Generalized Hyers-Ulam Stability for General Additive Functional Equations on Non-Archimedean Random Lie $C^\ast$-Algebras

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Abstract. In this paper, using the fixed point method, we prove some results related to the generalized Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random $C^\ast$-algebras and non-Archimedean random Lie $C^\ast$-algebras for the generalized additive functional equation

$$
\sum_{1 \leq i < j \leq n} f\left(\frac{x_i + x_j}{2}\right) + \sum_{i=1}^{n-2} x_i = \frac{(n-1)^2}{2} \sum_{i=1}^{n} f(x_i)
$$

where $n \in \mathbb{N}$ is a fixed integer with $n \geq 3$.

1. Introduction


In [34], Rassias and Kim introduced and investigated the following functional equation:

$$
\sum_{1 \leq i < j \leq n} f\left(\frac{x_i + x_j}{2}\right) + \sum_{i=1}^{n-2} x_i = \frac{(n-1)^2}{2} \sum_{i=1}^{n} f(x_i)
$$

where $n$ is a fixed integer with $n \geq 2$. We observe that in the case $n = 2$, the functional equation (1) yields the Jensen functional equation $2f((x+y)/2) = f(x) + f(y)$ and there are many interesting results concerning the
stability problems of the Jensen equation [19, 32, 33]. In [12], Jang and Saadati proved the generalized Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean C∗-algebras and non-Archimedean Lie C∗-algebras for the Jensen type functional equation \( f((x + y)/2) + f((x - y)/2) = f(x) \). For the case \( n = 3 \), Najati and Ranjbari [25] investigated homomorphisms between C∗-ternary algebras, and derivations on C∗-ternary algebras. In fact, in [34], the authors established the general solution of the functional equation (1) and investigated the generalized Hyers-Ulam stability problem of the functional equation (1) with \( n \geq 3 \) in quasi-\( \beta \)-normed spaces. In 2013, Kim et al. [18] proved some new Hyers-Ulam-Rassias stability results of \( n \)-Lie homomorphisms and Jordan \( n \)-Lie homomorphisms on \( n \)-Lie Banach algebras associated to the functional equation (1) using the fixed point method.

In this paper, using the fixed point method, we will investigate the generalized Hyers-Ulam stability results of homomorphisms and derivations in non-Archimedean random C∗-algebras and on non-Archimedean random Lie C∗-algebras for the additive functional equation (1) with \( n \geq 3 \).

2. Preliminaries

In this section, we adopt the usual terminology, notions and conventions of the theory of non-Archimedean random normed space as in [3–5, 16, 17, 20, 29, 36, 37]. Throughout this paper, \( \Delta^+ \) is the space of all probability distribution functions, i.e., the space of all mappings \( F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1] \) such that \( F \) is left-continuous and non-decreasing on \( \mathbb{R}, F(0) = 0 \) and \( F(+\infty) = 1 \). \( D^+ \) is a subset of \( \Delta^+ \) consisting of all functions \( F \in \Delta^+ \) for which \( \lim_{x \to +\infty} F(x) = 1 \), where \( \lim_{x \to +\infty} f(x) \) denotes the left limit of the function \( f \) at the point \( x \). That is, \( \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} f(t) \). The space \( \Delta^+ \) is partially ordered by the usual point-wise order of functions, i.e., \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \). The maximal element for \( \Delta^+ \) in this order is the distribution function \( \varepsilon_0 \) given by

\[
\varepsilon_0(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
1, & \text{if } t > 0.
\end{cases}
\]

**Definition 2.1.** (cf. [36]). A mapping \( T : [0, 1] \times [0, 1] \to [0, 1] \) is a continuous triangular norm (briefly, a continuous t-norm) if \( T \) satisfies the following conditions:

1. \( T \) is commutative and associative;
2. \( T \) is continuous;
3. \( T(a, 1) = a \) for all \( a \in [0, 1] \);
4. \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Typical examples of continuous t-norms are the Lukasiewicz t-norm \( T_L \), where \( T_L(a, b) = \max(a + b - 1, 0) \), \( \forall a, b \in [0, 1] \) and the t-norms \( T_P, T_M, T_D \), where \( T_P(a, b) := ab, T_M(a, b) := \min(a, b), T_D(a, b) := \begin{cases} \min(a, b), & \text{if } \max(a, b) = 1, \\
0, \quad \text{otherwise.}
\end{cases} \)

