,

(ii) $a$ is $(-q, p)$-invertible.

Corollary 3.5. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^*$. Then the following statements are equivalent:

(i) $a$ is image-kernel $(p, q)$-invertible,

(ii) $ap$ is image-kernel $(p, q)$-invertible,

(iii) $(1 - q)a$ is image-kernel $(p, q)$-invertible,

(iv) $(1 - q)ap$ is image-kernel $(p, q)$-invertible.

In addition, if one of the previous statements holds, then

$$a_{p,q}^\times = (ap)_{p,q}^\times = ((1 - q)a)_{p,q}^\times = ((1 - q)ap)_{p,q}^\times.$$

Corollary 3.6. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^*$. If $a$ is image-kernel $(p, q)$-invertible and $x, y \in \mathcal{R}$, then the following statements hold:

(i) $a + x(1 - p)$ is image-kernel $(p, q)$-invertible,

(ii) $a + qy$ is image-kernel $(p, q)$-invertible,

(iii) $a + x(1 - p) + qy$ is image-kernel $(p, q)$-invertible.

The set $\mathcal{R}_{p,q}^\times$ is fully described now.

Theorem 3.7. Let $p, q \in \mathcal{R}^*$.

(i) Then

$$\mathcal{R}_{p,q}^\times = \mathcal{R}^{(1-q)p} + q\mathcal{R}p + \mathcal{R}(1-p).$$

(ii) Also,

$$\mathcal{R}_{p,q}^\times = \mathcal{R}^{(1-q)p} + (1 - q)\mathcal{R}(1-p) + q\mathcal{R}.$$

We can get the next result as Theorem 2.9.

Corollary 3.8. Let $a, d, e \in \mathcal{R}$ and $p, q \in \mathcal{R}^*$.

(i) If $d$ is image-kernel $(p, 1 - p)$-invertible, then $a$ is image-kernel $(p, q)$-invertible if and only if $apd$ is image-kernel $(p, q)$-invertible. Moreover,

$$ (apd)_{p,q}^\times = a_{p,1-p}^\times p_{p,q}^\times, \quad \text{and} \quad a_{p,q}^\times = pd(a_{p,q}^\times).$$

(ii) If $e$ is image-kernel $(1 - q, q)$-invertible, $a$ is image-kernel $(p, q)$-invertible if and only if $e(1 - q)a$ is image-kernel $(p, q)$-invertible. Moreover,

$$ (e(1 - q)a)_{p,q}^\times = a_{p,1-q}^\times e_{1-q,q}^\times, \quad \text{and} \quad a_{p,q}^\times = (e(1 - q)a)_{p,q}^\times e(1 - q).$$

(iii) If $d$ is image-kernel $(p, 1 - p)$-invertible and $e$ is image-kernel $(1 - q, q)$-invertible, then $a$ is image-kernel $(p, q)$-invertible if and only if $e(1 - q)apd$ is image-kernel $(p, q)$-invertible. Moreover,

$$ (e(1 - q)apd)_{p,q}^\times = d_{p,1-p}^\times a_{p,q}^\times e_{1-q,q}^\times, \quad \text{and} \quad a_{p,q}^\times = pd(e(1 - q)apd)_{p,q}^\times e(1 - q).$$

As a consequence of Theorem 2.15, we have the following representation of $a_{p,q}^\times$.

Corollary 3.9. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^*$. If $a$ is image-kernel $(p, q)$-invertible and $x \in ((1 - q)ap)(1)$, then $a_{p,q}^\times = px(1 - q)$. 
References