By a non-Archimedean field we mean a field \( \mathbb{K} \) equipped with a function (valuation) \( | \cdot | \) from \( \mathbb{K} \) into \([0, \infty) \) such that \( |r| = 0 \) if and only if \( r = 0 \), \(|rs| = |r||s|\), and \(|r + s| \leq \max(|r|, |s|)\) for \( r, s \in \mathbb{K} \). Clearly \(|1| = |\infty| = 1\) and \(|n| \leq 1 \) for all \( n \in \mathbb{N} \). By the trivial valuation we mean the function \(| \cdot |\) taking everything but 0 into 1 and \(|0| = 0 \) (i.e., the function \(| | \) is called the trivial valuation if \(|r| = 1, \forall r \in \mathbb{R}, r \neq 0, \) and \(|0| = 0 \). Let \( X \) be a vector space over a field \( \mathbb{K} \) with a non-Archimedean non-trivial valuation \(| \cdot |\). A function \( \| \cdot \| : X \to [0, \infty) \) is called a non-Archimedean norm if it satisfies the following conditions:

(i) \( \|x\| = 0 \) if and only if \( x = 0 \);
(ii) For any \( r \in \mathbb{K} \) and \( x \in X, \|rx\| = |r||x| \);
(iii) For all \( x, y \in X, \|x + y\| \leq \max(\|x\|, \|y\|) \) (the strong triangle inequality).
Then \( (X, \| \cdot \|) \) is called a non-Archimedean normed space. Due to the fact that

\[
\|x_n - x_m\| \leq \max(\|x_{j+1} - x_j\|) : m \leq j \leq n - 1, \quad (n > m),
\]
a sequence \( \{x_n\} \) is Cauchy if and only if \( \{x_{n+1} - x_n\} \) converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

**Example 2.2.** (cf. [14]). For any non-zero rational number \( x \), there exists a unique integer \( n_x \in \mathbb{Z} \) such that \( x = \frac{a}{b} p^{-n_x} \), where \( a \) and \( b \) are integers not divisible by \( p \). Then \( x | p := p^{-n_x} \) defines a non-Archimedean norm on \( \mathbb{Q} \). The completion of \( \mathbb{Q} \) with respect to the metric \( d(x, y) = |x - y|_p \) is denoted by \( \mathbb{Q}_p \), which is called the \( p \)-adic number field.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra \( \mathcal{A} \) which satisfies \( ||ab|| \leq ||a|| ||b|| \) for all \( a, b \in \mathcal{A} \). For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [8, 38].

If \( \mathcal{U} \) is a non-Archimedean Banach algebra, then an involution on \( \mathcal{U} \) is a mapping \( t \to t^* \) from \( \mathcal{U} \) into \( \mathcal{U} \) which satisfies

<table>
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<tr>
<th>Condition</th>
<th>Description</th>
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<tbody>
<tr>
<td>(I) ( t^{**} = t ) for ( t \in \mathcal{U} );</td>
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<tr>
<td>(II) ( (as + \beta t)^* = \overline{a} s + \beta t^* );</td>
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<tr>
<td>(III) ( (st)^* = t^* s^* ) for ( s, t \in \mathcal{U} ).</td>
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If, in addition, \( ||t^* || = ||t||^2 \) for \( t \in \mathcal{U} \), then \( \mathcal{U} \) is a non-Archimedean \( C^* \)-algebra.

**Definition 2.3.** (cf. [14, 37]). A non-Archimedean random normed space (briefly, NA-RN-space) is a triple \((X, \mu, T)\), where \( X \) is a linear space over a non-Archimedean field \( \mathbb{K} \), \( T \) is a continuous \( t \)-norm, and \( \mu \) is a mapping from \( X \) into \( D^* \) such that the following conditions hold:

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<tr>
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<tr>
<td>(NA-RN1) ( \mu_x(t) = \varepsilon_0(t) ) for all ( t &gt; 0 ) if and only if ( x = 0 );</td>
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<tr>
<td>(NA-RN2) ( \mu_{xt}(t) = \mu_x \left( \frac{t}{</td>
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<td>(NA-RN3) ( \mu_{x+y}(\max(t, s)) \geq T(\mu_x(t), \mu_y(s)) ) for all ( x, y \in X ) and ( t, s \geq 0 );</td>
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It is easy to see that if (NA-RN3) holds, then

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<tr>
<td>(RN3) ( \mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s)) ).</td>
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**Example 2.4.** (cf. [26]). Let \( (X, \| \cdot \|) \) be a non-Archimedean normed linear space, and \( \alpha, \beta > 0 \). Define

\[
\mu_x(t) = \frac{\alpha t}{\alpha t + \beta \|x\|}
\]

for all \( x \in X \) and \( t > 0 \). Then \( (X, \mu, T_M) \) is a non-Archimedean RN-space.

**Proof.** (NA - RN1) is obviously true. Notice that for any \( t \in \mathbb{R}, t > 0 \) and \( c \neq 0 \)

\[
\mu_{cx}(t) = \frac{\alpha t}{\alpha t + \beta \|cx\|} = \frac{\alpha t}{\alpha t + \beta |c| \|x\|} = \frac{\alpha \cdot \frac{t}{|c|}}{\alpha \cdot \frac{t}{|c|} + \beta \|x\|} = \mu_x \left( \frac{t}{|c|} \right),
\]

which implies that (NA - RN2) holds.

To prove (NA - RN3). We assume that \( \mu_x(t) \leq \mu_y(s) \), thus we have

\[
\frac{\|y\|}{s} \leq \frac{\|x\|}{t}.
\]

Now, if \( \|x\| \geq \|y\| \) for all \( x, y \in X \), then we have by the strong triangle inequality

\[
t \|x + y\| \leq t \|x\| \leq (\max(t, s)) \|x\|.
\]

Therefore,

\[
\frac{\beta \|x + y\|}{\alpha (\max(t, s))} \leq \frac{\beta \|x\|}{\alpha t}
\]

and so

\[
1 + \frac{\beta \|x + y\|}{\alpha (\max(t, s))} \leq 1 + \frac{\beta \|x\|}{\alpha t}.
\]
which implies that \(\mu_{s+y}(\max(t, s)) \geq \mu_s(t)\).

if \(\|x\| \leq \|y\|\) for all \(x, y \in X\), then we also have

\[
\|x + y\| \leq \|y\| \leq t \cdot \frac{s}{t} \|x\| \leq (\max(t, s))\|x\|.
\]

By the same way to the above, we can also get \(\mu_{s+y}(\max(t, s)) \geq \mu_s(t)\). Hence, \(\mu_{s+y}(\max(t, s)) \geq T_M(\mu_s(t), \mu_y(s))\) for all \(x, y \in X\) and \(t, s \geq 0\). Then \((X, \mu, T_M)\) is a non-Archimedean RN-space. \(\Box\)

**Example 2.5.** (cf. [26]). Let \((X, \| \cdot \|)\) be a non-Archimedean normed linear space, let \(\beta > \alpha > 0\) and

\[
\mu_s(t) = \begin{cases} 
0, & t \leq \alpha\|x\|, \\
\frac{t}{\beta - \alpha}, & \alpha\|x\| < t \leq \beta\|x\|, \\
1, & t > \beta\|x\|.
\end{cases}
\]

Then \((X, \mu, T_M)\) is a non-Archimedean RN-space.

**Proof.** (NA – RN1) is obviously true. Notice that for \(c \neq 0\), if \(\mu_{cx}(t) = 1\), then \(t > \beta\|cx\|\), i.e. \(\frac{t}{\beta} > \beta\|x\|\) \(\Rightarrow \mu_{c}(\frac{t}{\beta}) = 1\).

Thus \(\mu_{cx}(t) = \mu_{c}(\frac{t}{\beta})\).

Again if \(\mu_{cx}(t) = \frac{t}{\beta - \alpha}\|cx\|\), then \(\alpha\|cx\| < t \leq \beta\|cx\|\), i.e. \(\alpha\|x\| < \frac{t}{\beta} \leq \beta\|x\|\), so we have

\[
\mu_{c}(\frac{t}{\beta}) = \frac{t}{\beta - \alpha}\|x\|,
\]

therefore, \(\mu_{cx}(t) = \mu_{c}(\frac{t}{\beta})\). Similarly, when \(\mu_{cx}(t) = 0\), then \(\mu_{cx}(t) = \mu_{c}(\frac{t}{\beta}) = 0\). Thus for \(c \neq 0\), \(\mu_{cx}(t) = \mu_{c}(\frac{t}{\beta})\)

which implies that (NA – RN2) holds.

Next, we have to show that

\[\mu_{s+y}(\max(t, s)) \geq T_M(\mu_s(t), \mu_y(s)).\]

If \(s = t = 0\), then in this case the relation is obvious. So we consider the case when \(t > 0, s > 0\).

If \(t > \beta\|x\|-\beta\|y\|\), \(\max(t, s) > \beta\|x\|\), \(\max(t, s) > \beta\|y\|\), and \(\mu_{s}(t) = 1, \mu_{y}(s) = 1\). Now, we have

\[\max(t, s) \geq \beta\|x\| (\text{ or } \beta\|y\|) \geq \beta(\|x\| + \|y\|)\]

Hence, we get

\[\mu_{s+y}(\max(t, s)) = 1 \Rightarrow \mu_{s+y}(\max(t, s)) \geq T_M(\mu_s(t), \mu_y(s)).\]

If \(t > \beta\|x\|\), and \(\alpha\|y\| < s \leq \beta\|y\|\), then \(\mu_{s}(t) = 1, \mu_{y}(s) = \frac{s}{\beta - \alpha}\|y\|\). Now, if \(\|x\| \geq \|y\|\), then we obtain

\[\max(t, s) \geq \beta\|x\| = \max(\beta\|x\|, \beta\|y\|) \geq \beta(\|x\| + \|y\|)\]

Hence, we have

\[\mu_{s+y}(\max(t, s)) = 1 \Rightarrow \mu_{s+y}(\max(t, s)) \geq T_M(\mu_s(t), \mu_y(s)).\]

Next, if \(\|y\| \geq \|x\|\). So we get

\[\max(t, s) \geq \alpha\|y\| = \max(\alpha\|x\|, \alpha\|y\|) = \alpha(\|x\| + \|y\|)\]

Hence, we get

\[\mu_{s+y}(\max(t, s)) = \frac{\max(t, s)}{\max(t, s) + (\beta - \alpha)\|x\| + \|y\|} \Rightarrow \mu_{s+y}(\max(t, s)) \geq T_M(\mu_s(t), \mu_y(s)).\]

If \(\alpha\|x\| < t \leq \beta\|x\|\), and \(\alpha\|y\| < s \leq \beta\|y\|\), then in this case the relation is similar to the proof of Example 2.4, and thus it is omitted. This completes the proof of the example. \(\Box\)
Definition 2.6. (cf. [14, 23]). A non-Archimedean random normed space \((X, \mu, T, T')\) is a non-Archimedean random normed space \((X, \mu, T)\) with an algebraic structure such that

\[ \mu_{xy}(t) \geq T'(\mu_x(t), \mu_y(t)) \text{ for all } x, y \in X \text{ and all } t > 0, \text{ in which } T' \text{ is a continuous } t\text{-norm.} \]

Example 2.7. (cf. [23]). Let \((X, \| \cdot \|)\) be a non-Archimedean normed algebra. Define

\[ \mu_x(t) = \begin{cases} 0, & x \neq 0, t \leq 0, \\ \frac{1}{t^{|t|}}, & x \neq 0, t > 0, \\ 1, & x = 0 \end{cases} \]

Then \((X, \mu, T_M)\) is a non-Archimedean RN-space. An easy computation shows that \(\mu_{xy}(t) \geq \mu_x(t)\mu_y(t)\) if and only if

\[ \|xy\| \leq \|x\|\|y\| + t\|y\| + t\|x\| \]

for all \(x, y \in X\) and \(t > 0\). It follows that \((X, \mu, T_M, T_P)\) is a non-Archimedean random normed algebra.

Definition 2.8. (cf. [14]). Let \((X, \mu, T, T')\) and \((Y, \nu, T', T'')\) be non-Archimedean random normed algebras.

(a) An \(R\)-linear mapping \(f : X \to Y\) is called a homomorphism if \(f(xy) = f(x)f(y)\) for all \(x, y \in X\).

(b) An \(R\)-linear mapping \(f : X \to Y\) is called a derivation if \(f(xy) = xf(y) + f(x)y\) for all \(x, y \in X\).

Definition 2.9. (cf. [14]). Let \((\mathcal{U}, \mu, T, T')\) be non-Archimedean random Banach algebra, then an involution on \(\mathcal{U}\)

is a mapping \(u \to u'\) from \(\mathcal{U}\) into \(\mathcal{U}\) which satisfies

(I') \(u'' = u\) for \(u \in \mathcal{U}\);

(II') \((au + bv)' = \bar{a}u' + \bar{b}v'\);

(III') \((uv)' = v'u'\) for \(u, v \in \mathcal{U}\).

If, in addition, \(\mu_{u'v'}(t) = T'(\mu_x(t), \mu_y(t))\) for \(u \in \mathcal{U}\) and \(t > 0\), then \(\mathcal{U}\) is a non-Archimedean random \(C^*\)-algebra.

Definition 2.10. (cf. [14]) Let \((X, \mu, T)\) be a non-Archimedean RN-space. Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is said to be convergent if there exists \(x \in X\) such that

\[ \lim_{n \to \infty} \mu_{x_n-x}(t) = 1, \]

for all \(t > 0\). In this case, \(x\) is called the limit of the sequence \(\{x_n\}\).

A sequence \(\{x_n\}\) in \(X\) is called Cauchy if for each \(\varepsilon > 0\) and \(t > 0\), there exists \(n_0\) such that for all \(n \geq n_0\) and all \(p > 0\) we have \(\mu_{x_n-x_p}(t) > 1 - \varepsilon\). Due to

\[ \mu_{x_n-x_p}(t) \geq \min\{\mu_{x_n-x_p-1}(t), \ldots, \mu_{x_n-1}(t)\}. \]

Therefore, the sequence \(\{x_n\}\) is Cauchy if for each \(\varepsilon \geq 0\) and \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\), we have \(\mu_{x_n-x}(t) > 1 - \varepsilon\).

If each Cauchy sequence is convergent, then the random norm is said to be complete, and the non-Archimedean RN-space is called a non-Archimedean random Banach space.

Definition 2.11. Let \(S\) be a set. A function \(d : S \times S \to [0, \infty]\) is called a generalized metric on \(S\) if \(d\) satisfies

1. \(d(x, y) = 0\) if and only if \(x = y\);
2. \(d(x, y) = d(y, x), \forall x, y \in S\);
3. \(d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in S.\)

The next Lemma 2.12 is due to Diaz and Margolis [6], which is extensively applied to the stability theory of functional equations.
Lemma 2.12. ([6]). Let $(S,d)$ be a complete generalized metric space and $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each fixed element $x \in S$, either
\[ d(J^n x, J^{n+1} x) = \infty \]
for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that
(i) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
(ii) the sequence $\{ J^n x \}$ is convergent to a fixed point $y^*$ of $J$;
(iii) $y^*$ is the unique fixed point of $J$ in the set $S^* := \{ y \in S : d(J^n x, y) < +\infty \}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y^*, y)$, $\forall y \in S^*$.

3. Stability of homomorphisms and derivations in non-Archimedean random $C^*$-algebras

In this section, assume that $\mathcal{A}$ is a non-Archimedean random $C^*$-algebra with the norm $\mu^\mathcal{A}$ and that $\mathcal{B}$ is a non-Archimedean random $C^*$-algebra with the norm $\mu^\mathcal{B}$. For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define
\[ D_{\lambda,f}(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} f \left( \frac{\lambda x_i + \lambda x_j}{2} + \sum_{l=1}^{n-2} \lambda x_k \right) - \frac{(n-1)^2}{2} \sum_{i=1}^{n} \lambda f(x_i) \]
for all $x_1, \ldots, x_n \in \mathcal{A}(n \geq 3)$ and $\lambda \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.

We need the following lemmas to prove the main results.

Lemma 3.1. (cf. [24]). Let $V$ and $W$ be linear spaces and let $n \geq 3$ be a fixed positive integer. A mapping $f : V \to W$ satisfies the functional equation (1) for all $x_1, \ldots, x_n \in V$ if and only if $f$ is an additive mapping.

Lemma 3.2. (cf. [28]). Let $f : \mathcal{A} \to \mathcal{A}$ be an additive mapping such that $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Then the mapping $f$ is $\mathbb{C}$-linear.

Note that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called homomorphism in non-Archimedean random $C^*$-algebras if $H$ satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$.

Now we are going to prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random $C^*$-algebras for the functional equation $D_{\lambda,f}(x_1, \ldots, x_n) = 0$.

Theorem 3.3. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$ and $\eta : \mathcal{A} \to D^+$ such that $|\rho| < 1$ is far from zero and
\[ \mu^\mathcal{B}_{D_{\lambda,f}(x_1, \ldots, x_n)}(t) \geq \varphi_{x_1, \ldots, x_n}(t) \]
(2)
\[ \mu^\mathcal{B}_{f(xy) - f(x)f(y)}(t) \geq \psi_{x,y}(t) \]
(3)
\[ \mu^\mathcal{B}_{f(x^*) - f(x)^*}(t) \geq \eta_x(t) \]
(4)
for all $\lambda \in \mathbb{T}^1$, $x_1, \ldots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists a constant $0 < L < 1$ such that
\[ \varphi_{\rho x_1, \ldots, \rho x_n}(lt) \geq \varphi_{x_1, \ldots, x_n}(t) \]
(5)
\[ \psi_{\rho x,y}(l^2 t) \geq \psi_{x,y}(t) \]
(6)
\[ \eta_{\rho x}(lt) \geq \eta_x(t) \]
(7)
for all $x, y, x_1, \ldots, x_n \in \mathcal{A}$ and $t > 0$, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that
\[ \mu^\mathcal{B}_{f(x) - H(x)}(t) \geq \varphi_{x_{\rho x}} \left( \frac{|\rho|^2 (1-L)}{2} t \right) \]
(8)
for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$. 

References

This implies that the inequality (8) holds. It follows from (2), (5) and (13) that
\[
\mu_{D_{\lambda,H}}(x_1, \ldots, x_n)(t) = \lim_{m \to \infty} \mu_{D_{\lambda,H}}(\rho^{m} x_1, \ldots, \rho^{m} x_n)(t)
\geq \lim_{m \to \infty} \varphi_{\rho^{m} x_1, \ldots, \rho^{m} x_n}(\rho^{m} t) = 1
\]
for all \( \lambda \in \mathbb{T}^1, x_1, \ldots, x_n \in \mathcal{A} \) and \( t > 0 \). Hence, we obtain
\[
D_{\lambda,H}(x_1, \ldots, x_n) = 0
\]
for all \( x_1, \ldots, x_n \in \mathcal{A} \). If we put \( \lambda = 1 \) in (15), then \( H \) is additive by Lemma 3.1. Also, letting \( x_1 = \cdots = x_n = x \) in the last equality, we obtain \( H(\lambda x) = \lambda H(x) \). Now by using Lemma 3.2, we infer that the mapping \( H \) is \( C \)-linear. On the other hand, it follows from (3), (6) and (13) that

\[
H_{[H(x) - H(y)]}(t) = \lim_{m \to \infty} H_{[H(x^m) - H(y^m)]}(|p|^{2m} t) \\
\geq \lim_{m \to \infty} \psi_{p^m, x^m, y^m}(|p|^{2m} t) = 1
\]

for all \( x, y \in \mathcal{A} \). So, \( H(xy) = H(x)H(y) \) for all \( x, y \in \mathcal{A} \). Thus \( H : \mathcal{A} \to \mathcal{B} \) is a homomorphism satisfying (8), as desired. Also, by (4), (7) and (13) and by a similar method, we have \( H(x^r) = H(x)^r \). This completes the proof of the theorem. \( \square \)

**Theorem 3.4.** Let \( f : \mathcal{A} \to \mathcal{B} \) be a mapping for which there are functions \( \varphi : \mathcal{A}^n \to D^* \), \( \psi : \mathcal{A}^2 \to D^* \) and \( \eta : \mathcal{A} \to D^* \) such that \( |p| < 1 \) is far from zero, and (2), (3) and (4) hold for all \( \lambda \in \mathbb{T}^1 \), \( x_1, \ldots, x_n, x, y \in \mathcal{A} \) and \( t > 0 \). If there exists a constant \( 0 < L < 1 \) such that

\[
\varphi_{x_1, \ldots, x_n}(L) \geq \varphi_{x_1, \ldots, x_n}(t) \quad (16)
\]

\[
\psi_{x, y}(L) \geq \psi_{x, y}(t) \quad (17)
\]

\[
\eta_x(L) \geq \eta_x(t) \quad (18)
\]

for all \( x, y, x_1, \ldots, x_n \in \mathcal{A} \) and \( t > 0 \), then there exists a unique homomorphism \( H : \mathcal{A} \to \mathcal{B} \) such that

\[
\mu_{f(x)}^{B_{[H(x)]}}(t) \geq \varphi_{x_1, \ldots, x_n}(\frac{|p||p|^2(1 - L)}{2|L|} t) \quad (19)
\]

for all \( x \in \mathcal{A} \) and \( t > 0 \), where \( p := n - 1 \).

**Proof.** Let \( \Omega \) and \( d \) be as in the proof of Theorem 3.3. Then \( (\Omega, d) \) becomes complete generalized metric space and the mapping \( \mathcal{J} : \Omega \to \Omega \) defined by

\[\mathcal{J} g(x) := \rho g \left( \frac{x}{\rho} \right), \text{ for all } g \in \Omega \text{ and } x \in \mathcal{A}.\]

Then, it is easy to see that \( d(\mathcal{J} g, \mathcal{J} h) \leq L d(g, h) \) for all \( g, h \in \mathcal{S} \). By (9) and (16), we obtain

\[
\mu_{f(x)}^{B_{[H(x)]}} \left( \frac{|2L|}{|p||p|^2} t \right) \geq \varphi_{x_1, \ldots, x_n}(L) \geq \varphi_{x_1, \ldots, x_n}(t)
\]

for all \( x \in \mathcal{A} \) and \( t > 0 \). So, we have \( d(f, \mathcal{J} f) \leq \frac{2L}{|p||p|^2} \).

The remaining assertion is similar to the corresponding part of Theorem 3.3. This completes the proof. \( \square \)

**Corollary 3.5.** Let \( \ell \in \{-1, 1\}, r 
eq 1 \) and \( \theta \) be nonnegative real numbers. Suppose that \( f : \mathcal{A} \to \mathcal{B} \) be a mapping such that

\[
\mu_{f(x), x_1, \ldots, x_n}^{B_{[H(x)]}}(t) \geq \frac{t}{t + \theta(|x_1| + \cdots + |x_n|)}
\]

\[
\mu_{f(x) - f(x), y}^{B_{[H(x)]}}(t) \geq \frac{t}{t + \theta |y|}
\]

\[
\mu_{f(x') - f(x), y}^{B_{[H(x)]}}(t) \geq \frac{t}{t + \theta |y|}
\]
for all $\lambda \in \mathbb{T}_1$, $x_1, \ldots, x_n, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that, if $\ell r > t$,

$$
\mu^B_{f(x), H(x)}(t) \geq \frac{\ell \rho((|\lambda| - |\rho|)\ell + \ell \rho(|\lambda| - |\rho|)\ell + \ell \rho \rho | \lambda^\prime | \rho} {\ell \rho((|\lambda| - |\rho|)\ell + \ell \rho(|\lambda| - |\rho|)\ell + \ell \rho \rho | \lambda^\prime | \rho} \tag{20}
$$

for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$.

**Proof.** The proof follows from Theorems 3.3 and 3.4 by taking

$$
\varphi_{x_1, \ldots, x_n}(t) = \frac{t}{l + \theta |x_1|_{\mathcal{A}} + |x_2|_{\mathcal{A}} + \cdots + |x_n|_{\mathcal{A}}},
$$
$$
\psi_{x, y}(t) = \frac{t}{l + \theta |x|_{\mathcal{A}} |y|_{\mathcal{A}}},
$$
$$
\eta_{x}(t) = \frac{t}{l + \theta |x|_{\mathcal{A}}}
$$

for all $x_1, \ldots, x_n, y \in \mathcal{A}$ and $t > 0$. We can choose $L = |\rho|^{(l-1)}$, we obtain the desired result. □

Note that a $\mathcal{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called derivation on $\mathcal{A}$ if $\delta$ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random $C^*$-algebras for the functional equation $D_{\lambda, f}(x_1, \ldots, x_n) = 0$.

**Theorem 3.6.** Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^*$, $\psi : \mathcal{A}^2 \to D^*$ and $\eta : \mathcal{A} \to D^*$ such that $|\rho| < 1$ is far from zero and

$$
\mu^\mathcal{A}_{D_{\lambda, f}(x_1, \ldots, x_n)}(t) \geq \varphi_{x_1, \ldots, x_n}(t)
$$

$$
\mu^\mathcal{A}_{f(xy) - f(x)y + f(x)y}(t) \geq \psi_{x, y}(t)
$$

$$
\mu^\mathcal{A}_{f(\lambda x) - f(x)}(t) \geq \eta_{x}(t)
$$

for all $\lambda \in \mathbb{T}_1$, $x_1, \ldots, x_n, y \in \mathcal{A}$ and $t > 0$. If there exits a constant $0 < L < 1$ such that (5), (6) and (7) hold, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$
\mu^\mathcal{A}_{f(x) - \delta(x)}(t) \geq \varphi_{x_1, \ldots, x_n}(t) \left(\frac{1 - L |\rho|^{1 - L} t}{|\rho|^{1 - L} t} \right)
$$

for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$.

**Proof.** By the same reasoning as in the proof of Theorem 3.3, the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by

$$
\delta(x) := \lim_{m \to \infty} \frac{1}{|\rho|^m} f(\rho^m x) \quad \forall x \in \mathcal{A}
$$

(25)

is a unique $\mathcal{C}$-linear mapping which satisfies (24). We show that $\delta$ is a derivation. By (22) and (25), we have

$$
\mu^\mathcal{A}_{\delta(xy) - \delta(x)y - \delta(y)}(t) = \lim_{m \to \infty} \mu^\mathcal{A}_{f(\rho^m xy - f(\rho^m x)y - f(\rho^m y)x)(|\rho|^{2mt})}
$$

$$
\geq \lim_{m \to \infty} \psi_{x, y}(t) |\rho|^{2mt} = 1
$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Hence we have $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. This means that $\delta$ is a derivation satisfying (24). This completes the proof. □
4. Stability of homomorphisms and derivations in non-Archimedean random Lie $C^*$-algebras

A non-Archimedean random $C^*$-algebra $C$, endowed with the Lie product $[x, y] = \frac{yx - xy}{2}$ on $C$, is called a non-Archimedean random Lie $C^*$-algebra.

**Definition 4.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean random Lie $C^*$-algebras. A $C$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a non-Archimedean random Lie $C^*$-algebra homomorphism if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathcal{A}$.

In this section, assume that $\mathcal{A}$ is a non-Archimedean random Lie $C^*$-algebra with the norm $\mu^A$ and that $\mathcal{B}$ is a non-Archimedean random Lie $C^*$-algebra with the norm $\mu^B$.

Now, we prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie $C^*$-algebras for the equation $D_{\lambda,f(x_1, \ldots, x_n)} = 0$.

**Theorem 4.2.** Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$ and $\eta : \mathcal{A} \to D^+$ such that $|\varphi| < 1$ is far from zero, (2) and (4) hold and

$$\mu^B_{\varphi(x,y)-\{f(x,y)\}}(f)(l) \geq \psi(x,y)(l)$$

for all $x, y \in \mathcal{A}$ and $t > 0$. If there exists a constant $0 < L < 1$ and (5), (6) and (7) hold, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that (8) holds for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$.

**Proof.** By the same reasoning as in the proof of Theorem 3.3, we can find the mapping $H : \mathcal{A} \to \mathcal{B}$ given by

$$H(x) := \lim_{m \to \infty} \frac{1}{|\varphi|^m} f(x^m)$$

for all $x \in \mathcal{A}$. It follows from (6), (26) and (27) that

$$\mu^B_{\varphi(x,y)-\{H(x),H(y)\}}(f)(l) = \lim_{m \to \infty} \mu^B_{\varphi(x,y)-\{f(x,y)\}}(f)(l) = 1$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then

$$H([x, y]) = [H(x), H(y)]$$

for all $x, y \in \mathcal{A}$. Thus, $H : \mathcal{A} \to \mathcal{B}$ is a Lie $C^*$-algebra homomorphism satisfying (8), as desired. \( \Box \)

**Theorem 4.3.** Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$ and $\eta : \mathcal{A} \to D^+$ such that $|\varphi| < 1$ is far from zero, and (2), (4) and (26) hold for all $\lambda \in T^1$, $x_1, \ldots, x_n, y \in \mathcal{A}$ and $t > 0$. If there exists a constant $0 < L < 1$ and (16), (17) and (18) hold, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that (19) holds for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$.

**Proof.** The proof follows from Theorem 3.4 and a method similar to Theorem 4.2. \( \Box \)

**Corollary 4.4.** Let $\ell \in [-1, 1]$, $r = \# \theta$ be nonnegative real numbers. Suppose that $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that

$$\mu^B_{D_{\lambda,f(x_1, \ldots, x_n)}}(l) \geq \frac{l}{t + \theta (\|x_1\|_{\mathcal{A}} + \|x_2\|_{\mathcal{A}} + \cdots + \|x_n\|_{\mathcal{A}})}$$

$$\mu^B_{f(x,y)-\{f(x,y)\}}(l) \geq \frac{l}{t + \theta \cdot (\|x\|_{\mathcal{A}} + \|y\|_{\mathcal{A}})}$$

$$\mu^B_{f(x)\cdot f(y)}(l) \geq \frac{l}{t + \theta \cdot \|x\|_{\mathcal{A}}}$$

for all $\lambda \in T^1$, $x_1, \ldots, x_n, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that (20) holds.
Proof. The proof follows from Theorems 4.2 and 4.3, and a method similar to Corollary 3.5. □

Definition 4.5. Let $\mathcal{A}$ be non-Archimedean random Lie $C^*$-algebra. A $C$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random Lie $C^*$-algebras for the functional equation $D_\rho(x, y) = L^\rho(x, y)(t) + \psi(y, t)x + \psi(x, t)y = 1$ for all $x, y \in \mathcal{A}$ and $t > 0$. If there exists a constant $0 < L < 1$ such that (5), (6) and (7) hold, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that (24) holds for all $x \in \mathcal{A}$ and $t > 0$, where $\rho := n - 1$.

Proof. By the same reasoning as in the proof of Theorem 4.2, we can find the mapping $\delta : \mathcal{A} \to \mathcal{B}$ given by

$$\delta(x) := \lim_{m \to \infty} \frac{1}{|\rho|^m} f(p^m x)$$

for all $x \in \mathcal{A}$. It follows from (6), (28) and (29) that

$$f_{\infty}(\rho, x) = \lim_{m \to \infty} f((\rho, x, \rho, x, \ldots, x, \rho, x))^m (\rho^m t) \geq \lim_{m \to \infty} \psi(y, t)x + \psi(x, t)y = 1$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathcal{A}$. Thus, $\delta : \mathcal{A} \to \mathcal{A}$ is a Lie derivation satisfying (24), as desired. □

Acknowledgements: This research work was done during 2015-16 while the first author studied at the University of Louisville as a Visiting Scholar from the Hubei University of Technology.

References


